International Mathematics Research Notices, 2024, **00**(0), 1–37
https://doi.org/10.1093/imrn/rnae197
Advance access publication date 00 Month 2024
Article

Affine Standard Lyndon Words: A-Type

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To the memory of Yulia Zdanovska

We generalize an algorithm of Leclerc [6] describing explicitly the bijection of Lalonde–Ram [5] from finite to affine Lie algebras. In type $A_n^{(1)}$, we compute all affine standard Lyndon words for any order of the simple roots and establish some properties of the induced orders on the positive affine roots.

1 Introduction

1.1 Summary

An interesting basis of the free Lie algebra generated by a finite family $\{e_i\}_{i\in I}$ was constructed in the 1950s using the combinatorial notion of *Lyndon* words (we recall these in Definitions 2.2–2.3). A few decades later, this was generalized in [5] to any finitely generated Lie algebra $\mathfrak{a}$. Explicitly, if $\mathfrak{a}$ is generated by $\{e_i\}_{i\in I}$, then any order on the finite alphabet I gives rise to the combinatorial basis $\mathrm{b}[\ell]$ as ℓ ranges through all *standard Lyndon* words (these will be recalled in Definition 2.11). Here, the standard bracketing $\mathrm{b}[\ell]$ is defined inductively with $\mathrm{b}[i] = e_i$ (see Definition 2.8).

The key application of [5] was to a simple finite-dimensional $\mathfrak{g}$, more precisely, to its maximal nilpotent subalgebra $\mathfrak{n}^+$. According to the root space decomposition:

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathbb{C}\cdot e_\alpha, \qquad \Delta^+ = \Big\{\text{positive roots}\Big\}. \tag{1.1}$$

We note that the one-dimensional direct summands above are canonical as they are distinct eigenspaces for the adjoint action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. However, picking a specific basis of *root vectors* $\{e_\alpha\}_{\alpha\in\Delta^+}$ is non-canonical. Appealing to an additional grading by the root lattice of $\mathfrak{g}$, [5] derived a natural bijection

$$\ell\colon \Delta^+ \xrightarrow{\sim} \Big\{\text{standard Lyndon words}\Big\}. \tag{1.2}$$

A decade later, this bijection played a pivotal role in [6], which studied the image of the dual canonical basis of $U_q(\mathfrak{n}^+)$, the positive half of a quantum group of $\mathfrak{g}$, under the embedding to the quantum shuffle algebra of [3, 10, 12]. To this end, [6] obtained an explicit algorithm (see Proposition 2.16) for the above bijection (1.2). The key ingredient that allows for the quantum group generalization is the fact (attributed to [11] in [6]) that the order on Δ^+ induced via (1.2) from a lexicographical order on words is *convex* in the sense of Definition 2.18 (see Proposition 2.20).

The motivation of the present note is to extend the above discussion to affine root systems. To this end, we recall an enigmatic remark from the very end of [5]: “*Preliminary computations seem to*

Received: January 16, 2024. **Revised:** August 14, 2024. **Accepted:** August 22, 2024

indicate that it will be very instructive to study root multiplicities for Kac–Moody Lie algebras by way of standard Lyndon words".

Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$, whose Dynkin diagram is obtained by extending the Dynkin diagram of $\mathfrak{g}$ with one vertex 0. Thus, on the combinatorial side, we consider the alphabet $\widehat{I} = I \sqcup \{0\}$. The corresponding *positive* subalgebra $\widehat{\mathfrak{n}}^+ \subset \widehat{\mathfrak{g}}$ still admits the root space decomposition $\widehat{\mathfrak{n}}^+ = \bigoplus_{\alpha\in\widehat{\Delta}^+} \widehat{\mathfrak{n}}^+_\alpha$, with $\widehat{\Delta}^+ = \{\text{positive affine roots}\}$. The key difference with (1.1) is that not all $\widehat{\mathfrak{n}}^+_\alpha$ are one-dimensional:

$$\dim \widehat{\mathfrak{n}}^+_\alpha = 1 \quad \forall \alpha \in \widehat{\Delta}^{+,\mathrm{re}}, \qquad \dim \widehat{\mathfrak{n}}^+_\alpha = |I| \quad \forall \alpha \in \widehat{\Delta}^{+,\mathrm{im}}. \tag{1.3}$$

Here, $\widehat{\Delta} = \widehat{\Delta}^{+,\mathrm{re}} \sqcup \widehat{\Delta}^{+,\mathrm{im}}$ is the decomposition into real and imaginary affine roots, with $\widehat{\Delta}^{+,\mathrm{im}} = \{k\delta | k \geq 1\}$. It is therefore natural to consider an extended set $\widehat{\Delta}^{+,\mathrm{ext}}$ of (5.1), counting imaginary roots with appropriate multiplicities. Then, the degree reasoning similar to the one used in [5] provides a natural analogue of (1.2):

$$\mathrm{SL}\colon \widehat{\Delta}^{+,\mathrm{ext}} \xrightarrow{\sim} \Big\{\text{affine standard Lyndon words}\Big\}. \tag{1.4}$$

Our first result (Proposition 3.4) is an inductive algorithm describing this bijection, slightly generalizing Leclerc's algorithm describing (1.2). As the first application, we use it to find all affine standard Lyndon words for the simplest case of $\widehat{\mathfrak{sl}}_2$.

Our major technical result is the explicit description of all affine standard Lyndon words for $\widehat{\mathfrak{sl}}_{n+1}$ ($n \geq 2$). To this end, we first straightforwardly treat the special order (4.1) in Theorem 4.2. We then derive a similar pattern for an arbitrary order in Theorem 4.7. The key feature is that all affine standard Lyndon words are determined by those of length $\leq n$. Furthermore, we crucially use Rosso's convexity result for $\mathfrak{sl}_{n+1}$ to obtain an explicit description of n affine standard Lyndon words in degree δ, which are key to establishing the general "periodicity" pattern.

The induced order (5.2) on $\widehat{\Delta}^{+,\mathrm{ext}}$ is quite different from the orders in the literature on affine quantum groups ([1, 4]). While for $\widehat{\mathfrak{sl}}_2$ one gets a usual order ([2])

$$\alpha_1 < \alpha_1 + \delta < \alpha_1 + 2\delta < \cdots < \cdots < 3\delta < 2\delta < \delta < \cdots < 2\delta + \alpha_0 < \delta + \alpha_0 < \alpha_0,$$

the imaginary roots are not placed consequently in other affine types. We use Theorem 4.7 to establish two properties of this order for $\widehat{\mathfrak{sl}}_{n+1}$, see Propositions 5.4 and 5.8.

1.2 Outline

The structure of the present paper is the following:

- In Section 2, we recall the notion of (standard) Lyndon words, their basic properties, and the application to simple Lie algebras, following [5] and [6].
- In Section 3, we generalize Leclerc's algorithm of [6] from simple Lie algebras to affine Lie algebras and illustrate its application in the simplest case of $A^{(1)}_1$.
- In Section 4, the heart of the paper, we compute affine standard Lyndon words for $A^{(1)}_n$ ($n \geq 2$) with any order on the corresponding alphabet $\widehat{I} = \{0, 1, \ldots, n\}$. The resulting set of affine standard Lyndon words is determined by a finite subset of those of length $\leq n$ as well as manifests a compelling periodicity pattern.
- In Section 5, we use the explicit formulas for affine standard Lyndon words from Theorem 4.7 to establish some properties of the order on $\widehat{\Delta}^{+,\mathrm{ext}}$, induced via (1.4) from the lexicographical order on the affine standard Lyndon words.

2 Lyndon Words Approach to Lie Algebras

In this section, we recall the results of [5] and [6] that provide a combinatorial construction of an important basis of finitely generated Lie algebras, with the main application to the maximal nilpotent subalgebra of a simple Lie algebra.

2.1 Lyndon words

Let I be a finite ordered alphabet, and let I^* be the set of all finite length words in the alphabet I. For $u = [i_1 \dots i_k] \in I^*$, we define its *length* by $|u| = k$. We introduce the *lexicographical order* on I^* in a standard way:

$$[i_1 \dots i_k] < [j_1 \dots j_l] \quad \text{if} \quad \begin{cases} i_1 = j_1, \dots, i_a = j_a, i_{a+1} < j_{a+1} \text{ for some } a \geq 0 \\ \text{or} \\ i_1 = j_1, \dots, i_k = j_k \text{ and } k < l \end{cases} .$$

Definition 2.2. A word $\ell = [i_1 \dots i_k]$ is called **Lyndon** if it is smaller than all of its cyclic permutations:

$$[i_1 \dots i_{a-1} i_a \dots i_k] < [i_a \dots i_k i_1 \dots i_{a-1}] \qquad \forall\, a \in \{2, \dots, k\}. \tag{2.1}$$

For a word $w = [i_1 \dots i_k] \in I^*$, the subwords

$$w_{a|} = [i_1 \dots i_a] \qquad \text{and} \qquad w_{|a} = [i_{a+1} \dots i_k] \tag{2.2}$$

with $0 \leq a \leq k$ will be called a *prefix* and a *suffix* of w, respectively. We call such a prefix or a suffix *proper* if $0 < a < k$. It is straightforward to show that Definition 2.2 is equivalent to the following one:

Definition 2.3. A word w is **Lyndon** if it is smaller than all of its proper suffixes:

$$w < w_{|a} \qquad \forall\, 0 < a < |w|. \tag{2.3}$$

As an immediate corollary, we obtain the following well-known result:

Lemma 2.4. If $\ell_1 < \ell_2$ are Lyndon, then $\ell_1\ell_2$ is also Lyndon, and so $\ell_1\ell_2 < \ell_2\ell_1$.

Proof. Let $\ell_1 = i_1 i_2 \dots i_k$ and $\ell_2 = i_{k+1} i_{k+2} \dots i_n$. Any cyclic permutation of the word $\ell_1\ell_2$ is of the form $u_j = i_j i_{j+1} \dots i_n i_1 i_2 \dots i_{j-1}$ with $1 < j \leq k$ or $k < j \leq n$.

- Case 1: $1 < j \leq k$. Since ℓ_1 is Lyndon, we have $\ell_{1|j-1} = i_j \dots i_k > \ell_1$ by (2.3). As $|\ell_1| > |\ell_{1|j-1}|$, there is $p \in \{j, j+1, \dots, k\}$ such that $i_1 = i_j, \dots, i_{p-j} = i_{p-1}$ and $i_{p-j+1} < i_p$. This immediately implies the desired inequality $\ell_1\ell_2 < u_j$.
- Case 2: $k < j \leq n$. Since ℓ_2 is Lyndon, we have $\ell_{2|j-k-1} = i_j \dots i_n \geq \ell_2$ by (2.3) and so $\ell_{2|j-k-1} = i_j \dots i_n > \ell_1$ as $\ell_2 > \ell_1$. If ℓ_1 is not a prefix of $\ell_{2|j-k-1}$, then $i_j = i_1, i_{j+1} = i_2, \dots, i_{j+p-2} = i_{p-1}$ and $i_{j+p-1} > i_p$ for some $1 \leq p \leq \mathbf{min}\{k, n-j+1\}$, so that $\ell_1\ell_2 < u_j$. On the other hand, if ℓ_1 is a prefix of $\ell_{2|j-k-1}$, then $\ell_{2|j-k-1} = \ell_1 i_{j+k} \dots i_n = \ell_1 \ell_{2|j-1}$. In the latter case, the desired inequality $\ell_1\ell_2 < u_j$ follows from $\ell_{2|j-1} > \ell_2$, a consequence of (2.3).

This completes the proof of the first claim that $\ell_1\ell_2$ is Lyndon. The second claim, the inequality $\ell_1\ell_2 < \ell_2\ell_1$, follows now from (2.1). ■

We recall the following two basic facts from the theory of Lyndon words:

Proposition 2.5. ([7, Proposition 5.1.3]) Any Lyndon word ℓ has a factorization

$$\ell = \ell_1\ell_2 \tag{2.4}$$

defined by the property that ℓ_2 is the longest proper suffix of ℓ, which is also a Lyndon word. Under these circumstances, ℓ_1 is also a Lyndon word.

The factorization (2.4) is called a **costandard factorization** of a Lyndon word.

Proposition 2.6. ([7, Proposition 5.1.5]) Any word w has a unique factorization

$$w = \ell_1 \dots \ell_k, \tag{2.5}$$

where $\ell_1 \geq \dots \geq \ell_k$ are all Lyndon words.

The factorization (2.5) is called a **canonical factorization**.

2.7 Standard bracketing

Let $\mathfrak{a}$ be a Lie algebra generated by a finite set $\{e_i\}_{i\in I}$ labelled by the alphabet I.

Definition 2.8. The standard bracketing of a Lyndon word ℓ is given inductively by:

- $\mathrm{b}[i] = e_i \in \mathfrak{a}$ for $i \in I$,
- $\mathrm{b}[\ell] = [\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] \in \mathfrak{a}$, where $\ell = \ell_1\ell_2$ is the costandard factorization (2.4).

The major importance of this definition is due to the following result of Lyndon:

Theorem 2.9. ([7, Theorem 5.3.1]) If $\mathfrak{a}$ is a free Lie algebra in the generators $\{e_i\}_{i\in I}$, then the set $\big\{\mathrm{b}[\ell] | \ell\text{-Lyndon word}\big\}$ provides a basis of $\mathfrak{a}$.

2.10 Standard Lyndon words

It is natural to ask if Theorem 2.9 admits a generalization to Lie algebras $\mathfrak{a}$ generated by $\{e_i\}_{i\in I}$ but with some defining relations. The answer was provided a few decades later in [5]. To state the result, define ${}_we, e_w \in U(\mathfrak{a})$ for any $w \in I^*$:

- For a word $w = [i_1 \dots i_k] \in I^*$, we set

$${}_we = e_{i_1} \dots e_{i_k} \in U(\mathfrak{a}) \tag{2.6}$$

- For a word $w \in I^*$ with a canonical factorization $w = \ell_1 \dots \ell_k$ of (2.5), we set

$$e_w = e_{\ell_1} \dots e_{\ell_k} \in U(\mathfrak{a}) \tag{2.7}$$

  with $e_\ell = \mathrm{b}[\ell] \in \mathfrak{a}$ for any Lyndon word ℓ, cf. Definition 2.8.

