— LECTURE 22 —

* Today: Jantzen–Kac–Kazhdan–Shapovalov determinant formula.

In Lecture 4, for a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ with abelian $\mathfrak{g}_0$ and an involutive automorphism $\omega:\mathfrak{g}\to\mathfrak{g}$ s.t. $\omega(\mathfrak{g}_n)=\mathfrak{g}_{-n}$, $\omega|_{\mathfrak{g}_0}=-\mathrm{Id}$, and any $\lambda\in\mathfrak{g}_0^*$, we endowed the Verma module $M_\lambda=M_\lambda^+$ with a unique symmetric bilinear map

$$\boxed{M_\lambda\times M_\lambda\xrightarrow{(\,,\,)_\lambda}\mathbb{C}\ \text{ s.t. } (v_\lambda,v_\lambda)_\lambda=1,\ (av,w)_\lambda+(v,\omega(a)w)_\lambda=0\quad\forall v,w\in M_\lambda,\ a\in\mathfrak{g}}$$

This form $(\,,\,)_\lambda$ is usually called the Shapovalov form.

Recall: M_λ-irreducible $\iff(\cdot,\cdot)_\lambda$-nondegenerate.

Moreover, M_λ is naturally $\mathbb{Z}$-graded $M_\lambda=\bigoplus_{n\geq 0}M_\lambda[-n]$ via $M_\lambda\simeq U(\mathfrak{n}_-)\simeq S(\mathfrak{n}_-)$ as graded v. spaces

By construction, $(M_\lambda[-n],M_\lambda[-m])_\lambda=0$ for $n\neq m$ $\Rightarrow$ suffices to study all restrictions

$$\boxed{(\,,\,)_{\lambda,n}:M_\lambda[-n]\times M_\lambda[-n]\to\mathbb{C}}$$

Thus, understanding $\det(\cdot,\cdot)_{\lambda,n}$ for all n is of crucial importance!

Example 1: $\mathfrak{g}=\mathfrak{sl}_2$ — was worked out in Lecture 4

Example 2: $\mathfrak{g}=Vir$ — was the subject of Lecture 11 (the proof was completed in Lecture 20).

Example 3: $\mathfrak{g}=\mathfrak{g}(A)$ - Kac–Moody — this is the subject of today's lecture.

Rmk 1: The formula was first obtained by Shapovalov (1972) for semisimple $\mathfrak{g}$. However, we shall follow the proof of Kac–Kazhdan (1979) which works for any Kac-Moody algebra and is crucially based on Jantzen's filtration.

To treat the case of $\mathfrak{g}=\mathfrak{g}(A)$, we shall slightly change the setting in two ways:

1) we will use the involutive anti-automorphism $\sigma:\mathfrak{g}(A)\circlearrowleft$ such that $e_i\overset{\sigma}{\mapsto}f_i$, $f_i\overset{\sigma}{\mapsto}e_i$, $\sigma|_{\mathfrak{h}}=\mathrm{Id}_{\mathfrak{h}}$

2) we will treat the pairing $\langle\,,\,\rangle$ with values in $U(\mathfrak{h})$ which shall give rise to $\langle\,,\,\rangle_\lambda$ $\forall\lambda\in\mathfrak{h}^*$.

Recall: $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+\Rightarrow U(\mathfrak{g})\underset{\substack{\text{as}\\ \text{vector spaces}}}{\simeq}U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}_+)\Rightarrow\boxed{U(\mathfrak{g})\underset{\substack{\text{as vector}\\ \text{spaces}}}{\simeq}U(\mathfrak{h})\oplus(\mathfrak{n}_-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_+)}$

The latter gives rise to the projection onto the first summand $\boxed{\pi:U(\mathfrak{g})\twoheadrightarrow U(\mathfrak{h})}$

