— LECTURE 21 —

* Finish last lecture: (← THIS took most of the lecture!)

- construction of Virasoro irr. repr-s at discrete series via the coset construction applied to $\widehat{\mathfrak{sl}}_2$-modules L_1 and L_λ ($\lambda \in P_+$) and thus proving their unitarity!;
- completion of the proof of the Shapovalov determinant formula for Virasoro by proving $\det_m(c, h_{r,s}(c)) = 0$ for $m \geq rs$

* Last time, all our computations were crucially based on the Weyl-Kac character f-la and the explicit description of the Weyl gp and its action for $\mathfrak{g}(A) = \widehat{\mathfrak{sl}}_2$

$\boxed{Q}$: How does it generalize to any untwisted Kac-Moody alg. $\mathfrak{g}(A) = \widehat{\mathfrak{g}_0}$ ($\mathfrak{g}_0$-simple f.d.)?

It turns out that in this case $\boxed{W = W_0 \ltimes Q^\vee}$ - the semidirect product of the Weyl gp W_0 of $\mathfrak{g}_0$ and the dual root lattice $Q^\vee = \bigoplus_i \mathbb{Z}\alpha_i^\vee$, where $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$ (α_i-simple root of $\mathfrak{g}_0$) which gets identified with h_i under isom. $\mathfrak{h}_0^* \simeq \mathfrak{h}_0$ (here $\mathfrak{h}_0$- Cartan of $\mathfrak{g}_0$). Note that W_0 - finite gp!

Let us now recall the Weyl-Kac denominator formula:

$$\boxed{\sum_{w\in W} \det(w) e^{w\rho - \rho} = \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} \quad (\star)$$

For $\mathfrak{g} = \mathfrak{g}(A) = \widehat{\mathfrak{g}_0}$, we have

$$\boxed{\Delta_+ = \{\alpha \mid \alpha \in (\Delta_0)_+\} \cup \{n\delta + \alpha \mid n \geq 1, \alpha \in \Delta_0 \cup \{0\}\}}$$

$(\Delta_0)_+$: positive roots for $\mathfrak{g}_0$; Δ_0: all roots for $\mathfrak{g}_0$

Moreover, we have

$$\boxed{\dim \mathfrak{g}_{n\delta} = \dim \mathfrak{h}_0 = r, \quad \dim \mathfrak{g}_\beta = 1 \text{ for other positive } \beta}$$

Then, if we set $q := e^{-\delta}$, then RHS of $(\star)$ equals

$$(1) \quad \boxed{\prod_{\alpha\in(\Delta_0)_+} (1-e^{-\alpha}) \cdot \prod_{n=1}^{\infty} \Big\{(1-q^n)^r \cdot \prod_{\alpha\in\Delta_0} (1-q^n e^{\alpha})\Big\}}$$

← formal series in $q, z_1, \ldots, z_r$ where $z_i := e^{-\alpha_i}$

Let us now compute the LHS of $(\star)$. As a matter of fact $\det(w_0, y) = \det w_0 \ \forall w_0 \in W_0, y \in Q^\vee$.

So: $\text{LHS} = \sum_{w_0\in W_0} \sum_{y\in Q^\vee} \det(w_0) \cdot e^{w_0 y \rho - \rho}$

After some careful computations, see [Feigin-Zelevinsky, Section 6.16], one gets the following for LHS:

$$(2) \quad \boxed{e^{-\rho_0} \cdot \sum_{\mu\in\frac{1}{2}Q^\vee} q^{(\mu+2\rho_0,\mu)} \cdot J_{\mu+\rho_0} \quad \text{with } J_\lambda := \sum_{w_0\in W_0} \det(w_0) e^{w_0\lambda}} \quad (\rho_0 \text{ - denotes } \rho \text{ for } \mathfrak{g}_0)$$

Exercise*: Work this out!

Prop 1: $\boxed{\prod_{\alpha\in(\Delta_0)_+} (1-e^{-\alpha}) \cdot \prod_{n=1}^{\infty} \{(1-q^n)^r \cdot \prod_{\alpha\in\Delta_0} (1-q^n e^{\alpha})\} = e^{-\rho_0} \sum_{\mu\in\frac{1}{2}Q^\vee} q^{(\mu+2\rho_0,\mu)} \cdot J_{\mu+\rho_0}.}$

► Follows from $(\star)$, (1), (2) ∎

Cor 1: $\boxed{\varphi(q)^{\dim(\mathfrak{g})} = \sum_{\mu\in\frac{1}{2}Q^\vee} q^{(\mu+2\rho_0,\mu)} \cdot \prod_{\alpha\in(\Delta_0)_+} \frac{(\alpha, \mu+\rho_0)}{(\alpha,\rho_0)}}.$

► Divide both sides in Prop 1 by $\prod_{\alpha\in(\Delta_0)_+}(1-e^{-\alpha})$ and specialize all $z_i \mapsto 1$. One needs to apply Weyl char. f-la for $\mathfrak{g}_0$-integrable module L_μ: $\mathrm{ch}\, L_\mu = \frac{\sum_{w_0\in W_0} \det(w_0) e^{w_0(\mu+\rho_0)-\rho_0}}{\prod_{\alpha\in(\Delta_0)_+}(1-e^{-\alpha})}$, evaluate it at $t\cdot\rho_0$ and take $t\to 0$ limit to get $\dim(L_\mu) = \prod_{\alpha\in(\Delta_0)_+} {}^{(\alpha,\mu+\rho_0)}/_{(\alpha,\rho_0)}$. ∎