It is well-known that the elements (2.6) and (2.7) are connected by the following triangularity property:

$$e_w = \sum_{v \geq w} c_w^v \cdot {}_ve \qquad \text{with} \quad c_w^v \in \mathbb{Z} \quad \text{and} \quad c_w^w = 1. \tag{2.8}$$

The following definition is due to [5]:

Definition 2.11. (a) A word w is called **standard** if ${}_we$ cannot be expressed as a linear combination of ${}_ve$ for various $v > w$, with ${}_we$ as in (2.6).
(b) A Lyndon word ℓ is called **standard Lyndon** if e_ℓ cannot be expressed as a linear combination of e_m for various Lyndon words $m > \ell$, with $e_\ell = \mathrm{b}[\ell]$ as above.

The following result is nontrivial and justifies the above terminology:

Proposition 2.12. ([5]) A Lyndon word is standard iff it is standard Lyndon.

The major importance of this definition is due to the following result:

Theorem 2.13. ([5, Theorem 2.1]) For any Lie algebra $\mathfrak{a}$ generated by a finite collection $\{e_i\}_{i\in I}$, the set $\{\mathrm{b}[\ell] | \ell\text{-standard Lyndon word}\}$ provides a basis of $\mathfrak{a}$.

2.14 Application to simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra with a root system $\Delta = \Delta^+ \sqcup \Delta^-$. Let $\{\alpha_i\}_{i\in I} \subset \Delta^+$ be the simple roots, and $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the root lattice. We endow Q with the symmetric pairing $(\cdot,\cdot)$ so that the Cartan matrix $(a_{ij})_{i,j\in I}$ of $\mathfrak{g}$ is given by $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. The Lie algebra $\mathfrak{g}$ admits the standard **root space decomposition**:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\,, \quad \mathfrak{h} \subset \mathfrak{g} - \text{Cartan subalgebra}, \tag{2.9}$$

with $\mathbf{dim}(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Delta$. We pick *root vectors* $e_\alpha \in \mathfrak{g}_\alpha$ so that $\mathfrak{g}_\alpha = \mathbb{C}\cdot e_\alpha$.

Consider the *positive* Lie subalgebra $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ of $\mathfrak{g}$. Explicitly, $\mathfrak{n}^+$ is generated by $\{e_i\}_{i\in I}$ subject to the classical *Serre* relations:

$$\underbrace{[e_i,[e_i,\ldots,[e_i,e_j]\ldots]]}_{1-a_{ij}\ \text{Lie brackets}} = 0 \qquad \forall\, i \neq j. \tag{2.10}$$

Let $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. The Lie algebra $\mathfrak{n}^+$ is naturally Q^+-graded via $\mathbf{deg}(e_i) = \alpha_i$.

Fix any order on the set I. According to Theorem 2.13, $\mathfrak{n}^+$ has a basis consisting of the e_ℓ's, as ℓ ranges over all standard Lyndon words. Evoking the above Q^+-grading of the Lie algebra $\mathfrak{n}^+$, it is natural to define the grading of words as follows:

$$\mathbf{deg}[i_1\ldots i_k] = \alpha_{i_1} + \cdots + \alpha_{i_k} \in Q^+. \tag{2.11}$$

Due to the decomposition (2.9) and the fact that the root vectors $\{e_\alpha\}_{\alpha\in\Delta^+} \subset \mathfrak{n}^+$ all live in distinct degrees $\alpha \in Q^+$, we conclude that there exists a bijection [5]:

$$\ell \colon \Delta^+ \xrightarrow{\sim} \{\text{standard Lyndon words}\} \tag{2.12}$$

such that $\mathbf{deg}\,\ell(\alpha) = \alpha$ for all $\alpha \in \Delta^+$. We call (2.12) the *Lalonde–Ram*'s bijection.

2.15 Results of Leclerc and Rosso

The Lalonde–Ram's bijection (2.12) was described explicitly in [6]. To state the result, we recall that for a root $\gamma = \sum_{i\in I} n_i\alpha_i \in \Delta^+$, its *height* is $\mathrm{ht}(\gamma) = \sum_{i\in I} n_i$.

Proposition 2.16. ([6, Proposition 25]) The bijection ℓ is inductively given by:

- for simple roots, we have $\ell(\alpha_i) = [i]$
- for other positive roots, we have the following *Leclerc's* algorithm:

$$\ell(\alpha) = \max\left\{\ell(\gamma_1)\ell(\gamma_2)\middle|\alpha = \gamma_1+\gamma_2\,,\ \gamma_1,\gamma_2 \in \Delta^+\,,\ \ell(\gamma_1) < \ell(\gamma_2)\right\}. \tag{2.13}$$

Formula (2.13) recovers $\ell(\alpha)$ once we know $\ell(\gamma)$ for all $\{\gamma \in \Delta^+ \,|\, \mathrm{ht}(\gamma) < \mathrm{ht}(\alpha)\}$.

Remark 2.17. While Lalonde–Ram computed explicitly the standard Lyndon words for any simple $\mathfrak{g}$ and a specific order in [5, Theorem 3.4], the above Leclerc's algorithm allows to find standard Lyndon words for any simple $\mathfrak{g}$ and any ordering of its simple roots. Moreover, this algorithm is easy to program on a computer.

We shall also need one more important property of ℓ. To the end, let us recall:

Definition 2.18. A total order on the set of positive roots Δ^+ is **convex** if

$$\alpha < \alpha + \beta < \beta \tag{2.14}$$

for all $\alpha < \beta \in \Delta^+$ such that $\alpha + \beta$ is also a root.

Remark 2.19. It is well-known ([9]) that convex orders on Δ^+ are in bijection with reduced decompositions of the longest element in the Weyl group of $\mathfrak{g}$.

The following result is [6, Proposition 26], where it is attributed to the preprint of Rosso [11] (a detailed proof can be found in [8, Proposition 2.34]):

Proposition 2.20. Consider the order on Δ^+ induced from the lexicographical order on standard Lyndon words:

$$\alpha < \beta \quad \Longleftrightarrow \quad \ell(\alpha) < \ell(\beta) \text{ lexicographically.} \tag{2.15}$$

This order is convex.

Remark 2.21. We note that both Proposition 2.16 and Proposition 2.20 are of crucial importance for the further application to quantum groups $U_q(\mathfrak{g})$, see [6].

3 Generalization to Affine Lie Algebras

In this section, we generalize Proposition 2.16 to the case of affine Lie algebras $\mathfrak{g}$. As an example, we compute all affine standard Lyndon words for $\mathfrak{g}$ of type $A_1^{(1)}$.

3.1 Affine Lie algebras

In this section, we consider the next simplest class of Kac–Moody Lie algebras after the simple ones, the affine Lie algebras. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $\{\alpha_i\}_{i\in I}$ be the simple roots, and $\theta \in \Delta^+$ be the highest root (with the maximal value of $\mathrm{ht}(\theta)$). We define $\widehat{I} = I \sqcup \{0\}$. Consider the affine root lattice $\widehat{Q} = Q \times \mathbb{Z}$ with the generators $\{(\alpha_i, 0)\}_{i\in I}$ and $\alpha_0 := (-\theta, 1)$. We endow $\widehat{Q}$ with the symmetric pairing defined by:

$$\big((\alpha, n), (\beta, m)\big) = (\alpha, \beta) \qquad \forall\, \alpha, \beta \in Q\,,\ n, m \in \mathbb{Z}. \tag{3.1}$$

This leads to the affine Cartan matrix $(a_{ij})_{i,j\in\widehat{I}}$ and the **affine Lie algebra** $\widehat{\mathfrak{g}}$. The associated affine root system $\widehat{\Delta} = \widehat{\Delta}^+ \sqcup \widehat{\Delta}^-$ has the following explicit description:

$$\widehat{\Delta}^+ = \big\{\Delta^+ \times \mathbb{Z}_{\geq 0}\big\} \sqcup \big\{0 \times \mathbb{Z}_{>0}\big\} \sqcup \big\{\Delta^- \times \mathbb{Z}_{>0}\big\}, \tag{3.2}$$

$$\widehat{\Delta}^- = \big\{\Delta^- \times \mathbb{Z}_{\leq 0}\big\} \sqcup \big\{0 \times \mathbb{Z}_{<0}\big\} \sqcup \big\{\Delta^+ \times \mathbb{Z}_{<0}\big\}, \tag{3.3}$$

where $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0}$, $\mathbb{Z}_{<0}$ denote the obvious subsets of $\mathbb{Z}$. Here, $\delta = \alpha_0 + \theta = (0, 1) \in Q \times \mathbb{Z}$ is the minimal *imaginary root* of the affine root system $\widehat{\Delta}$. With this notation, we have the following root space decomposition, cf. (2.9):

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha\in\widehat{\Delta}} \widehat{\mathfrak{g}}_\alpha\,, \quad \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}} \text{ -- Cartan subalgebra.} \tag{3.4}$$

Let us now recall another realization of $\widehat{\mathfrak{g}}$. To this end, consider the Lie algebra

$$\begin{gathered} \widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot c \quad \text{with a Lie bracket given by} \\ [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n,-m}(x, y) \cdot c \quad \text{and} \quad [c, x \otimes t^n] = 0 \end{gathered} \tag{3.5}$$

where $x, y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$, and $(\cdot,\cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a non-degenerate invariant pairing.

The rich theory of affine Lie algebras is mainly based on the following key result:

Claim 3.2. There exists a Lie algebra isomorphism

$$\widehat{\mathfrak{g}} \xrightarrow{\sim} \widetilde{\mathfrak{g}} \tag{3.6}$$

determined on the generators by the following formulas:

$$\begin{aligned} & e_i \mapsto e_i \otimes t^0 && f_i \mapsto f_i \otimes t^0 && h_i \mapsto h_i \otimes t^0 \qquad \forall\, i \in I\,, \\ & e_0 \mapsto f_\theta \otimes t^1 && f_0 \mapsto e_\theta \otimes t^{-1} && h_0 \mapsto [f_\theta, e_\theta] \otimes t^0 + (f_\theta, e_\theta) \cdot c\,, \end{aligned}$$

where e_θ and f_θ are root vectors of degrees θ and $-\theta$, respectively.

In view of this result, we can explicitly describe the root subspaces from (3.4):

$$\widehat{\mathfrak{g}}_{(\alpha,k)} = \mathfrak{g}_\alpha \otimes t^k \quad \text{for } (\alpha, k) \in \widehat{\Delta}^{+,\mathrm{re}} := \left\{\Delta^+ \times \mathbb{Z}_{\geq 0}\right\} \sqcup \left\{\Delta^- \times \mathbb{Z}_{>0}\right\}, \tag{3.7}$$

$$\widehat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes t^k \quad \text{for } k\delta \in \widehat{\Delta}^{+,\mathrm{im}} := \left\{0 \times \mathbb{Z}_{>0}\right\}. \tag{3.8}$$

As $\mathbf{dim}(\mathfrak{g}_\alpha) = 1$ for any $\alpha \in \Delta$ and $\mathbf{dim}(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}) = |I|$, we thus obtain

$$\mathbf{dim}(\widehat{\mathfrak{g}}_\alpha) = 1 \quad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{re}}\,, \qquad \mathbf{dim}(\widehat{\mathfrak{g}}_\alpha) = |I| \quad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{im}}. \tag{3.9}$$

Notation: In what follows, we shall always simply write xt^n instead of $x \otimes t^n$.

3.3 Affine standard Lyndon words

It is natural to ask if the above results can be generalized to affine Lie algebras $\widehat{\mathfrak{g}}$. On the Lie algebraic side, we consider only the *positive* subalgebra $\widehat{\mathfrak{n}}^+ = \bigoplus_{\alpha \in \widehat{\Delta}^+} \widehat{\mathfrak{g}}_\alpha$. Thus, $\widehat{\mathfrak{n}}^+$ is generated by $\{e_i\}_{i \in \widehat{I}}$ subject to the Serre relations (2.10) for $i \neq j \in \widehat{I}$. On the combinatorial side, we consider the finite alphabet $\widehat{I}$ with any order on it, which allows to define Lyndon and standard Lyndon words (with respect to $\widehat{\mathfrak{n}}^+$). We shall use the term **affine standard Lyndon words** in the present setup.

The key difference with the case of simple $\mathfrak{g}$ is that some root subspaces are not one-dimensional, see (3.9). Thus, we do not get such a simple bijection as (2.12) for simple Lie algebras. However, the degree reasoning as in Subsection 2.14 implies that there is a unique affine standard Lyndon word in each real degree $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, denoted by $\mathrm{SL}(\alpha)$, and $|I|$ affine standard Lyndon words in each imaginary degree $\alpha \in \widehat{\Delta}^{+,\mathrm{im}}$, denoted by $\mathrm{SL}_1(\alpha), \ldots, \mathrm{SL}_{|I|}(\alpha)$, listed in the decreasing order.

The main result of this section is the following *generalized Leclerc's* algorithm:

Proposition 3.4. The affine standard Lyndon words (with respect to $\widehat{\mathfrak{n}}^+$) are determined inductively by the following rules:

(a) For simple roots, we have $\mathrm{SL}(\alpha_i) = [i]$. For other real $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, we have:

$$\mathrm{SL}(\alpha) = \mathbf{max}\left\{ \mathrm{SL}_*(\gamma_1)\mathrm{SL}_*(\gamma_2) \,\middle|\, \substack{\alpha = \gamma_1 + \gamma_2,\ \gamma_k \in \widehat{\Delta}^+ \\ \mathrm{SL}_*(\gamma_1) < \mathrm{SL}_*(\gamma_2) \\ [\mathrm{b}[\mathrm{SL}_*(\gamma_1)], \mathrm{b}[\mathrm{SL}_*(\gamma_2)]] \neq 0} \right\}, \tag{3.10}$$

where $\mathrm{SL}_*(\gamma)$ denotes $\mathrm{SL}(\gamma)$ for $\gamma \in \widehat{\Delta}^{+,\mathrm{re}}$ and any of $\{\mathrm{SL}_k(\gamma)\}_{k=1}^{|I|}$ for $\gamma \in \widehat{\Delta}^{+,\mathrm{im}}$.

(b) For imaginary $\alpha \in \widehat{\Delta}^{+,\mathrm{im}}$, the corresponding $|I|$ affine standard Lyndon words $\{\mathrm{SL}_k(\alpha)\}_{k=1}^{|I|}$ are the $|I|$ lexicographically largest words from the list as in the right-hand side of (3.10) whose standard bracketings are linearly independent.

Remark 3.5. Since $[\widehat{\mathfrak{g}}_{a\delta}, \widehat{\mathfrak{g}}_{b\delta}] = 0$ for any $a, b > 0$, we shall assume that $\gamma_1, \gamma_2 \in \widehat{\Delta}^{+,\mathrm{re}}$ when applying part (b). Thus, $\mathrm{SL}_1(\alpha)$ is given precisely by (3.10), $\mathrm{SL}_2(\alpha)$ is the next largest word among the above concatenations whose bracketing is not a multiple of $\mathrm{b}[\mathrm{SL}_1(\alpha)]$, and so on, up to $\mathrm{SL}_{|I|}(\alpha)$, which is the largest of the remaining concatenations whose standard bracketing is linearly independent with $\{\mathrm{b}[\mathrm{SL}_k(\alpha)]\}_{k=1}^{|I|-1}$.