Def 1: Consider an $U(\mathfrak{h})$-valued bilinear form $\langle\cdot,\cdot\rangle:U(\mathfrak{g})\times U(\mathfrak{g})\to U(\mathfrak{h})$ defined via

$$\boxed{\langle x,y\rangle:=\pi(\sigma(x)y)}$$

Note:

a) $\pi(\sigma(z))=\pi(z)$ for $z\in U(\mathfrak{g})\Rightarrow\langle x,y\rangle=\langle y,x\rangle\Rightarrow\boxed{\langle\cdot,\cdot\rangle\text{-symmetric}}$

b) If $x\in U(\mathfrak{g})_{\mu_1}$, $y\in U(\mathfrak{g})_{\mu_2}$, $\mu_1\neq\mu_2$ ($\in Q$) $\Rightarrow\langle x,y\rangle=0$

c) If $y\in U(\mathfrak{g})\mathfrak{n}_+\Rightarrow\langle x,y\rangle=0$ $\forall x$. Same holds if $x\in U(\mathfrak{g})\mathfrak{n}_+$ due to symmetry.

$\Rightarrow$ (a)–(c) $\Rightarrow$ $\langle\cdot,\cdot\rangle$ is uniquely determined by its restrictions $\boxed{\langle\cdot,\cdot\rangle^\eta:U(\mathfrak{n}_-)_{-\eta}\times U(\mathfrak{n}_-)_{-\eta}\to U(\mathfrak{h})}$ $\forall\eta\in Q_+$ ($=\bigoplus\mathbb{Z}_{\geq0}\alpha_i$)

For any $\lambda \in \mathfrak{h}^*$, the pairing $\langle\cdot,\cdot\rangle^\eta$ gives rise to the pairing $\boxed{\langle\cdot,\cdot\rangle^\eta_\lambda : \mathcal{U}(\mathfrak{n}_-)_{-\eta} \times \mathcal{U}(\mathfrak{n}_-)_{-\eta} \to \mathbb{C}}$ via identification $\mathcal{U}(\mathfrak{h}) \simeq S(\mathfrak{h}) \simeq \mathbb{C}[\mathfrak{h}^*]$, hence, evaluation at $\lambda \in \mathfrak{h}^*$, denoted (λ), makes sense!

Def 2: Define the bilinear form $\langle\cdot,\cdot\rangle_\lambda : M_\lambda \times M_\lambda \to \mathbb{C}$ via

$$\langle u_1 v_\lambda, u_2 v_\lambda \rangle_\lambda = \langle u_1, u_2 \rangle(\lambda) \quad \forall u_1, u_2 \in \mathcal{U}(\mathfrak{n}_-)$$

(here we recall that $\mathcal{U}(\mathfrak{n}_-) \to M_\lambda$, $u \mapsto u v_\lambda$, is a vector space isom.)

Exercise: a) Prove $\langle a v, w \rangle_\lambda = \langle v, \sigma(a) w \rangle_\lambda \quad \forall v, w \in M_\lambda,\ a \in \mathcal{U}(\mathfrak{g})$

b) Verify $\langle M_\lambda[\lambda-\mu_1], M_\lambda[\lambda-\mu_2] \rangle_\lambda = 0$ for $\mu_1 \neq \mu_2$.

c) $\mathrm{Ker}(M_\lambda \twoheadrightarrow L_\lambda) = \mathrm{Ker}(\langle\cdot,\cdot\rangle_\lambda)$

d) Relate this pairing $\langle\cdot,\cdot\rangle_\lambda$ to the one $(\cdot,\cdot)_\lambda$ we started from (as in Lecture 4).

Theorem 1 (Shapovalov, Jantzen, Kac–Kazhdan): Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac–Moody algebra (or, more generally, symmetrizable contragredient algebra). Then up to a nonzero constant factor:

$$\boxed{\det(\langle\cdot,\cdot\rangle^\eta) = \prod_{\alpha>0} \prod_{n\geq 1} \left(h_\alpha + \rho(h_\alpha) - \frac{n(\alpha,\alpha)}{2}\right)^{P(\eta - n\alpha)}}$$

Here: * $P(\mu) :=$ Kostant partition function$(\mu) = \dim \mathcal{U}(\mathfrak{n}_-)_{-\mu}$.