Proof of Proposition 3.4. (a) Consider the costandard factorization $\mathrm{SL}(\alpha) = \ell_1\ell_2$ as in (2.4). Then, $\ell_1 = \mathrm{SL}_*(\gamma_1), \ell_2 = \mathrm{SL}_*(\gamma_2)$ for some $\gamma_1, \gamma_2 \in \widehat{\Delta}^+$ and $\ell_1 < \ell_2$. Finally, $\mathrm{b}[\mathrm{SL}(\alpha)] \neq 0$ implies that $[\mathrm{b}[\mathrm{SL}_*(\gamma_1)], \mathrm{b}[\mathrm{SL}_*(\gamma_2)]] \neq 0$. Therefore, $\ell_1\ell_2$ is an element from the right-hand side of (3.10). It thus remains to show that $\mathrm{SL}(\alpha)$ is $\geq$ any concatenation $\mathrm{SL}_*(\gamma_1)\mathrm{SL}_*(\gamma_2)$ featuring in the right-hand side of (3.10).

The proof of the latter is completely analogous to that of [8, Proposition 2.23]. Consider any $\gamma_1, \gamma_2 \in \widehat{\Delta}^+$ such that $\gamma_1 + \gamma_2 = \alpha$. Let us write $\ell_1 = \mathrm{SL}_*(\gamma_1)$, $\ell_2 = \mathrm{SL}_*(\gamma_2)$, $\ell = \mathrm{SL}(\alpha)$. We may assume, without loss of generality, that $\ell_1 < \ell_2$. Evoking the notations of Subsection 2.10, we have

$$\mathrm{b}[\ell_k] = e_{\ell_k} = \sum_{v_k \geq \ell_k} c_{\ell_k}^{v_k} \cdot {}_{v_k}e \tag{3.11}$$

$\forall k \in \{1, 2\}$, due to the triangularity property (2.8). Thus, due to the degree reasons (see [8, Footnote 2]), we get

$$\mathrm{b}[\ell_1]\mathrm{b}[\ell_2] = e_{\ell_1}e_{\ell_2} = \sum_{v \geq \ell_1\ell_2} x_v \cdot {}_v e \tag{3.12}$$

for some coefficients x_v. As a consequence of $\ell_2\ell_1 > \ell_1\ell_2$ (Lemma 2.4), we also get

$$\mathrm{b}[\ell_2]\mathrm{b}[\ell_1] = e_{\ell_2}e_{\ell_1} = \sum_{v \geq \ell_1\ell_2} x'_v \cdot {}_v e \tag{3.13}$$

for some coefficients x'_v. Hence, we obtain the following formula for the commutator:

$$[\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] = [e_{\ell_1}, e_{\ell_2}] = \sum_{v \geq \ell_1\ell_2} y_v \cdot {}_v e \tag{3.14}$$

for various coefficients y_v. Furthermore, we may restrict the sum above to standard v's, since by the very definition of this notion, any ${}_v e$ can be inductively written as a linear combination of ${}_u e$'s for standard $u \geq v$. By the same reason, we may restrict the right-hand side of (2.8) to standard v's and conclude that $\{e_w\}_{w-\text{standard}}$ provide a basis of $U(\widehat{\mathfrak{n}}^+)$, which is upper triangular in terms of the basis $\{{}_w e\}_{w-\text{standard}}$. With the above observations in mind, (3.14) implies

$$[\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] = [e_{\ell_1}, e_{\ell_2}] = \sum_{\substack{v \geq \ell_1\ell_2 \\ v-\text{standard}}} z_v \cdot e_v \tag{3.15}$$

for various z_v. Meanwhile, the assumption $[\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] \neq 0$ and $\widehat{\mathfrak{g}}_\alpha = \mathbb{C} \cdot \mathrm{b}[\ell]$ imply

$$[\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] = [e_{\ell_1}, e_{\ell_2}] \in \mathbb{C}^\times \cdot e_\ell. \tag{3.16}$$

As $\{e_v\}_{v-\text{standard}}$ is a basis of $U(\widehat{\mathfrak{n}}^+)$, comparing (3.15, 3.16) we obtain $\ell \geq \ell_1\ell_2$, precisely as claimed above.

(b) The proof of part (b) is completely analogous to that of part (a), with the only difference that we need to find $|I|$ affine standard Lyndon words. Thus, we just use Definition 2.11(b) to complement the above argument in the present setup. ■

3.6 Affine standard Lyndon words in type $A_1^{(1)}$

As the first simplest example, let us compute affine standard Lyndon words in the simplest case of $A_1^{(1)}$, which corresponds to the affinization $\widehat{\mathfrak{sl}}_2$ of the unique rank 1 simple Lie algebra $\mathfrak{sl}_2$. In this case, there are two simple roots α_0, α_1 and $\delta = \alpha_0 + \alpha_1$. The set of positive roots is $\widehat{\Delta}^+ = \{k\delta + \alpha_1, k\delta + \alpha_0, (k+1)\delta | k \geq 0\}$. Without loss of generality, we can assume that $1 < 0$, due to the $0 \leftrightarrow 1$ symmetry.

Proposition 3.7. The affine standard Lyndon words for $\widehat{\mathfrak{sl}}_2$ with the order $1 < 0$ on the corresponding alphabet $\widehat{I} = \{0, 1\}$ are

- For $k \geq 1$, we have

$$\mathrm{SL}(k\delta+\alpha_1) = 1\underbrace{10}_{k\text{ times}}, \tag{3.17}$$

$$\mathrm{SL}(k\delta+\alpha_0) = \underbrace{10}_{k\text{ times}}0, \tag{3.18}$$

$$\mathrm{SL}((k+1)\delta) = 1\underbrace{10}_{k\text{ times}}0. \tag{3.19}$$

- For the remaining roots, we have

$$\mathrm{SL}(\alpha_1) = 1\,, \quad \mathrm{SL}(\alpha_0) = 0\,, \quad \mathrm{SL}(\delta) = 10. \tag{3.20}$$

Proof. Formulas (3.20) are obvious, while the proof of (3.17)–(3.19) will proceed by induction on k. The base $k = 1$ case is easy. We shall now prove the induction step, just by using the generalized Leclerc's algorithm from Proposition 3.4.

1) Root $\alpha = k\delta+\alpha_1$. Any decomposition $\alpha = \gamma_1+\gamma_2$ has the following form: $\{\gamma_1, \gamma_2\} = \{a\delta, b\delta+\alpha_1 \mid a+b = k, 1 \leq a \leq k\}$. By the induction hypothesis:

$$\mathrm{SL}(b\delta+\alpha_1) = 1\underbrace{10}_{b\text{ times}} < 1\underbrace{10}_{(a-1)\text{ times}}0 = \mathrm{SL}(a\delta).$$

Following (3.10), consider the lexicographically largest word among all possible concatenations $1\underbrace{10}_{b\text{ times}}1\underbrace{10}_{(a-1)\text{ times}}0$, which is $1\underbrace{10}_{k\text{ times}}$. Let us show by induction on k that its standard bracketing is $(-2)^k E_{12} t^k$, thus completing the proof of (3.17):

$$\mathrm{b}[1\underbrace{10}_{k\text{ times}}] = [\mathrm{b}[1\underbrace{10}_{(k-1)\text{ times}}], \mathrm{b}[10]] = [(-2)^{k-1}E_{12}t^{k-1}, (E_{11}-E_{22})t] = (-2)^k E_{12} t^k.$$

2) Root $\alpha = k\delta+\alpha_0$. Any decomposition $\alpha = \gamma_1+\gamma_2$ has the following form: $\{\gamma_1, \gamma_2\} = \{a\delta, b\delta+\alpha_0 \mid a+b = k, 1 \leq a \leq k\}$. As in 1), one combines the inductive hypothesis with (3.10) to find: $\mathrm{SL}(\alpha) = \underbrace{10}_{k\text{ times}}0$ with the standard bracketing

$$\mathrm{b}[\underbrace{10}_{k\text{ times}}0] = (-2)^k E_{21} t^{k+1}.$$

3) Let us now treat the imaginary root $\alpha = (k+1)\delta$. As $\mathrm{rank}(\mathfrak{sl}_2) = 1$, there is only one affine standard Lyndon word in degree α, which can be found by (3.10). Any decomposition $\alpha = \gamma_1+\gamma_2$ that contributes into $\mathrm{SL}(\alpha)$ is of the form: $\{\gamma_1, \gamma_2\} = \{a\delta+\alpha_1, b\delta+\alpha_0 \mid a+b = k, 0 \leq a \leq k\}$. By the induction hypothesis:

$$\mathrm{SL}(a\delta+\alpha_1) = 1\underbrace{10}_{a\text{ times}} < \underbrace{10}_{b\text{ times}}0 = \mathrm{SL}(b\delta+\alpha_0).$$

Following (3.10), consider the lexicographically largest word among all the corresponding concatenations $\mathrm{SL}(a\delta+\alpha_1)\mathrm{SL}(b\delta+\alpha_0) = 1\underbrace{10}_{k\text{ times}}0$, which completes the proof of (3.19). Let us evaluate its standard bracketing:

$$\mathrm{b}[1\underbrace{10}_{k\text{ times}}0] = [\mathrm{b}[1], \mathrm{b}[\underbrace{10}_{k\text{ times}}0]] = [E_{12}, (-2)^k E_{21} t^{k+1}] = (-2)^k (E_{11}-E_{22}) t^{k+1}.$$

This completes the proof of the induction step. ∎

4 Affine Standard Lyndon Words in Type $A_n^{(1)}$ for $n \geq 2$

In this section, we describe affine standard Lyndon words in affine type $A_n^{(1)}$ for $n \geq 2$ and any order on $\widehat{I} = \{0, 1, 2, \ldots, n\}$. First, we treat the simplest case (of the *standard order*) to which Proposition 3.4 can be easily applied. We then crucially utilize the convexity property of Proposition 2.20 to derive the structure of affine standard Lyndon words for an arbitrary order on $\widehat{I}$.

4.1 Standard order

We start by computing all affine standard Lyndon words for type $A_n^{(1)}$ with

$$\text{the } \textit{standard} \text{ order on } \widehat{I}: \quad 1 < 2 < 3 < \cdots < n < 0. \tag{4.1}$$

There are $n+1$ simple roots $\alpha_0, \alpha_1, \ldots, \alpha_n$, and $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$. It is convenient to place the letters of the alphabet $\widehat{I} = \{0, 1, 2, \ldots, n\}$ on a circle counterclockwise. For any counterclockwise oriented arch from i to j, we define

$$\alpha_{i \to j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \in Q. \tag{4.2}$$

Using this notation, the positive affine roots can be explicitly described as follows:

$$\widehat{\Delta}^+ = \left\{ k\delta + \alpha_{i \to j}, (k+1)\delta \,\middle|\, k \geq 0\,,\ i, j \in \widehat{I},\ j \neq \overline{i-1} \right\}. \tag{4.3}$$

Here, for any $k \in \mathbb{Z}$ we define $\overline{k} \in \widehat{I}$ via

$$\overline{k} := k \bmod (n+1). \tag{4.4}$$

We also use $[i \to j)$ to denote all letters on the arch from i (included) to j (excluded)

$$[i \to j) := \left\{ i, \overline{i+1}, \ldots, \overline{j-1} \right\}. \tag{4.5}$$

Theorem 4.2. The affine standard Lyndon words for $\widehat{\mathfrak{sl}}_{n+1}$ with the standard order $1 < 2 < \cdots < n < 0$ on the corresponding alphabet $\widehat{I} = \{0, 1, \ldots, n\}$ are as follows:

- For $k \geq 1$, we have

$$\mathrm{SL}(k\delta + \alpha_{i \to j}) = \underbrace{10n \ldots i\, 23 \ldots \overline{i-1}}_{k \text{ times}}\, i\, \overline{i+1} \ldots j\,, \quad \text{for } 2 < i \leq j \leq 0, \tag{4.6}$$

$$\mathrm{SL}(k\delta + \alpha_2) = \underbrace{10n \ldots 32}_{k \text{ times}}\, 2, \tag{4.7}$$

$$\mathrm{SL}(k\delta + \alpha_{2 \to j}) = \begin{cases} \underbrace{10n \ldots 32}_{\frac{k}{2} \text{ times}}\, 2\, \underbrace{10n \ldots 32}_{\frac{k}{2} \text{ times}}\, 34 \ldots j & \text{if } 2 \mid k \\ \underbrace{10n \ldots 32}_{\frac{k+1}{2} \text{ times}}\, 34 \ldots j\, \underbrace{10n \ldots 32}_{\frac{k-1}{2} \text{ times}}\, 2 & \text{if } 2 \nmid k \end{cases}, \quad \text{for } 2 < j \leq 0, \tag{4.8}$$

$$\mathrm{SL}(k\delta + \alpha_{1 \to i}) = 123 \ldots n\, \underbrace{1023 \ldots n}_{(k-1) \text{ times}}\, 1023 \ldots i\,, \quad \text{for } 1 \leq i < 0, \tag{4.9}$$

$$\mathrm{SL}(k\delta + \alpha_{j \to i}) = \mathrm{SL}(k\delta + \alpha_{j \to 0} + \alpha_{1 \to i}) = \qquad \text{for } i < \overline{i+1} < j$$

$$10n \ldots j\, 23 \ldots \overline{j-2}\, \underbrace{10n \ldots \overline{j-1}\, 23 \ldots \overline{j-2}}_{(k-1) \text{ times}}\, 10n \ldots \overline{j-1}\, 23 \ldots i, \tag{4.10}$$

$$\mathrm{SL}_n((k+1)\delta) = 123\ldots n\,\underbrace{1023\ldots n}_{k\text{ times}}0,$$

$$\mathrm{SL}_r((k+1)\delta) = 10n\ldots\overline{r+2}\,23\ldots r\,\underbrace{10n\ldots(r+1)23\ldots r}_{k\text{ times}}(r+1)\,, \text{ for } r<n. \tag{4.11}$$

- For the remaining roots, we have

$$\mathrm{SL}(\alpha_{i\to j}) = i(i+1)\ldots j\,, \quad \text{for } i\le j \text{ and } (i,j)\ne(1,0), \tag{4.12}$$

$$\mathrm{SL}(\alpha_{j\to i}) = \mathrm{SL}(\alpha_{j\to 0}+\alpha_{1\to i}) = 10n\ldots j\,23\ldots i\,, \quad \text{for } i<\overline{i+1}<j, \tag{4.13}$$

$$\mathrm{SL}_r(\delta) = 10\ldots\overline{r+2}\,23\ldots\overline{r+1}\,, \quad \text{for } 1\le r\le n. \tag{4.14}$$

Proof. The proof will proceed by induction on the height $\mathrm{ht}(\alpha)$. Let $h=\mathrm{ht}(\delta)=n+1$ be the *Coxeter number* of $\mathfrak{sl}_{n+1}$. The base of induction is $\mathrm{ht}(\alpha)<2h$, that is, $k=0,1$ cases for real roots $k\delta+\alpha_{i\to j}$ and $k=0$ case for imaginary roots $(k+1)\delta$.

Base of Induction (part I)

First, let us verify (4.12)–(4.14) and find bracketings of the corresponding words.

• Proof of (4.12).

Consider the costandard factorization $\ell=\ell_1\ell_2$ of any Lyndon word ℓ with $\mathbf{deg}\,\ell=\alpha_{i\to j}$. As $i<i+1$ are the two smallest letters of ℓ, the word ℓ_1 starts with i and ℓ_2 starts with $i+1$. If furthermore ℓ is standard Lyndon, so is ℓ_1, hence, $\mathbf{deg}\,\ell_1\in\widehat{\Delta}^+$. For degree reasons, this is only possible if $\ell_1=i$ and $\mathbf{deg}\,\ell_2=\alpha_{(i+1)\to j}$. Arguing by induction on the height of $\alpha_{i\to j}$, we thus immediately derive the desired formula (4.12). Moreover, we also inductively get the explicit formula for the corresponding standard bracketing:

$$\mathrm{b}[\mathrm{SL}(\alpha_{i\to j})] = \mathrm{b}[i(i+1)\ldots j] = [\mathrm{b}[i],\mathrm{b}[(i+1)\ldots j]] = \begin{cases} E_{i,j+1}t^0 & \text{if } j\le n \\ E_{i,1}t & \text{if } j=0 \end{cases}.$$

Notation: Henceforth, we shall use the matrix $E_{0,p}$ to denote $E_{n+1,p}$.

• Proof of (4.13) for $i=1$.

In this case, we shall rather use (3.10) and argue by induction on the height of $\alpha_{j\to 1}$ (i.e., a descending induction of $j\in\widehat{I}$). The possible decompositions of $\alpha_{j\to 1}$ into the (unordered) sum of two positive roots are as follows:

$$\alpha_{j\to 1}=\alpha_{j\to k}+\alpha_{\overline{k+1}\to 1}\quad (j\le k\le n)\,,\qquad \alpha_{j\to 1}=\alpha_{j\to 0}+\alpha_1.$$

Combining the induction hypothesis with formula (4.12), we get the following list of concatenated words featuring in the right-hand side of (3.10) for $\alpha=\alpha_{j\to 1}$:

$$10n\ldots\overline{k+1}\,j\,\overline{j+1}\ldots k\quad (j\le k\le 0). \tag{4.15}$$

Clearly, $10n\ldots j$ is the lexicographically largest word from this list (4.15). Let us evaluate its standard bracketing:

$$\mathrm{b}[10n\ldots j] = [\mathrm{b}[10n\ldots\overline{j+1}],\mathrm{b}[j]] = [(-1)^{n-j-1}E_{j+1,2}t, E_{j,j+1}] = (-1)^{n-j}E_{j,2}t,$$

where we use the induction hypothesis for the value of $\mathrm{b}[10n\ldots(j+1)]$. We thus obtain $\mathrm{SL}(\alpha_{j\to 1})=10n\ldots j$ as claimed in (4.13), since the bracketing is nonzero.

• Proof of (4.13) for $i>1$.

In the present case, we can argue alike in our verification of (4.12). Consider the costandard factorization $\mathrm{SL}(\alpha_{j\to i})=\ell_1\ell_2$. Since $1<2$ are the two smallest letters, ℓ_1 starts with 1 and ℓ_2 starts with 2.

Moreover, we have $\mathbf{deg}\,\ell_1, \mathbf{deg}\,\ell_2 \in \widehat{\Delta}^+$. For degree reasons, this is only possible if $\mathbf{deg}\,\ell_1 = \alpha_{j\to 1}$ and $\mathbf{deg}\,\ell_2 = \alpha_{2\to i}$. We thus have $\ell_1 = 10n\ldots j$ and $\ell_2 = 23\ldots i$ by above, and (4.13) follows. Furthermore,

$$\mathrm{b}[\mathrm{SL}(\alpha_{j\to i})] = \mathrm{b}[10n\ldots j\,23\ldots i] = [\mathrm{b}[10n\ldots j], \mathrm{b}[23\ldots i]] = (-1)^{n-j}E_{j,i+1}t.$$

• Proof of (4.14).

Let us now treat the case of the smallest imaginary root δ. The possible decompositions of δ into the (unordered) sum of two positive roots are as follows:

$$\delta = \alpha_{1\to i} + \alpha_{\overline{i+1}\to 0}\ \ (1\le i\le n)\,, \qquad \delta = \alpha_{i\to j} + \alpha_{\overline{j+1}\to(i-1)}\ \ (2\le i\le j\le n).$$

Using already verified formulas (4.12) and (4.13), we thus get the following list of concatenated words featuring in the right-hand side of (3.10) for $\alpha = \delta$:

$$12\ldots i\,\overline{i+1}\ldots n0\,, \qquad 10n\ldots\overline{j+1}\,23\ldots(i-1)i(i+1)\ldots j\quad (2\le j\le n).$$

Since this list contains exactly n different words (we note the independence of i), all of them are precisely $\mathrm{SL}_1(\delta),\ldots,\mathrm{SL}_n(\delta)$. Ordering them lexicographically, we derive the desired formula (4.14). Let us compute their standard bracketings:

$$\begin{aligned}
\mathrm{b}[\mathrm{SL}_r(\delta)] &= \mathrm{b}[10\ldots\overline{r+2}\,23\ldots\overline{r+1}] = [\mathrm{b}[10\ldots\overline{r+2}], \mathrm{b}[23\ldots\overline{r+1}]] = \\
&\quad [(-1)^{n-r}E_{r+2,2}t, E_{2,r+2}] = (-1)^{n-r+1}(E_{22} - E_{r+2,r+2})t \quad \text{if } r\le n-1, \\
\mathrm{b}[\mathrm{SL}_n(\delta)] &= \mathrm{b}[123\ldots n0] = [\mathrm{b}[1], \mathrm{b}[23\ldots n0]] = (E_{11} - E_{22})t.
\end{aligned} \tag{4.16}$$

<u>Base of Induction (part II)</u>

As a continuation of the induction base, let us now verify (4.6)–(10) for $k = 1$.

• Proof of (4.6) for $k = 1$.

We verify the formula for $\mathrm{SL}(\delta+\alpha_{i\to j})$ with $2 < i \le j$ by induction on $\mathrm{ht}(\alpha_{i\to j})$. (1) The base of induction is $i = j$. The possible decompositions of $\delta + \alpha_i$ into the (unordered) sum of two positive roots are as follows:

$$\delta + \alpha_i = (\delta) + (\alpha_i)\,, \qquad \delta + \alpha_i = \alpha_{i\to \jmath} + \alpha_{\overline{\jmath+1}\to i} \quad (\jmath \ne i, \overline{i-1}). \tag{4.17}$$

Using already verified formulas (4.12)–(4.14), we get the following list of concatenated words featuring in the right-hand side of (3.10) for $\alpha = \delta + \alpha_i$:

$$\begin{aligned}
&10n\ldots i\,23\ldots\overline{i-1}\,i\,, \quad 10n\ldots\overline{i+1}\,23\ldots i\,i, \\
&10n\ldots\overline{\jmath+1}\,23\ldots i\,i(i+1)\ldots \jmath \quad \text{for } i < \jmath \le n, \\
&10n\ldots i\,23\ldots \jmath\,(\jmath+1)\ldots i \quad \text{for } 1\le \jmath < \overline{i-1}, \\
&12\ldots i\,i\,\overline{i+1}\ldots 0.
\end{aligned} \tag{4.18}$$

Here, the two words in the first line correspond to the fact that $[\mathrm{b}[\mathrm{SL}_r(\delta)], \mathrm{b}[i]] \ne 0$ only for $\overline{r+2} = i, i-1$, due to (4.16), while the last three lines just correspond to the cases $i < \jmath \le n$, $1 \le \jmath < \overline{i-1}$, and $\jmath = 0$ in (4.17). Clearly, $10n\ldots i\,23\ldots\overline{i-1}\,i$ is the lexicographically largest word from the list (4.18). Therefore, $\mathrm{SL}(\delta+\alpha_i)$ is indeed given by (4.6) as the corresponding standard bracketing does not vanish:

$$\mathrm{b}[\mathrm{SL}(\delta+\alpha_i)] = \mathrm{b}[10n\ldots i\,23\ldots\overline{i-1}\,i] = \begin{cases} (-1)^{n-i}E_{i,i+1}t & \text{if } 2 < i \le n \\ -E_{n+1,1}t^2 & \text{if } i = 0 \end{cases}.$$

(2) Let us now prove the induction step: compute $\mathrm{SL}(\delta+\alpha_{i\to j})$ for $\mathrm{ht}(\alpha_{i\to j}) = p+1$ using the formulas for $\mathrm{SL}(\delta+\alpha_{\imath\to \jmath})$ with $\mathrm{ht}(\alpha_{\imath\to\jmath}) \le p$. The possible decompositions of $\delta+\alpha_{i\to j}$ into the (unordered) sum of two

positive roots are as follows:

$$\begin{aligned}
&\delta+\alpha_{i\to j}=(\delta)+(\alpha_{i\to j})\\
&\delta+\alpha_{i\to j}=(\delta+\alpha_{i\to \jmath})+(\alpha_{\overline{\jmath+1}\to j}) \quad \text{for } \jmath\in[i\to j)\\
&\delta+\alpha_{i\to j}=(\alpha_{i\to \jmath})+(\delta+\alpha_{\overline{\jmath+1}\to j}) \quad \text{for } \jmath\in[i\to j)\\
&\delta+\alpha_{i\to j}=(\alpha_{i\to \jmath})+(\alpha_{\overline{\jmath+1}\to j}) \quad \text{for } \jmath\in[\overline{j+1}\to(i-1)).
\end{aligned} \tag{4.19}$$

The corresponding list of concatenations is as follows:

$$\begin{aligned}
&10n\dots i\,23\dots(i-1)i\dots j\,,\quad 10n\dots\overline{j+1}\,23\dots j\,i\dots j,\\
&10n\dots i\,23\dots(i-1)i\dots \jmath\,\overline{\jmath+1}\dots j \quad \text{for } \jmath\in[i\to j),\\
&10n\dots\overline{\jmath+1}\,23\dots\dots j\,i\,\overline{i+1}\dots \jmath \quad \text{for } \jmath\in[i\to j),\\
&10n\dots\overline{\jmath+1}\,23\dots j\,i(i+1)\dots j\,\overline{j+1}\dots \jmath \quad \text{for } j<\jmath\le n,\\
&10n\dots i\,23\dots \jmath\,\overline{\jmath+1}\dots j \quad \text{for } 1\le \jmath<i-1,\\
&123\dots j\,i\,\overline{i+1}\dots 0.
\end{aligned} \tag{4.20}$$

The two words in the first line correspond to the fact that $[b[SL_r(\delta)], b[SL(\alpha_{i\to j})]] \neq 0$ only when $\overline{r+2}=i$, $\overline{j+1}$, while the words from the last three lines correspond to the cases $j<\jmath\le n$, $1\le\jmath<i-1$, and $\jmath=0$ in the last decomposition of (4.19). Clearly, $10n\dots i\,23\dots j$ is the lexicographically largest word from the list (4.20). Therefore, $SL(\delta+\alpha_{i\to j})$ is indeed given by (4.6) as the corresponding standard bracketing does not vanish:

$$b[SL(\delta+\alpha_{i\to j})]=b[10n\dots i\,23\dots j]=\begin{cases}(-1)^{n-i}E_{i,j+1}t & \text{if } 2<i<j\le n\\ (-1)^{n-i}E_{i,1}t^2 & \text{if } 2<i<j=0\end{cases}.$$

• Proof of (4.7) for $k=1$.

The possible decompositions of $\delta+\alpha_2$ into the (unordered) sum of two positive roots are as follows:

$$\delta+\alpha_2=(\delta)+(\alpha_2)\,,\qquad \delta+\alpha_2=\alpha_{2\to \jmath}+\alpha_{\overline{\jmath+1}\to 2}\quad(\jmath\neq 1,2). \tag{4.21}$$

Thus, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_2$ are

$$\begin{aligned}
&10n\dots\overline{r+2}\,23\dots\overline{r+1}\,2 \quad \text{for } 1\le r\le n,\\
&10n\dots\overline{\jmath+1}\,223\dots \jmath \quad (2<\jmath\le n)\,,\qquad 1223\dots n0.
\end{aligned} \tag{4.22}$$

Here, the n words in the first line correspond to the fact that $[b[SL_r(\delta)], b[2]]\neq 0$ for all $1\le r\le n$, according to (4.16). Clearly, $10n\dots 322$ is the lexicographically largest word from the list (4.22). Therefore, $SL(\delta+\alpha_2)$ is indeed given by (4.7) as the corresponding standard bracketing does not vanish:

$$b[SL(\delta+\alpha_2)]=b[10n\dots 322]=[b[10n\dots 32], b[2]]=2(-1)^nE_{23}t.$$

• Proof of (4.8) for $k=1$.

Let us prove by induction on j that:

$$SL(\delta+\alpha_{2\to j})=10n\dots 3234\dots j2 \quad \text{for } 2\le j\le 0. \tag{4.23}$$

(1) The base of induction is $j=2$, for which the result was just proved above.