* the roots α are counted with multiplicities!

The proof will proceed in several steps.

Step 1: Computation of the leading term.

Lemma 1: The leading term of $\det(\langle\cdot,\cdot\rangle^\eta)$ is equal up to a nonzero constant factor to:

$$\boxed{\prod_{\alpha>0} \prod_{n\geq 1} h_\alpha^{P(\eta - n\alpha)}}$$ (again, the roots α are counted with multiplicities!)

Exercise: Prove this!

Compare to [Lecture 11, Theorem 1]; explain details in the class...

Def 3: $\beta \in Q_+ \setminus \{0\}$ is called a quasiroot if $\exists \alpha \in \Delta$ s.t. β is proportional to α.

Step 2: First approximation.

Lemma 2: $\det(\langle\cdot,\cdot\rangle^\eta)$ is equal up to a nonzero constant factor to a product of linear factors of the form

$$\boxed{h_\beta + \rho(h_\beta) - \tfrac{1}{2}(\beta,\beta),\ \beta\text{-quasiroot}}$$

▶ If M_λ is not irreducible $\Rightarrow \exists \beta \in Q_+ \setminus \{0\}$ s.t. $M_{\lambda-\beta} \hookrightarrow M_\lambda$. But comparing the action of the Casimir operator implies $(\lambda, \lambda+2\rho) = (\lambda-\beta, \lambda-\beta+2\rho) \Leftrightarrow (\lambda+\rho, \beta) = \frac{1}{2}(\beta,\beta) \Leftrightarrow (\lambda+\rho)(h_\beta) = \frac{1}{2}(\beta,\beta)$.

Thus: If $\forall \beta \in Q_+ \setminus \{0\}$: $(\lambda+\rho)(h_\beta) \neq \frac{1}{2}(\beta,\beta) \Rightarrow M_\lambda$-irred $\Rightarrow \langle\cdot,\cdot\rangle_\lambda$-nondegenerate

$\Rightarrow \det(\langle\cdot,\cdot\rangle^\eta)$ viewed as an element of $\mathbb{C}[\mathfrak{h}^*]$ has zeros in the union of the hyperplanes in $\mathfrak{h}^*$ given by $\{(\lambda+\rho)(h_\beta) - \frac{1}{2}(\beta,\beta) = 0\}_{\beta \in Q_+ \setminus \{0\}}$

Exercise: Deduce that $\det(\langle\cdot,\cdot\rangle^\eta)$ must be a product of these linear factors.

Using Lemma 1, we see that β must be a quasiroot! ■

In what follows, we will need also the following simple result:

Lemma 3: Let β be a quasiroot such that $(\lambda+\rho)(h_\beta)=\frac{1}{2}(\beta,\beta)$. We also assume that $(\lambda+\rho)(h_\gamma)\neq\frac{1}{2}(\gamma,\gamma)$ for $\gamma\in Q_+\setminus\{0,\beta\}$ and $(\lambda+\rho-\beta)(h_\gamma)\neq\frac{1}{2}(\gamma,\gamma)$ for $\gamma\in Q_+\setminus\{0\}$.

Then: $\mathrm{Ker}\,(M_\lambda\twoheadrightarrow L_\lambda)$ is a direct sum of a finite number of $M_{\lambda-\beta}$.