(2) Let us now prove the induction step: prove (4.23) for $\mathrm{SL}(\delta+\alpha_{2\to j})$ utilizing the same formula for $\mathrm{SL}(\delta+\alpha_{2\to \jmath})$ with $2\leq \jmath<j$. The possible decompositions of $\delta+\alpha_{2\to j}$ into the (unordered) sum of two positive roots are as follows:

$$
\begin{aligned}
\delta+\alpha_{2\to j} &= (\delta)+(\alpha_{2\to j}) \\
\delta+\alpha_{2\to j} &= (\delta+\alpha_{2\to \jmath})+(\alpha_{\overline{\jmath+1}\to j}) \quad \text{for } \jmath\in[2\to j) \\
\delta+\alpha_{2\to j} &= (\delta+\alpha_{\overline{\jmath+1}\to j})+(\alpha_{2\to \jmath}) \quad \text{for } \jmath\in[2\to j) \\
\delta+\alpha_{2\to j} &= (\alpha_{2\to \jmath})+(\alpha_{\overline{\jmath+1}\to j}) \quad \text{for } \jmath\in[\overline{j+1}\to 1)
\end{aligned}
\tag{4.24}
$$

Thus, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_{2\to j}$ are

$$
\begin{aligned}
&10n\ldots\overline{r+2}\,23\ldots\overline{r+1}\,23\ldots j\,, \quad \text{for } 1\leq r\leq n, \\
&10n\ldots 3234\ldots \jmath\, 2\,\overline{\jmath+1}\ldots j \quad \text{for } \jmath\in[2\to j), \\
&10n\ldots\overline{\jmath+1}\,23\ldots j\,23\ldots \jmath \quad \text{for } \jmath\in[2\to j), \\
&10n\ldots\overline{\jmath+1}\,23\ldots j\,23\ldots \jmath \quad (j<\jmath\leq n)\,, \qquad 12\ldots j\,23\ldots n0.
\end{aligned}
\tag{4.25}
$$

The n words in the first line correspond to the fact that $[\mathrm{b}[\mathrm{SL}_r(\delta)],\mathrm{b}[\mathrm{SL}(\alpha_{2\to j})]]\neq 0$ for all $1\leq r\leq n$, according to (4.16). Clearly, $10n\ldots 3234\ldots j2$ is the lexicographically largest word from the list (4.25). Therefore, $\mathrm{SL}(\delta+\alpha_{2\to j})$ is indeed given by (4.23) as the corresponding standard bracketing does not vanish:

$$
\mathrm{b}[\mathrm{SL}(\delta+\alpha_{2\to j})]=\mathrm{b}[10n\ldots 3234\ldots j2]=\begin{cases}(-1)^n E_{2,j+1}t & \text{if } 2<j\leq n\\ (-1)^n E_{21}t^2 & \text{if } j=0\end{cases}.
$$

• Proof of (4.9) for $k=1$.

Let us prove by induction on i that:

$$
\mathrm{SL}(\delta+\alpha_{1\to i})=123\ldots n\,1023\ldots i \quad \text{for } 1\leq i\leq n. \tag{4.26}
$$

(1) The base of induction is $i=1$. The possible decompositions of $\delta+\alpha_1$ into the (unordered) sum of two positive roots are as follows:

$$
\delta+\alpha_1=(\delta)+(\alpha_1)\,, \qquad \delta+\alpha_1=(\alpha_{1\to \jmath})+(\alpha_{\overline{\jmath+1}\to 1}) \quad (\jmath\neq 0,1). \tag{4.27}
$$

Thus, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_1$ are

$$
\begin{aligned}
&1\,10n\ldots\overline{r+2}\,23\ldots\overline{r+1} \quad \text{for } 1\leq r\leq n, \\
&123\ldots \jmath\,10n\ldots(\jmath+1) \quad (1<\jmath<n)\,, \qquad 123\ldots n\,10.
\end{aligned}
\tag{4.28}
$$

Here, the n words in the first line correspond to the fact that $[\mathrm{b}[\mathrm{SL}_r(\delta)],\mathrm{b}[1]]\neq 0$ for all $1\leq r\leq n$, according to (4.16). Clearly, $123\ldots n\,10$ is the lexicographically largest word from the list (4.28). Therefore, $\mathrm{SL}(\delta+\alpha_1)$ is indeed given by (4.26) as the corresponding standard bracketing does not vanish:

$$
\mathrm{b}[\mathrm{SL}(\delta+\alpha_1)]=\mathrm{b}[123\ldots n\,10]=[\mathrm{b}[123\ldots n],\mathrm{b}[10]]=-E_{12}t.
$$

(2) Let us now prove the induction step: prove (4.26) for $\mathrm{SL}(\delta+\alpha_{1\to i})$ utilizing the same formula for $\mathrm{SL}(\delta+\alpha_{1\to \imath})$ with $1\leq \imath<i$. The possible decompositions of $\delta+\alpha_{1\to i}$ into the (unordered) sum of two

positive roots are as follows:

$$\begin{aligned}
&\delta+\alpha_{1\to \mathrm{i}} = (\delta) + (\alpha_{1\to \mathrm{i}}) \\
&\delta+\alpha_{1\to \mathrm{i}} = (\delta+\alpha_{1\to\iota}) + (\alpha_{(\iota+1)\to \mathrm{i}}) \quad \text{for } \iota\in[1\to \mathrm{i}) \\
&\delta+\alpha_{1\to \mathrm{i}} = (\delta+\alpha_{(\iota+1)\to \mathrm{i}}) + (\alpha_{1\to\iota}) \quad \text{for } \iota\in[1\to \mathrm{i}) \\
&\delta+\alpha_{1\to \mathrm{i}} = (\alpha_{1\to\iota}) + (\alpha_{\overline{\iota+1}\to \mathrm{i}}) \quad \text{for } \iota\in[\overline{\mathrm{i}+1}\to 0)
\end{aligned} \tag{4.29}$$

Thus, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_{1\to \mathrm{i}}$ are

$$\begin{aligned}
&123\ldots i\,10n\ldots\overline{i+1}\,23\ldots i\,, \qquad 123\ldots i\,123\ldots n0, \\
&123\ldots n\,1023\ldots\iota\,(\iota+1)\ldots i \quad \text{for } 1\le\iota<i, \\
&123\ldots\iota\,10n\ldots(\iota+1)\,23\ldots i \quad \text{for } 1<\iota<i\,, \qquad 1\,10n\ldots 3234\ldots i2, \\
&123\ldots\iota\,10n\ldots\overline{\iota+1}\,23\ldots i \quad \text{for } i<\iota\le n.
\end{aligned} \tag{4.30}$$

The two words in the first line correspond to the fact that $[\mathrm{b}[\mathrm{SL}_r(\delta)],\mathrm{b}[\mathrm{SL}(\alpha_{1\to \mathrm{i}})]]\neq 0$ only when $r=i-1,n$ (for $1<i\le n$), while the words in the third line correspond to the cases $1<\iota<i$ and $\iota=1$ in the third line of (4.29). Clearly, $123\ldots n\,1023\ldots i$ is the lexicographically largest word from the list (4.30). Therefore, $\mathrm{SL}(\delta+\alpha_{1\to \mathrm{i}})$ is indeed given by (4.26) as the corresponding standard bracketing does not vanish:

$$\mathrm{b}[\mathrm{SL}(\delta+\alpha_{1\to \mathrm{i}})] = \mathrm{b}[123\ldots n\,1023\ldots \mathrm{i}] = [\mathrm{b}[123\ldots n],\mathrm{b}[1023\ldots \mathrm{i}]] = -E_{1,\mathrm{i}+1}t.$$

- Proof of (10) for $k=1$.
  Let us prove by induction on $\mathrm{ht}(\alpha_{\mathrm{j}\to \mathrm{i}})$ that

$$\mathrm{SL}(\delta+\alpha_{\mathrm{j}\to \mathrm{i}}) = 10n\ldots j\,23\ldots\overline{j-2}\,10n\ldots\overline{j-1}\,23\ldots i \quad \text{for } i<\overline{i+1}<j. \tag{4.31}$$

(1) The base of induction is $(\mathrm{j},\mathrm{i})=(0,1)$. The possible decompositions of $\delta+\alpha_{0\to1}$ into the (unordered) sum of two positive roots are as follows:

$$\begin{aligned}
&\delta+\alpha_{0\to1} = (\delta)+(\alpha_{0\to1}), \\
&\delta+\alpha_{0\to1} = (\delta+\alpha_0)+(\alpha_1)\,, \qquad \delta+\alpha_{0\to1} = (\delta+\alpha_1)+(\alpha_0), \\
&\delta+\alpha_{0\to1} = (\alpha_{0\to\iota})+(\alpha_{(\iota+1)\to1}) \quad \text{for } 1<\iota<n.
\end{aligned} \tag{4.32}$$

Thus, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_{0\to1}$ are

$$\begin{aligned}
&1010\ldots\overline{r+2}\,23\ldots(r+1) \quad \text{for } 1<r\le n-1\,, \qquad 123\ldots n010, \\
&11023\ldots n0\,, \qquad 123\ldots n100, \\
&1023\ldots\iota\,10n\ldots(\iota+1) \quad \text{for } 1<\iota<n.
\end{aligned} \tag{4.33}$$

Here, the n words in the first line correspond to the fact that $[\mathrm{b}[\mathrm{SL}_r(\delta)],\mathrm{b}[10]]\neq 0$ for all $1\le r\le n$, according to (4.16). Clearly, $1023\ldots(n-1)10n$ is the lexicographically largest word from the list (4.33). Therefore, $\mathrm{SL}(\delta+\alpha_{0\to1})$ is indeed given by (4.31) as the corresponding standard bracketing does not vanish:

$$\mathrm{b}[\mathrm{SL}(\delta+\alpha_{0\to1})] = \mathrm{b}[1023\ldots(n-1)10n] = [\mathrm{b}[1023\ldots(n-1)],\mathrm{b}[10n]] = -E_{n+1,2}t^2.$$

(2) Let us now prove the induction step: prove (4.31) for $\mathrm{SL}(\delta+\alpha_{\mathrm{j}\to \mathrm{i}})$ utilizing the same formula for $\mathrm{SL}(\delta+\alpha_{\jmath\to\iota})$ with $[\jmath\to\iota)\subsetneq[\mathrm{j}\to \mathrm{i})$. The possible decompositions of $\delta+\alpha_{\mathrm{j}\to \mathrm{i}}$ into the (unordered) sum of

two positive roots are as follows:

$$\begin{aligned}
\delta+\alpha_{j\to i} &= (\delta+\alpha_{j\to \jmath})+(\alpha_{\overline{\jmath+1}\to i}) \quad \text{for } \jmath\in[j\to i)\\
\delta+\alpha_{j\to i} &= (\alpha_{j\to \jmath})+(\delta+\alpha_{\overline{\jmath+1}\to i}) \quad \text{for } \jmath\in[j\to i)\\
\delta+\alpha_{j\to i} &= (\alpha_{j\to \jmath})+(\alpha_{\overline{\jmath+1}\to i}) \quad \text{for } \jmath\in[(i+1)\to \overline{j-1})
\end{aligned} \tag{4.34}$$

as well as

$$\delta+\alpha_{j\to i} = (\delta)+(\alpha_{j\to i}). \tag{4.35}$$

The concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_{j\to i}$ arising through (4.34) are

$$\begin{aligned}
&10n\ldots\overline{\jmath+1}\,23\ldots i\,10n\ldots j\,23\ldots \jmath \quad \text{for } j\le \jmath\le 0,\\
&10n\ldots j\,23\ldots\overline{j-2}\,10n\ldots\overline{j-1}\,23\ldots \jmath\,(\jmath+1)\ldots i \quad \text{for } 1\le \jmath< i,\\
&10n\ldots\overline{\jmath+1}\,23\ldots(\jmath-1)10n\ldots \jmath\,23\ldots ij(j+1)\ldots \jmath \quad \text{for } j\le \jmath\le n,\\
&123\ldots n\,1023\ldots ij(j+1)\ldots n\,0,\\
&10n\ldots j\,10n\ldots 32\,34\ldots i\,2,\\
&10n\ldots j\,23\ldots \jmath\,10n\ldots(\jmath+1)\,23\ldots i \quad \text{for } 2\le \jmath< i,\\
&10n\ldots j\,23\ldots \jmath\,10n\ldots(\jmath+1)\,23\ldots i \quad \text{for } \jmath\in[(i+1)\to\overline{j-1}),
\end{aligned} \tag{4.36}$$

where the words in the first two lines of (4.36) correspond to the first line of (4.34), depending on whether $\jmath\ge j$ or $\jmath<i$, while the words in the third–sixth lines of (4.36) correspond to the second line of (4.34), depending on whether $j\le \jmath<0$, $\jmath=0$, $\jmath=1$, or $1<\jmath<i$. Meanwhile, the concatenated words in the right-hand side of (3.10) for $\alpha=\delta+\alpha_{j\to i}$ arising through the decomposition (4.35) depend on whether $i=1$ or $i\ne 1$:

$$10n\ldots j\,23\ldots i\,10n\ldots j\,23\ldots\overline{j-1}\,,\ 10n\ldots j\,23\ldots i\,10n\ldots(i+1)23\ldots i \tag{4.37}$$

if $i\ne 1$, and

$$\begin{aligned}
&10n\ldots\overline{r+2}\,23\ldots\overline{r+1}\,10n\ldots j \quad \text{for } j-2<r\le n,\\
&10n\ldots j\,10n\ldots(r+2)23\ldots(r+1) \quad \text{for } 1\le r\le j-2
\end{aligned} \tag{4.38}$$

if $i=1$. It is easy to see that $10n\ldots j\,23\ldots\overline{j-2}\,10n\ldots\overline{j-1}\,23\ldots i$ is the lexicographically largest word from the above lists (4.36)–(4.38). Thus, $\mathrm{SL}(\delta+\alpha_{j\to i})$ is indeed given by (4.31) as the corresponding standard bracketing does not vanish:

$$\mathrm{b}[\mathrm{SL}(\delta+\alpha_{j\to i})]=[\mathrm{b}[10n\ldots j\,23\ldots\overline{j-2}],\mathrm{b}[10n\ldots\overline{j-1}\,23\ldots i]]=-E_{j,i+1}t^2.$$

Step of Induction

Let us now prove the step of induction, proceeding by the height of a root. We shall thus verify the stated formulas for affine standard Lyndon words $\mathrm{SL}_*(\alpha)$ with

$$(d+1)h\le \mathrm{ht}(\alpha)<(d+2)h\,, \qquad \text{where } h=n+1=\mathrm{ht}(\delta), \tag{4.39}$$

assuming the validity of the stated formulas for all $\mathrm{SL}_*(\beta)$ with $\mathrm{ht}(\beta)<\mathrm{ht}(\alpha)$. In other words, we verify (11) for $k=d$ and formulas (4.6)–(10) for $k=d+1$.

When evaluating the standard bracketings $\mathrm{b}[\cdots]$ below, we will only need their values up to nonzero scalar factors. To this end, we shall use the following notation:

$$A \doteq B \quad \text{if} \quad A = c \cdot B \quad \text{for some } c \in \mathbb{C}\backslash\{0\}. \tag{4.40}$$

• Proof of (11) for $k = d$.

The possible decompositions of $(d+1)\delta$ into the (unordered) sum of two positive real roots are as follows:

$$(d+1)\delta = (a\delta + \alpha_1) + ((d-a)\delta + \alpha_{2\to 0}), \tag{4.41}$$

$$(d+1)\delta = (a\delta + \alpha_{1\to j}) + ((d-a)\delta + \alpha_{\overline{j+1}\to 0}) \quad \text{for } 2 \le j \le n, \tag{4.42}$$

$$(d+1)\delta = (a\delta + \alpha_{2\to j}) + ((d-a)\delta + \alpha_{\overline{j+1}\to 1}) \quad \text{for } 2 \le j \le n, \tag{4.43}$$

$$(d+1)\delta = (a\delta + \alpha_{i\to j}) + ((d-a)\delta + \alpha_{\overline{j+1}\to(i-1)}) \quad \text{for } 2 < i \le j \le n, \tag{4.44}$$

with $0 \le a \le d$. By the induction hypothesis, we get the following concatenations:

$$\ell_0^{(a)} = \begin{cases} 1\,\underbrace{10n\ldots 32}_{\frac{d}{2}\text{ times}}\,2\,\underbrace{10n\ldots 32}_{\frac{d}{2}\text{ times}}\,34\ldots n0 & \text{if } a = 0, 2 \mid d \\ 1\,\underbrace{10n\ldots 32}_{\frac{d+1}{2}\text{ times}}\,34\ldots n0\,\underbrace{10n\ldots 32}_{\frac{d-1}{2}\text{ times}}\,2 & \text{if } a = 0, 2 \nmid d \\ 12\ldots n\,\underbrace{1023\ldots n}_{(a-1)\text{ times}}\,10\,\underbrace{10n\ldots 32}_{\frac{d-a}{2}\text{ times}}\,2\,\underbrace{10n\ldots 32}_{\frac{d-a}{2}\text{ times}}\,34\ldots n0 & \text{if } 0 < a < d, 2 \mid (d-a) \\ 12\ldots n\,\underbrace{1023\ldots n}_{(a-1)\text{ times}}\,10\,\underbrace{10n\ldots 32}_{\frac{d-a+1}{2}\text{ times}}\,34\ldots n0\,\underbrace{10n\ldots 32}_{\frac{d-a-1}{2}\text{ times}}\,2 & \text{if } 0 < a < d, 2 \nmid (d-a) \\ 12\ldots n\,\underbrace{1023\ldots n}_{d\text{ times}}\,0 & \text{if } a = d \end{cases} \tag{4.45}$$

for the decompositions (4.41),

$$\ell_{1j}^{(a)} = \begin{cases} 123\ldots n\,\underbrace{1023\ldots n}_{(a-1)\text{ times}}\,1023\ldots j\,\underbrace{10n\ldots\overline{j+1}\,23\ldots j}_{(d-a)\text{ times}}\,\overline{j+1}\ldots 0 & \text{if } 1 \le a \le d \\ 123\ldots j\,\underbrace{10n\ldots\overline{j+1}\,23\ldots j}_{d\text{ times}}\,\overline{j+1}\ldots 0 & \text{if } a = 0 \end{cases} \tag{4.46}$$

for the decompositions (4.42) with $2 \le j \le n$,

$$\ell_{2j}^{(a)} = \begin{cases} 10n\ldots\overline{j+1}\,23\ldots(j-1)\,\underbrace{10n\ldots j\,23\ldots(j-1)}_{d\text{ times}}\,j & \text{if } a = 0 \\ \begin{array}{l} 10n\ldots\overline{j+1}\,23\ldots(j-1)\,\underbrace{10n\ldots j\,23\ldots(j-1)}_{(d-a-1)\text{ times}}\,10n\ldots j \\ \qquad\qquad \underbrace{10n\ldots 32}_{\frac{a}{2}\text{ times}}\,2\,\underbrace{10n\ldots 32}_{\frac{a}{2}\text{ times}}\,34\ldots j \end{array} & \text{if } 0 < a < d, 2 \mid a \\ \begin{array}{l} 10n\ldots\overline{j+1}\,23\ldots(j-1)\,\underbrace{10n\ldots j\,23\ldots(j-1)}_{(d-a-1)\text{ times}}\,10n\ldots j \\ \qquad\qquad \underbrace{10n\ldots 32}_{\frac{a+1}{2}\text{ times}}\,34\ldots j\,\underbrace{10n\ldots 32}_{\frac{a-1}{2}\text{ times}}\,2 \end{array} & \text{if } 0 < a < d, 2 \nmid a \\ 10n\ldots\overline{j+1}\,\underbrace{10n\ldots 32}_{\frac{d}{2}\text{ times}}\,2\,\underbrace{10n\ldots 32}_{\frac{d}{2}\text{ times}}\,34\ldots j & \text{if } a = d - \text{even} \\ 10n\ldots\overline{j+1}\,\underbrace{10n\ldots 32}_{\frac{d+1}{2}\text{ times}}\,34\ldots j\,\underbrace{10n\ldots 32}_{\frac{d-1}{2}\text{ times}}\,2 & \text{if } a = d - \text{odd} \end{cases} \tag{4.47}$$

for the decompositions (4.43) with $2 \le j \le n$, and

$$\ell^{(a)}_{3ji} = \begin{cases} 10n\ldots\overline{j+1}\,23\ldots(j-1)\,\underbrace{10n\ldots j\,23\ldots(j-1)}_{d\text{ times}}\,j & \text{if } a=0 \\ 10n\ldots\overline{j+1}\,23\ldots(j-1)\,\underbrace{10n\ldots j\,23\ldots(j-1)}_{(d-a-1)\text{ times}}\,10n\ldots j & \\ \qquad 23\ldots(i-1)\,\underbrace{10n\ldots i\,23\ldots(i-1)}_{a\text{ times}}\,i(i+1)\ldots j & \text{if } 0<a<d \\ 10n\ldots\overline{j+1}\,23\ldots(i-1)\,\underbrace{10n\ldots i\,23\ldots(i-1)}_{d\text{ times}}\,i(i+1)\ldots j & \text{if } a=d \end{cases} \tag{4.48}$$

for the decompositions (4.44) with $2 < i \le j \le n$.

Clearly, the lexicographically largest word from the lists (4.45)–(4.48) is

$$\ell^{(0)}_{22} = 10n\ldots 3\,\underbrace{10n\ldots 2}_{d\text{ times}}\,2,$$

which coincides with the word in the right-hand side of (11) for $k = d$ and $r = 1$. Let us compute its standard bracketing

$$\mathrm{b}[\ell^{(0)}_{22}] = [\mathrm{b}[10n\ldots 3], \mathrm{b}[\underbrace{10n\ldots 2}_{d\text{ times}}\,2]] \doteq [E_{32}t, E_{23}t^d] \doteq (E_{22} - E_{33})t^{d+1},$$

where we use the induction hypothesis in the second equality. Moreover, a similar argument also implies that

$$\mathrm{b}[\ell^{(a)}_{22}] \doteq (E_{22} - E_{33})t^{d+1} \doteq \mathrm{b}[\ell^{(0)}_{22}] \qquad \forall\, 0 < a \le d. \tag{4.49}$$

The next lexicographically largest word from the lists (4.45)–(4.48), with the words $\{\ell^{(a)}_{22}\}_{a=0}^{d}$ excluded due to (4.49), is

$$\ell^{(0)}_{23} = \ell^{(0)}_{333} = 10n\ldots 42\,\underbrace{10n\ldots 32}_{d\text{ times}}\,3,$$

which coincides with the word in the right-hand side of (11) for $k = d$ and $r = 2$. Let us compute its standard bracketing:

$$\mathrm{b}[\ell^{(0)}_{23}] = [\mathrm{b}[10n\ldots 42], \mathrm{b}[\underbrace{10n\ldots 32}_{d\text{ times}}\,3]] \doteq [E_{43}t, E_{34}t^d] \doteq (E_{33} - E_{44})t^{d+1},$$

where we use the induction hypothesis in the second equality. Moreover, a similar argument also applies to the remaining words $\ell^{(a)}_{23}$ and $\ell^{(a)}_{333}$ with $0 < a \le d$:

$$\mathrm{b}[\ell^{(a)}_{23}], \mathrm{b}[\ell^{(a)}_{333}] \in \operatorname{span}\big\{(E_{22} - E_{33})t^{d+1}, (E_{33} - E_{44})t^{d+1}\big\} = \operatorname{span}\big\{\mathrm{b}[\ell^{(0)}_{22}], \mathrm{b}[\ell^{(0)}_{23}]\big\}.$$

Proceeding further with the same line of reasoning we find that the $(n-1)$ lexicographically largest words from the above lists with linearly independent standard bracketings are: $\ell^{(0)}_{22}, \ell^{(0)}_{23}, \ldots, \ell^{(0)}_{2n}$. This proves (11) for $k = d$ and $1 \le r \le n-1$.

The lexicographically largest word among the remaining lists (4.45)–(4.46) is

$$\ell^{(0)}_{1n} = \ell^{(d)}_{0} = 123\ldots n\,\underbrace{1023\ldots n}_{d\text{ times}}\,0.$$

Let us evaluate its standard bracketing:

$$\mathrm{b}[\ell_{1n}^{(0)}] = [\mathrm{b}[123\dots n], \mathrm{b}[\underbrace{1023\dots n}_{d\text{ times}} 0]] \doteq [E_{1,n+1}, E_{n+1,1}t^{d+1}] = (E_{11} - E_{n+1,n+1})t^{d+1}.$$

As this expression is linear independent with $\{\mathrm{b}[\ell_{2j}^{(0)}]\}_{j=2}^{n}$ computed above, we get $\mathrm{SL}_n((d+1)\delta) = \ell_{1n}^{(0)}$. This completes our proof of (11) for $k = d$ and proves

$$\mathrm{b}[\mathrm{SL}_r((d+1)\delta)] \doteq \begin{cases} (E_{r+1,r+1} - E_{r+2,r+2})t^{d+1} & \text{if } 1 \le r \le n-1 \\ (E_{11} - E_{n+1,n+1})t^{d+1} & \text{if } r = n \end{cases}.$$

• Proof of (4.6)–(10) for $k = d+1$.

The case of real roots is treated precisely as in our part II of the induction base. Let us present the proof of (4.8), leaving the other ones to the interested reader.

Instead of listing all possible decompositions of $(d+1)\delta + \alpha_{2\to j}$, we start by noting that the word $\ell(d+1, j)$ from the right-hand side of (4.8) for $k = d+1$ corresponds to the decomposition $(d+1)\delta + \alpha_{2\to j} = (\lfloor \frac{d+1}{2} \rfloor \delta + \alpha_2) + (\lceil \frac{d+1}{2} \rceil \delta + \alpha_{3\to j})$. Since $\ell(d+1, j) > 10n\dots 32 = \mathrm{SL}_1(\delta)$, it suffices to consider in (3.10) only those decompositions $(d+1)\delta + \alpha_{2\to j} = \gamma_1 + \gamma_2$ such that each word $\mathrm{SL}_*(\gamma_1)$, $\mathrm{SL}_*(\gamma_2)$ is either $> 10n\dots 32$ or is a prefix of $10n\dots 32$. By the induction hypothesis, this restricts us to the following list:

$$\begin{aligned} &(d+1)\delta + \alpha_{2\to j} = (\delta) + (d\delta + \alpha_{2\to j}), \\ &(d+1)\delta + \alpha_{2\to j} = (a\delta + \alpha_2) + ((d+1-a)\delta + \alpha_{3\to j}), \quad 0 \le a \le d+1, \\ &(d+1)\delta + \alpha_{2\to j} = ((d+1)\delta + \alpha_{2\to \jmath}) + (\alpha_{(\jmath+1)\to j}), \quad 2 < \jmath < j. \end{aligned} \tag{4.50}$$

We therefore get the following list of concatenated words:

$$\begin{cases} 10n\dots 32\, \underbrace{10n\dots 32}_{\frac{d}{2}\text{ times}}\, 2\, \underbrace{10n\dots 32}_{\frac{d}{2}\text{ times}}\, 34\dots j & \text{if } 2 \mid d \\ 10n\dots 32\, \underbrace{10n\dots 32}_{\frac{d+1}{2}\text{ times}}\, 34\dots j\, \underbrace{10n\dots 32}_{\frac{d-1}{2}\text{ times}}\, 2 & \text{if } 2 \nmid d \end{cases},$$

$$\begin{cases} \underbrace{10n\dots 32}_{a\text{ times}}\, 2\ \underbrace{10n\dots 32}_{(d+1-a)\text{ times}}\ 34\dots j & \text{if } \frac{d+1}{2} \le a \le d+1 \\ \underbrace{10n\dots 32}_{(d+1-a)\text{ times}}\ 34\dots j\, \underbrace{10n\dots 32}_{a\text{ times}}\, 2 & \text{if } 0 \le a < \frac{d+1}{2} \end{cases}, \tag{4.51}$$

$$\begin{cases} \underbrace{10n\dots 32}_{\frac{d+2}{2}\text{ times}}\, 34\dots \jmath\, \underbrace{10n\dots 32}_{\frac{d}{2}\text{ times}}\, 2\, (\jmath+1)\dots j & \text{if } 2 \mid d \\ \underbrace{10n\dots 32}_{\frac{d+1}{2}\text{ times}}\, 2\, \underbrace{10n\dots 32}_{\frac{d+1}{2}\text{ times}}\, 34\dots \jmath\, (\jmath+1)\dots j & \text{if } 2 \nmid d \end{cases}.$$

It is easy to see that the word $\ell(d+1, j)$ is the lexicographically largest word from the list (4.51). Let us evaluate its standard bracketing:

$$\mathrm{b}[\ell(d+1,j)] \doteq [\mathrm{b}[\underbrace{10n\dots 32}_{\lfloor \frac{d+1}{2} \rfloor\text{ times}}\, 2], \mathrm{b}[\underbrace{10n\dots 32}_{\lceil \frac{d+1}{2} \rceil\text{ times}}\, 34\dots j]] \doteq$$

$$\begin{cases} [E_{23}t^{\lfloor \frac{d+1}{2} \rfloor}, E_{3,j+1}t^{\lceil \frac{d+1}{2} \rceil}] & \text{if } 2 < j \le n \\ [E_{23}t^{\lfloor \frac{d+1}{2} \rfloor}, E_{31}t^{\lceil \frac{d+3}{2} \rceil}] & \text{if } j = 0 \end{cases} \doteq \begin{cases} E_{2,j+1}t^{d+1} & \text{if } 2 < j \le n \\ E_{21}t^{d+2} & \text{if } j = 0 \end{cases},$$

where we use the induction hypothesis for $\mathrm{b}[\mathrm{SL}(\lfloor \frac{d+1}{2} \rfloor \delta + \alpha_2)], \mathrm{b}[\mathrm{SL}(\lceil \frac{d+1}{2} \rceil \delta + \alpha_{3\to j})]$.