As $(\lambda+\rho-\beta)(h_\gamma)\neq\frac{1}{2}(\gamma,\gamma)$ $\forall\gamma\in Q_+\setminus\{0\}$, the proof of Lemma 2 yields $M_{\lambda-\beta}$-irreducible.
Hence if $M_\lambda(\lambda-\beta)^{\mathrm{sing}}\subseteq M_\lambda(\lambda-\beta)$ denotes a subspace $\left(\begin{smallmatrix}\text{it is fin.dim}\\ \text{of dimension } N\end{smallmatrix}\right)$ of singular vectors, then the $\mathfrak{g}$-module $\mathcal{U}$ they generate is isomorphic to $M_{\lambda-\beta}^{\oplus N}$. It remains to show that $L:=M_\lambda/\mathcal{U}$ is irreducible.
If L is not irreducible $\Rightarrow\exists\gamma\in Q_+\setminus\{0\}$ and $v\in L(\lambda-\gamma)^{\mathrm{sing}}\Rightarrow$ nonzero morphism $M_{\lambda-\gamma}\to L$
Comparing the action of the Casimir operator as in Lemma 2 $\Rightarrow(\lambda+\rho)(h_\gamma)=\frac{1}{2}(\gamma,\gamma)\Rightarrow\gamma\in\{0,\beta\}$.
But $L(\lambda-\beta)^{\mathrm{sing}}=0$ (by construction) $\Rightarrow\gamma\neq\beta\Rightarrow\gamma=0\Rightarrow$ Contradiction! ∎

Step 3: Jantzen's filtration

The following construction plays the key role in the proof!
Define $\boxed{\widehat{U}(\mathfrak{g}):=U(\mathfrak{g})\otimes_{\mathbb{C}}\mathbb{C}[t],\ \widehat{\mathfrak{h}}^*:=\mathfrak{h}^*\otimes_{\mathbb{C}}\mathbb{C}[t]}$ ← extension of the ground ring to $\mathbb{C}[t]$.
Fix $\nu\in\mathfrak{h}^*$ such that $\nu(h_\alpha)\neq0$ $\forall\alpha\in Q_+\setminus\{0\}$. Define $\boxed{\tilde\lambda:=\lambda+t\cdot\nu\in\widehat{\mathfrak{h}}^*}$. Then, we likewise have Verma $\widehat{U}(\mathfrak{g})$-module $\boxed{\widehat{M}_{\tilde\lambda}}$. We may also endow M_λ with an $\widehat{U}(\mathfrak{g})$-module structure by $\widehat{U}(\mathfrak{g})\xrightarrow{t\mapsto0}U(\mathfrak{g})$.
Then, we have a $\widehat{U}(\mathfrak{g})$-homomorphism $\widehat{M}_{\tilde\lambda}\twoheadrightarrow M_\lambda$.
We extend $\sigma: U(\mathfrak{g})\circlearrowleft$ to $\sigma:\widehat{U}(\mathfrak{g})\circlearrowleft$ via $a\otimes F(t)\mapsto\sigma(a)\otimes F(t)$ $\forall a\in U(\mathfrak{g})$, $F(t)\in\mathbb{C}[t]$.
Hence, we have a bilinear $\overset{\mathbb{C}[\widehat{\mathfrak{h}}^*]\text{-valued}}{\text{form}}$ $\langle\cdot,\cdot\rangle$ on $\widehat{U}(\mathfrak{g})$ and a bilinear $\mathbb{C}[t]$-valued form $\langle\cdot,\cdot\rangle^\eta_{\tilde\lambda}$, $\eta\in Q_+$.
Consider the following $\widehat{U}(\mathfrak{g})$-module filtration of $\widehat{M}_{\tilde\lambda}$:

$$\boxed{\widehat{M}_{\tilde\lambda}=\widehat{M}^{(0)}\supseteq\widehat{M}^{(1)}\supseteq\widehat{M}^{(2)}\supseteq\ldots,\quad \widehat{M}^{(k)}:=\{v\in\widehat{M}_{\tilde\lambda}\mid\langle v,w\rangle\,\vdots\,t^k\ \ \forall w\in\widehat{M}_{\tilde\lambda}\}}$$

Def 4: Jantzen's $U(\mathfrak{g})$-module filtration of M_λ is obtained as the image of the above $\widehat{M}^{(\bullet)}$ under $\widehat{M}_{\tilde\lambda}\twoheadrightarrow M_\lambda$

$$\boxed{M_\lambda=M^{(0)}\supseteq M^{(1)}\supseteq M^{(2)}\supseteq\ldots}$$

Rmk 2: a) $M^{(1)}=\mathrm{Ker}\,(\langle\cdot,\cdot\rangle_\lambda)$, while $M^{(i)}$ keeps track of higher order zeros of $\langle\cdot,\cdot\rangle_\lambda$

b) For a fixed $\eta\in Q_+$ $\exists i$ such that $M_\lambda(\lambda-\eta)\cap M^{(i)}=0$.