This completes our proof of (4.8) for $k = d+1$. ■

4.3 General order

We now compute affine standard Lyndon words for $\widehat{\mathfrak{sl}}_{n+1}$ with an arbitrary order $<$ on $\widehat{I} = \{0, 1, \ldots, n\}$. The key feature is that all affine standard Lyndon words are determined by those of length $\leq n$. Furthermore, the explicit description of the degree δ affine standard Lyndon words is instrumental for the general pattern.

Notation: To distinguish from $<$, we shall now use $\prec$ for the standard order on $\widehat{I}$:

$$1 \prec 2 \prec 3 \prec \cdots \prec n \prec 0\,.$$

We start with the following simple result:

Lemma 4.4. Consider two arches $[a \to \overline{b+1}) \subsetneq [a' \to \overline{b'+1})$ such that $b' \neq a' - 1$ and $\min[a' \to \overline{b'+1}) \in [a \to \overline{b+1})$. Then: $\mathrm{SL}(\alpha_{a\to b}) < \mathrm{SL}(\alpha_{a'\to b'})$.

Proof. We note that this result is a property of the Lalonde–Ram's bijection ℓ (2.12) for the simple Lie algebra $\mathfrak{sl}_{\mathrm{ht}(\alpha_{a'\to b'})+1}$ with simple roots labelled by $[a' \to \overline{b'+1})$.

If $b \neq b'$, consider roots $\gamma_1 = \alpha_{a\to b}$ and $\gamma_2 = \alpha_{\overline{b+1}\to b'}$ whose sum is $\alpha = \gamma_1 + \gamma_2 = \alpha_{a\to b'}$. In view of the remark made above (reduction to a finite case), the convexity of Proposition 2.20 implies that $\mathrm{SL}(\alpha)$ is "sandwiched" between $\mathrm{SL}(\gamma_1)$ and $\mathrm{SL}(\gamma_2)$. But by our assumption the minimal letter of $\mathrm{SL}(\gamma_1)$ is $\min[a' \to \overline{b'+1})$, which is smaller than the minimal letter of $\mathrm{SL}(\gamma_2)$. Thus, we get $\mathrm{SL}(\gamma_1) < \mathrm{SL}(\alpha) < \mathrm{SL}(\gamma_2)$.

By a similar argument, we also conclude that $\mathrm{SL}(\alpha_{a\to b'}) < \mathrm{SL}(\alpha_{a'\to b'})$ if $a \neq a'$. This completes our proof of the desired inequality $\mathrm{SL}(\alpha_{a\to b}) < \mathrm{SL}(\alpha_{a'\to b'})$. ■

Due to the D_{n+1}-symmetry of $\widehat{I}$ and $\widehat{\Delta}^+$, where D_{n+1} denotes a dihedral group, we can assume, without loss of generality, that

$$1 = \min\left\{a \,\middle|\, a \in \widehat{I}\right\} \quad \text{and} \quad \mathrm{i} := \min\left\{a \,\middle|\, a \in \widehat{I} \setminus \{1\}\right\} \neq 0, \tag{4.52}$$

where $\min$ is taken with respect to our order $<$ on $\widehat{I}$.

Lemma 4.5. For $c \in \widehat{I} \setminus \{1\} = \{2, \ldots, n, 0\}$, define the degree δ word $\ell_c(\delta) \in \widehat{I}^*$ via

$$\ell_c(\delta) := \mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})c. \tag{4.53}$$

Then, we have

1) $\ell_a(\delta) > \ell_b(\delta)$ whenever $\mathrm{i} \preceq a \prec b \preceq 0$,
2) $\ell_a(\delta) < \ell_b(\delta)$ whenever $1 \prec a \prec b \preceq \mathrm{i}$,

so that $\ell_2(\delta) < \ell_3(\delta) < \cdots < \ell_{\mathrm{i}}(\delta) > \ell_{\mathrm{i}+1}(\delta) > \cdots > \ell_0(\delta)$.

We need a simple fact about Lalonde–Ram's bijection (2.12) for a finite type A:

Claim 4.6. (1) If $b = \min\{a, \overline{a+1}, \ldots, \overline{b-1}, b\}$, then $\mathrm{SL}(\alpha_{a\to b}) = b\,\overline{b-1}\ldots\overline{a+1}\,a$.
(2) If $a = \min\{a, \overline{a+1}, \ldots, \overline{b-1}, b\}$, then $\mathrm{SL}(\alpha_{a\to b}) = a\,\overline{a+1}\ldots\overline{b-1}\,b$.

Proof of Lemma 4.5. The proof is based on the more explicit formulas for $\ell_c(\delta)$:

• Case 1: $1 \prec c \prec \mathrm{i}$.

Consider the costandard factorization $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}}) = \ell_{1,c}\ell_{2,c}$. As $\ell_{1,c}$ starts with 1, $\ell_{2,c}$ starts with i, $\deg \ell_{1,c}, \deg \ell_{2,c} \in \widehat{\Delta}^+$, and $\deg \ell_{1,c} + \deg \ell_{2,c} = \alpha_{\overline{c+1}\to\overline{c-1}}$, we see that $\ell_{2,c} = \mathrm{SL}(\alpha_{\overline{c+1}\to e})$ and $\ell_{1,c} = \mathrm{SL}(\alpha_{\overline{e+1}\to\overline{c-1}})$ for some $e \succeq \mathrm{i}$. For $e \succ \mathrm{i}$, we have $\mathrm{SL}(\alpha_{\overline{e+1}\to\overline{c-1}}) < \mathrm{SL}(\alpha_{\overline{\mathrm{i}+1}\to\overline{c-1}})$ by Lemma 4.4. Therefore, we have:

$$\mathrm{SL}(\alpha_{\overline{e+1}\to\overline{c-1}})\mathrm{SL}(\alpha_{\overline{c+1}\to e}) < \mathrm{SL}(\alpha_{\overline{\mathrm{i}+1}\to\overline{c-1}})\mathrm{SL}(\alpha_{\overline{c+1}\to\mathrm{i}}) \qquad \forall\, e \succ \mathrm{i}\,.$$

Since the word $\mathrm{SL}(\alpha_{\overline{i+1}\to\overline{c-1}})\mathrm{SL}(\alpha_{\overline{c+1}\to i})$ is Lyndon (as it starts with the smallest letter 1, which appears only once) and its bracketing is clearly nonzero, we conclude

$$\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})=\mathrm{SL}(\alpha_{\overline{i+1}\to\overline{c-1}})\mathrm{SL}(\alpha_{\overline{c+1}\to i})=\mathrm{SL}(\alpha_{\overline{i+1}\to\overline{c-1}})\,i\,\overline{i-1}\ldots\overline{c+1}\,,$$

with the last equality due to Claim 4.6. Thus, we obtain

$$\ell_c(\delta)=\mathrm{SL}(\alpha_{\overline{i+1}\to\overline{c-1}})\,i\,\overline{i-1}\ldots\overline{c+1}\,c \qquad \forall\, 1\prec c\preceq i\,. \tag{4.54}$$

The desired inequality $\ell_a(\delta)<\ell_b(\delta)$ for $1\prec a\prec b\preceq i$ follows now from Lemma 4.4.

- Case 2: $i\prec c\preceq 0$.

Arguing as in the previous case, we see that the costandard factorization $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})=\ell_{1,c}\ell_{2,c}$ has the form $\ell_{2,c}=\mathrm{SL}(\alpha_{e\to\overline{c-1}})$ and $\ell_{1,c}=\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{e-1}})$ for some $1\prec e\preceq i$. For $1\prec e\prec i$, we have $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{e-1}})<\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})$ by Lemma 4.4, and so $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{e-1}})\mathrm{SL}(\alpha_{e\to\overline{c-1}})<\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})\mathrm{SL}(\alpha_{i\to\overline{c-1}})$. As the word $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})\mathrm{SL}(\alpha_{i\to\overline{c-1}})$ is Lyndon (as it starts with the smallest letter 1, which appears only once) and clearly has a nonzero bracketing, we conclude

$$\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})=\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})\mathrm{SL}(\alpha_{i\to\overline{c-1}})=\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})\,i\,\overline{i+1}\ldots\overline{c-1}$$

with the last equality due to Claim 4.6. Thus, we obtain

$$\ell_c(\delta)=\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{i-1}})\,i\,\overline{i+1}\ldots\overline{c-1}\,c \qquad \forall\, i\prec c\preceq 0\,. \tag{4.55}$$

The desired inequality $\ell_a(\delta)>\ell_b(\delta)$ for $i\preceq a\prec b$ follows from Lemma 4.4 again. ■

For $a,b\in\widehat{I}$, we introduce $sgn(a-b)\in\{-1,0,1\}$ via

$$sgn(a-b):=\begin{cases}1 & \text{if } a\succ b\\ -1 & \text{if } a\prec b\\ 0 & \text{if } a=b\end{cases}. \tag{4.56}$$

The following generalization of Theorem 4.2 is the main result of this section:

Theorem 4.7. The affine standard Lyndon words for $\widehat{\mathfrak{sl}}_{n+1}$ $(n\geq 2)$ with any order $<$ on $\widehat{I}=\{0,1,\ldots,n\}$ satisfying (4.52) are described by the formulas below ($k\geq 1$):

$$\Big\{\mathrm{SL}_1(k\delta),\ldots,\mathrm{SL}_n(k\delta)\Big\}=\Big\{\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\underbrace{\ell_{c+sgn(i-c)}(\delta)}_{(k-1)\text{ times}}c\;\Big|\;c\in\widehat{I}\setminus\{1\}\Big\}, \tag{4.57}$$

$$\mathrm{SL}(k\delta+\alpha_{a\to b})=\underbrace{\ell_{b+1}(\delta)}_{k\text{ times}}b(b-1)\ldots a\,,\quad \text{for } 1\prec a\preceq b\prec i, \tag{4.58}$$

$$\mathrm{SL}(k\delta+\alpha_{a\to b})=\underbrace{\ell_{\overline{a-1}}(\delta)}_{k\text{ times}}a\,\overline{a+1}\ldots b\,,\quad \text{for } i\prec a\preceq b\preceq 0, \tag{4.59}$$

$$
\begin{aligned}
\mathrm{SL}(k\delta+\alpha_{a\to b}) = {} & \text{for } 1 \prec a \prec \mathrm{i} \prec b \\
& \begin{cases}
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a & \text{if } 3 \mid k \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-2}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b & \text{if } 3 \mid \overline{k+1} \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+2}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a & \text{if } 3 \mid \overline{k+2}
\end{cases}, \quad \text{if } \overline{\mathrm{i}-1} < \overline{\mathrm{i}+1} \\
& \begin{cases}
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b & \text{if } 3 \mid k \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-2}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a & \text{if } 3 \mid \overline{k+1} \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+2}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b & \text{if } 3 \mid \overline{k+2}
\end{cases}, \quad \text{if } \overline{\mathrm{i}-1} > \overline{\mathrm{i}+1}
\end{aligned}
\tag{4.60}
$$

$$
\mathrm{SL}(k\delta+\alpha_{\mathrm{i}\to b}) = \begin{cases}
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{2}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{2}\text{ times}}\ \overline{\mathrm{i}+1}\dots b & \text{if } 2 \mid k \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{2}\text{ times}}\ \overline{\mathrm{i}+1}\dots b\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{2}\text{ times}}\ \mathrm{i} & \text{if } 2 \nmid k
\end{cases}, \quad \text{for } \mathrm{i} \prec b \preceq 0 \tag{4.61}
$$

$$
\mathrm{SL}(k\delta+\alpha_{a\to \mathrm{i}}) = \begin{cases}
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{2}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k}{2}\text{ times}}\ \overline{\mathrm{i}-1}\dots a & \text{if } 2 \mid k \\
\underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k+1}{2}\text{ times}}\ \overline{\mathrm{i}-1}\dots a\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{k-1}{2}\text{ times}}\ \mathrm{i} & \text{if } 2 \nmid k
\end{cases}, \quad \text{for } 1 \prec a \prec \mathrm{i} \tag{4.62}
$$

$$
\mathrm{SL}(k\delta+\alpha_{\mathrm{i}}) = \underbrace{\ell_{\mathrm{i}}(\delta)}_{k\text{ times}}\ \mathrm{i} \tag{4.63}
$$

and finally a slightly less explicit formula

$$
\begin{gathered}
\mathrm{SL}(k\delta+\alpha_{b\to a}) = \ell_1\ \underbrace{\ell_{b\to a}(\delta)}_{(k-1)\text{ times}}\ \ell_2\,, \quad \text{for } 1 \in [b \to \overline{a+1}) \\
\text{where } \mathrm{SL}(\delta+\alpha_{b\to a}) = \ell_1\ell_2 \text{ is the costandard factorization (2.4)} \\
\text{and } \ell_{b\to a}(\delta) \text{ is one of } \ell_c(\delta) \text{ such that } \mathrm{SL}(2\delta+\alpha_{b\to a}) = \ell_1\ell_{b\to a}(\delta)\ell_2.
\end{gathered}
\tag{4.64}
$$

Remark 4.8. (a) The implicit words ℓ_1 and ℓ_2 providing the costandard factorization of $\mathrm{SL}(\delta+\alpha_{b\to a})$ in (4.64) can actually be described explicitly (see Lemma 4.11):

$$
\begin{aligned}
\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}}) \quad &\text{and} \quad \ell_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to a}) \quad \text{if} \quad \mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}})\,, \\
\ell_1 = \mathrm{SL}(\alpha_{\overline{a+2}\to a}) \quad &\text{and} \quad \ell_2 = \mathrm{SL}(\alpha_{b\to\overline{a+1}}) \quad \text{if} \quad \mathrm{SL}(\alpha_{\overline{b-1}\to a}) < \mathrm{SL}(\alpha_{b\to\overline{a+1}})\,.
\end{aligned}
$$

(b) Likewise, the word $\ell_{b\to a}(\delta)$ featuring in (4.64) can be characterized as the lexicographically largest among those $\ell_c(\delta)$ that satisfy $[\mathrm{b}[\ell_1], \mathrm{b}[\ell_c(\delta)]] \neq 0$. Explicitly, as follows from the proof below, we have (cf. part (a) above):

$$
\ell_{b\to a}(\delta) = \begin{cases}
\ell_{b-1+sgn(\mathrm{i}-(b-1))}(\delta) & \text{if } \mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}}) \\
\ell_{a+1+sgn(\mathrm{i}-(a+1))}(\delta) & \text{if } \mathrm{SL}(\alpha_{\overline{b-1}\to a}) < \mathrm{SL}(\alpha_{b\to\overline{a+1}})
\end{cases}. \tag{4.65}
$$

(c) Let us also record the explicit order between the words $\ell_1, \ell_2, \ell_{b\to a}(\delta)$, cf. (4.80):

$$\ell_1 < \ell_2 \leq \ell_{b\to a}(\delta)\,.$$

(d) For the standard order (4.1), we clearly recover the formulas from our previous Theorem 4.2. We also note that the proof below significantly simplifies when $i = 2$.
(e) Finally, we note $\mathrm{SL}(\alpha_{a\to b})$ can be easily reconstructed using either of the algorithms presented before Lemma 4.11, with 1 replaced by $\min\{a, \overline{a+1}, \ldots, \overline{b-1}, b\}$.