Step 4: Computation of powers

By definition, if β is a quasiroot then $\exists\alpha\in\Delta$ such that $\beta=r\cdot\alpha$ and we must have $r\in\mathbb{Q}$.
According to Lemma 2, $\det(\langle\cdot,\cdot\rangle^\eta)$ is a product of linear terms $h_\beta+\rho(h_\beta)-\frac{(\beta,\beta)}{2}$.
If $(\alpha,\alpha)=0\Rightarrow$ all these linear factors are the same for different r, while the total power should match the power of h_α in Lemma 1.

So: If $(\alpha,\alpha)=0$, $\alpha\in\Delta_+$, then the power of $h_\alpha+\rho(h_\alpha)$ in $\det(\langle\cdot,\cdot\rangle^\eta)$ is exactly $\sum_{n\geq1}P(\eta-n\alpha)$. As claimed in Theorem 1.

Let us now treat the case of $(\alpha,\alpha)\neq0$, in which Lemma 1 yields the sum of powers over all possible r.

Exercise: If $\begin{cases}\beta\in Q_\alpha\cap(Q_+\setminus\{0\}) \\ (\alpha,\alpha)\neq 0\end{cases}$, then $\exists\, \lambda\in\mathfrak{h}^*$ such that

$$(\lambda+\rho)(h_\beta)=\tfrac{1}{2}(\beta,\beta)\ ,\ (\lambda+\rho)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma) \text{ for } \gamma\in Q_+\setminus\{0,\beta\},\ (\lambda-\alpha+\rho)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma) \text{ for } \gamma\in Q_+\setminus\{0\}$$

Pick λ as in the exercise and consider the Jantzen filtration of the Verma module M_λ:

$$\boxed{M_\lambda=M^{(0)}\supsetneq M^{(1)}\supseteq M^{(2)}\supseteq\ldots}$$

According to Lemma 3: $M^{(1)}\simeq M_{\lambda-\beta}^{\oplus N_1}\Rightarrow \forall i\geq 1:\ M^{(i)}\simeq M_{\lambda-\beta}^{\oplus N_i}$ (here we use irreducibility of $M_{\lambda-\beta}$).
Set $\boxed{N_\beta:=\sum_{i\geq 1}N_i}$.

By the very definition of the filtration $\widetilde{M}^{(\cdot)}$, the determinant $\det(\widetilde{\langle\cdot,\cdot\rangle}_\lambda^\eta)$ is divisible exactly by $\sum_{i\geq 1}\dim M^{(i)}(\lambda-\eta)=N_\beta \mathcal{P}(\eta-\beta)$-th power of $t\Rightarrow \det(\langle\cdot,\cdot\rangle^\eta)$ is divisible exactly by $N_\beta\mathcal{P}(\eta-\beta)$-th power of $h_\beta+\rho(h_\beta)-\frac{1}{2}(\beta,\beta)$.

So: the multiplicity of h_α in the leading term of $\det(\langle\cdot,\cdot\rangle^\eta)$ equals $\sum_{\beta\in Q_+\alpha\cap(Q_+\setminus\{0\})}N_\beta\cdot\mathcal{P}(\eta-\beta)$.

But due to Lemma 1, this power also equals $\sum_{n\geq 1}\mathcal{P}(\eta-n\alpha)$.

Exercise: The functions $\{\phi_\beta(\eta)=\mathcal{P}(\eta-\beta)\}_{\beta\in Q_+\alpha\cap(Q_+\setminus\{0\})}$ are linearly independent

Thus: * $N_\beta=0$ if $\beta\in Q_+\alpha\setminus\mathbb{Z}\alpha$

* $N_{n\alpha}=\dim\mathfrak{g}_\alpha$

This completes our proof of Theorem 1. ▨

Theorem 1 yields the following criteria for irreducibility of M_λ:

Corollary 1: M_λ - irreducible iff $(\lambda+\rho)(h_\alpha)\neq\frac{n}{2}(\alpha,\alpha)$ for any $\alpha\in\Delta_+$, $n\in\mathbb{Z}_{>0}$.

This refines [Homework 9, Problem 2]

Another interesting question is to understand for which (μ,λ), there exist non-trivial (as we know also injective) $\mathfrak{g}$-homomorphisms $M_\mu\to M_\lambda$? Equivalently, when L_μ is an irreducible subquotient of M_λ?

Theorem 2 (BGG – Jantzen – KK): L_μ is an irreducible subquotient of M_λ iff

(*) $\boxed{\exists\ \beta_1,\beta_2,\ldots,\beta_k\in\Delta_+ \text{ and } n_1,\ldots,n_k\in\mathbb{Z}_{>0} \text{ such that } (\lambda+\rho-n_1\beta_1-\ldots-n_{i-1}\beta_{i-1})(h_{\beta_i})=\tfrac{n_i}{2}(\beta_i,\beta_i)\ \forall 1\leq i\leq k \text{ and } \lambda-\mu=n_1\beta_1+\ldots+n_k\beta_k}$

▶ The proof shall again utilize the Jantzen filtration as follows:

(J) $\boxed{\mathrm{ord}_t\det(\widetilde{\langle\cdot,\cdot\rangle}_\lambda^\eta):=\text{maximal power of } t \text{ which divides } \det(\widetilde{\langle\cdot,\cdot\rangle}_\lambda^\eta)=\sum_{i\geq 1}\dim M^{(i)}(\lambda-\eta)}$

For a given λ, set

$$\boxed{\Delta_+(\lambda):=\{\alpha\in\Delta_+\mid(\lambda+\rho)(h_\alpha)=\tfrac{n}{2}(\alpha,\alpha)\text{ for some } n\in\mathbb{Z}_{>0}\}\ ,\ Z_\alpha:=\{n\in\mathbb{Z}_{>0}\mid(\lambda+\rho)(h_\alpha)=\tfrac{n}{2}(\alpha,\alpha)\}}$$

($|Z_\alpha|=1$ if $(\alpha,\alpha)\neq 0$)

We will use the following simple result:

Exercise: Prove that any irreducible subquotient of M_λ is of the form L_μ, and that M_λ admits a unique up to a permutation Jordan – Hölder series.

(Continuation of the proof of Theorem 2)

For the Jantzen filtration $M_\lambda = M^{(0)} \supsetneq M^{(1)} \supseteq M^{(2)} \supseteq \dots$, due to (J), we have:

$$\sum_{i\geq 1} ch\, M^{(i)} = \sum_{\eta\in Q_+} \Big(\sum_{i\geq 1} \dim M^{(i)}(\lambda-\eta)\Big) e^{\lambda-\eta} = \sum_{\eta\in Q_+} \mathrm{ord}_t\big(\det \widetilde{\langle\cdot,\cdot\rangle}^{\eta}_{\tilde\lambda}\big)\cdot e^{\lambda-\eta}.$$

But, due to Theorem 1, we have $\mathrm{ord}_t\big(\det \widetilde{\langle\cdot,\cdot\rangle}^{\eta}_{\tilde\lambda}\big) = \sum_{\alpha\in\Delta_+(\lambda)} \sum_{n\in \mathbb{Z}_\alpha} P(\eta - n\alpha)$ $\Rightarrow$

(the easiest way to see it is to apply Theorem 1 to Verma module $\widetilde{M}_{\tilde\lambda}$)

$$\Rightarrow \sum_{i\geq 1} ch\, M^{(i)} = \sum_{\alpha\in\Delta_+(\lambda)} \sum_{n\in\mathbb{Z}_\alpha} \sum_{\eta\in Q_+} P(\eta-n\alpha) e^{\lambda-\eta} = \sum_{\alpha\in\Delta_+(\lambda)} \sum_{n\in\mathbb{Z}_\alpha} ch\, M_{\lambda-n\alpha}$$

So:
$$\boxed{\sum_{i\geq 1} ch\, M^{(i)} = \sum_{\alpha\in\Delta_+(\lambda)} \sum_{n\in\mathbb{Z}_\alpha} ch\, M_{\lambda-n\alpha}} \qquad (C)$$

"$\Rightarrow$": Let L_μ be an irreducible subquotient of M_λ. Then $\mu = \lambda-\eta$, $\eta\in Q_+$.

The proof of $(\star)$ proceeds by induction on $|\eta|\in\mathbb{Z}_{\geq 0}$.

Base: $|\eta| = 0 \Rightarrow \eta = 0 \Rightarrow \mu=\lambda$. The claim is obvious.

Induction Step: As $M_\lambda/M^{(1)} = M^{(0)}/M^{(1)} \simeq L_\lambda$, the above exercise implies that L_μ is a subquotient of $M^{(1)}$.

Applying now (C) together with an obvious fact that $ch\, L_{\lambda-\nu}$ are linearly independent, we see that $ch\, L_{\mu=\lambda-\eta}$ occurs in one of $ch\, M_{\lambda-n\alpha}$ ($\alpha\in\Delta_+(\lambda)$, $n\in\mathbb{Z}_\alpha$)

Thus: L_μ is also a subquotient of this $M_{\lambda-n\alpha}$

As $|\lambda-n\alpha-\mu| < |\lambda-\mu|$, the induction assumption applies to μ and $\lambda' := \lambda-n\alpha$, i.e.

$\exists\, \beta_2,\dots,\beta_k\in\Delta_+$, $n_2,\dots,n_k\in\mathbb{Z}_{\geq 0}$ s.t. $(\lambda'+\rho-n_2\beta_2-\dots-n_{i-1}\beta_{i-1})(h_{\beta_i}) = \frac{n_i}{2}(\beta_i,\beta_i)$ $\forall\, 2\leq i\leq k$ and $\lambda'-\mu = n_2\beta_2+\dots+n_k\beta_k$

Then: Picking $\beta_1 := \alpha$, $n_1 := n$, we get $(\star)$ for the pair (μ,λ).

"$\Leftarrow$": Let us now prove that if the pair (μ,λ) satisfies $(\star)$, then L_μ is a subquotient of M_λ.

Set $\eta := \lambda-\mu\in Q_+$. If $\eta=0$, the claim is obvious. If $\eta>0$, set $\alpha:=\beta_1$, $n:=n_1$, so that $\alpha\in\Delta_+(\lambda)$, $n\in\mathbb{Z}_\alpha$, and the pair $(\mu, \lambda':=\lambda-n\alpha)$ also satisfies $(\star)$. The latter implies (by the inductive assumption) that L_μ is a subquotient of $M_{\lambda-n\alpha}$.

Appealing again to (C), the latter yields that L_μ is a subquotient of M_λ. ∎

Exercise: a) Show that if $\mathfrak{g}=\mathfrak{g}(A)$-simple f.d., then $\forall \mu,\lambda$: $\dim_{\mathbb{C}} \mathrm{Hom}(M_\mu, M_\lambda)\leq 1$.

Hint: Use the fact that $\dim_{\mathbb{C}} M_\lambda(\lambda-\eta)$ is bounded by a polynomial in η as well as any nontrivial of homom. $M_\mu\to M_\lambda$ is an embedding

b) Show that a) does not hold for Kac-Moody $\mathfrak{g}(A)$ in general.