Remark 4.9. (a) In the base of induction below we prove that

$$\big\{\mathrm{SL}_1(\delta), \ldots, \mathrm{SL}_n(\delta)\big\} = \big\{\ell_c(\delta)\,|\, c \in \widehat{I} \setminus \{1\}\big\}. \tag{4.66}$$

As easily follows from (4.54, 4.55), their standard bracketings are

$$\mathrm{b}[\ell_c(\delta)] \doteq \begin{cases} (\mathrm{E}_{\mathrm{i}+1,\mathrm{i}+1} - \mathrm{E}_{c,c})\mathrm{t} & \text{if } 1 \prec c \preceq i \\ (\mathrm{E}_{\mathrm{i},\mathrm{i}} - \mathrm{E}_{c+1,c+1})\mathrm{t} & \text{if } i \prec c \preceq 0 \end{cases}. \tag{4.67}$$

(b) The standard bracketing $\mathrm{b}[\mathrm{SL}(\alpha_{a\to b})]$ for $1, i \notin [a \to \overline{b+1})$ is a nonzero multiple of $E_{a,b+1}$ if $b \neq 0$, $E_{a1}\mathrm{t}$ if $a \prec b = 0$, $E_{n+1,1}\mathrm{t}$ if $a = b = 0$. Thus, the lexicographically largest word among $\mathrm{SL}_*(\delta)$ whose bracketing $\mathrm{b}[\mathrm{SL}_*(\delta)]$ does not commute with $\mathrm{b}[\mathrm{SL}(\alpha_{a\to b})]$ is $\ell_{\overline{b+1}}(\delta)$ if $a \prec i$ and $\ell_{\overline{a-1}}(\delta)$ if $a \succ i$, due to Lemma 4.5 and (4.67).

Proof of Theorem 4.7. The proof proceeds by induction on k.

<u>Base of Induction</u>

The base of induction is $k = 1$. In this case, the nontrivial cases are formulas (4.57) for $\mathrm{SL}_*(\delta)$ and (4.58)–(4.63) for $\mathrm{SL}(\delta + \alpha_{a\to b})$ with $1 \notin [a \to \overline{b+1})$.

• Proof of (4.57) for $k = 1$.

For any $1 \leq r \leq n$, consider the costandard factorization $\mathrm{SL}_r(\delta) = \ell_1\ell_2$. For degree reasons, we have $\ell_1 = \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}})$, $\ell_2 = \mathrm{SL}(\alpha_{a\to b})$ for some $b \neq \overline{a-1}$ such that $1 \in [\overline{b+1} \to a)$ and $i \in [a \to \overline{b+1})$. If $b = i$, then $1 \prec a \preceq i$ and

$$\mathrm{SL}_r(\delta) = \mathrm{SL}(\alpha_{\overline{\mathrm{i}+1}\to\overline{a-1}})\mathrm{SL}(\alpha_{a\to\mathrm{i}}) = \mathrm{SL}(\alpha_{\overline{\mathrm{i}+1}\to\overline{a-1}})\,\mathrm{i}\,\overline{\mathrm{i}-1}\ldots a = \ell_a(\delta)\,,$$

due to (4.54) and Claim 4.6. Likewise, if $a = i$, then $i \prec b$ and

$$\mathrm{SL}_r(\delta) = \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{\mathrm{i}-1}})\mathrm{SL}(\alpha_{\mathrm{i}\to b}) = \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{\mathrm{i}-1}})\,\mathrm{i}\,\overline{\mathrm{i}+1}\ldots b = \ell_b(\delta)\,,$$

due to (4.55) and Claim 4.6. Finally, if $1 \prec a \prec i \prec b$, then $\mathrm{SL}_r(\delta) < \ell_c(\delta)$ for any $c \in [a \to \overline{b+1})$, due to Lemma 4.4 and explicit formulas (4.54, 4.55). On the other hand, $\mathrm{b}[\mathrm{SL}_r(\delta)] = [\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]] \doteq (E_{a,a} - \mathrm{E}_{b+1,b+1})\mathrm{t}$, while the standard bracketing $\mathrm{b}[\ell_c(\delta)]$ is given by (4.67). Hence, $\mathrm{b}[\mathrm{SL}_r(\delta)]$ is a linear combination of standard bracketings of the larger words $\ell_a(\delta), \ell_{\mathrm{i}}(\delta), \ell_b(\delta)$, a contradiction with $\mathrm{SL}_r(\delta)$ being standard. Thus, any degree δ affine standard Lyndon word is of the form $\ell_c(\delta)$ for $c \neq 1$. This completes the proof of (4.66), as we have n such words.

• Proof of (4.58)–(4.63) for $k = 1$.

We skip these proofs as they coincide with those in the step of induction below.

<u>Step of Induction</u>

Let us now prove the step of induction, proceeding by the height of a root. We thus verify formulas (4.57)–(4.64) for affine standard Lyndon words $\mathrm{SL}_*(\alpha)$ with $k = r + 1$ assuming the validity of these formulas for $\mathrm{SL}_*(\beta)$ with $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$.

Notation: In what follows, we shall denote $[a \to \overline{b+1})$ from (4.5) simply by $[a; b]$:

$$[a; b] := \big\{a, \overline{a+1}, \ldots, \overline{b-1}, b\big\}\,.$$

• Proof of (4.57) for $k = r + 1$.

We consider only decompositions of the form $(r+1)\delta = (r_1\delta + \alpha_{a\to b}) + ((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}})$, due to Remark 3.5. We may further assume that $1 \in [\overline{b+1}; \overline{a-1}]$. We start with the following useful result (which will be strengthened in Lemma 4.11):

Claim 4.10. If $\ell_1\ell_2$ is the costandard factorization (2.4) of $\mathrm{SL}(\delta + \alpha_{\overline{b+1}\to\overline{a-1}})$ and $1 \in [\overline{b+1}; \overline{a-1}]$, then both words ℓ_1 and ℓ_2 contain all the letters located on the (counterclockwise oriented) arch $[\overline{b+1}; \overline{a-1}]$.

Proof of Claim 4.10. First, we note that both ℓ_1, ℓ_2 start with 1. If ℓ_1 does not contain all the letters from $[\overline{b+1}; \overline{a-1}]$, then it consists only of letters from c to d, where $1 \in [c; d] \subsetneq [\overline{b+1}; \overline{a-1}]$. Thus, $\ell_1 < \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}})$ by Lemma 4.4, hence

$$\mathrm{SL}(\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) = \ell_1\ell_2 < \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}})\ell_{e(i;a,b)}(\delta)\,, \tag{4.68}$$

with $e(i; a, b) := a$ if $a \preceq i$ and $e(i; a, b) := b$ if $i \prec a \preceq b$. However, $\mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}}) < \ell_{e(i;a,b)}(\delta)$ by Lemma 4.4 and their standard bracketings do not commute by (4.67): $[\mathrm{b}[\mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}})], \mathrm{b}[\ell_{e(i;a,b)}(\delta)]] \neq 0$. Thus, the concatenated word $\mathrm{SL}(\alpha_{\overline{b+1}\to\overline{a-1}})\ell_{e(i;a,b)}(\delta)$ appears in the set from the right-hand side of (3.10) for the root $\alpha = \delta + \alpha_{\overline{b+1}\to\overline{a-1}}$, contradicting (4.68).

If ℓ_2 does not contain all the letters from $[\overline{b+1}; \overline{a-1}]$, then we apply precisely the same argument to $\ell_2\ell_1$ and use the inequality $\ell_1\ell_2 < \ell_2\ell_1$ to get a contradiction. ∎

For $r_1 < r$, we have $\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) = \ell_1 \underbrace{\ell_{\overline{b+1}\to\overline{a-1}}(\delta)}_{(r-r_1-1)\text{ times}} \ell_2$ by the induction hypothesis, where $\ell_1\ell_2$ is the costandard factorization of $\mathrm{SL}(\delta+\alpha_{\overline{b+1}\to\overline{a-1}})$. According to Claim 4.10: $\mathrm{b}[\ell_1] \doteq E_{\overline{b+1},c}t^{1-\delta_{\overline{b+1},1}}$ for some $c \in [a; b]$ or $\mathrm{b}[\ell_1] \doteq E_{c,a}t$ for some $c \in [a+1; b]$. For any $d \in [a; b]$, one of the roots $\mathbf{deg}\,\ell_1, \mathbf{deg}\,\ell_2 \in \widehat{\Delta}^+$ does not contain α_d, which together with $\ell_1 < \ell_2$, Lemma 4.4, and Claim 4.10 implies

$$\ell_1 \leq \mathrm{SL}(\alpha_{\overline{d+1}\to\overline{d-1}})\,. \tag{4.69}$$

Moreover, the equality in (4.69) does hold only for $d = b$ if $\mathrm{SL}(\alpha_{b\to\overline{a-1}}) > \mathrm{SL}(\alpha_{\overline{b+1}\to a})$ and for $d = a$ if $\mathrm{SL}(\alpha_{b\to\overline{a-1}}) < \mathrm{SL}(\alpha_{\overline{b+1}\to a})$, according to Lemma 4.11.

Thus, if $a \neq b$ and $\mathrm{SL}(\alpha_{b\to\overline{a-1}}) > \mathrm{SL}(\alpha_{\overline{b+1}\to a})$, then for $d \in [a; \overline{b-1}]$ we have

$$\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}})\mathrm{SL}(r_1\delta + \alpha_{a\to b}) < \mathrm{SL}(\alpha_{\overline{d+1}\to\overline{d-1}}) < \mathrm{SL}(\alpha_{\overline{d+1}\to\overline{d-1}})\underbrace{\ell_{d+\mathrm{sgn}(i-d)}(\delta)}_{r\text{ times}}\,d.$$

In the remaining case $d = b$ (with $a \neq b$ and $\mathrm{SL}(\alpha_{b\to\overline{a-1}}) > \mathrm{SL}(\alpha_{\overline{b+1}\to a})$), we have

$$\begin{aligned}
&\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}})\mathrm{SL}(r_1\delta + \alpha_{a\to b}) = \\
&\qquad \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{b-1}})\underbrace{\ell_{\overline{b+1}\to\overline{a-1}}(\delta)}_{(r-r_1-1)\text{ times}}\ell_2\,\mathrm{SL}(r_1\delta + \alpha_{a\to b}) < \\
&\qquad\qquad \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{b-1}})\underbrace{\ell_{\overline{b+1}\to\overline{a-1}}(\delta)}_{r\text{ times}}\,b = \mathrm{SL}(\alpha_{\overline{b+1}\to\overline{b-1}})\underbrace{\ell_{b+\mathrm{sgn}(i-b)}(\delta)}_{r\text{ times}}\,b\,,
\end{aligned}$$

cf. (4.65), with the inequality implied by $\ell_2 < \ell_{\overline{b+1}\to\overline{a-1}}(\delta)$, due to (4.80) and $a \neq b$. The case of $a \neq b$ and $\mathrm{SL}(\alpha_{b\to\overline{a-1}}) < \mathrm{SL}(\alpha_{\overline{b+1}\to a})$ is treated completely analogously.

On the other hand, if $a = b = d$ and $r_1 \geq 0$, then

$$\mathrm{SL}(r_1\delta + \alpha_{a\to b}) = \mathrm{SL}(r_1\delta + \alpha_a) = \underbrace{\ell_{a+\mathrm{sgn}(i-a)}(\delta)}_{r_1\text{ times}}\,a$$

by the induction hypothesis (applying (4.58) if $a < i$, (4.59) if $a > i$, (4.63) if $a = i$) and $\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) = \mathrm{SL}((r-r_1)\delta + \alpha_{\overline{a+1}\to\overline{a-1}})$ is given by

$$\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{a+1}\to\overline{a-1}}) = \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\underbrace{\ell_{a+sgn(i-a)}(\delta)}_{(r-r_1)\text{ times}}. \tag{4.70}$$

To prove the latter claim, we first note that $\ell_1 = \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})$ and $\ell_2 = \mathrm{SL}_?(\delta)$, while the lexicographically largest word $\mathrm{SL}_?(\delta)$ whose bracketing does not commute with $\mathrm{b}[\mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})] \doteq E_{a+1,a}t^{1-\delta_{a,0}}$ is precisely $\ell_{a+sgn(i-a)}(\delta)$, due to (4.67) and Lemma 4.5. Therefore, $\ell_2 = \ell_{a+sgn(i-a)}(\delta)$. Second, we also claim that $\ell_{\overline{a+1}\to\overline{a-1}}(\delta)$ equals $\ell_2 = \ell_{a+sgn(i-a)}(\delta)$. To this end, recall that for $\alpha = 2\delta + \alpha_{\overline{a+1}\to\overline{a-1}}$ we have

$$\mathrm{SL}(\alpha) = \ell_1\ell_{\overline{a+1}\to\overline{a-1}}(\delta)\ell_2 = \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{\overline{a+1}\to\overline{a-1}}(\delta)\ell_{a+sgn(i-a)}(\delta). \tag{4.71}$$

∘ If $\ell_{\overline{a+1}\to\overline{a-1}}(\delta) < \ell_{a+sgn(i-a)}(\delta)$, then $\mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{\overline{a+1}\to\overline{a-1}}(\delta)\ell_{a+sgn(i-a)}(\delta) < \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{a+sgn(i-a)}(\delta)$ $\ell_{a+sgn(i-a)}(\delta) =: \widetilde{\ell}$ and the bracketing of the latter is

$$\mathrm{b}[\widetilde{\ell}] = [\mathrm{b}[\mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{a+sgn(i-a)}(\delta)], \mathrm{b}[\ell_{a+sgn(i-a)}(\delta)]] \doteq [\mathrm{b}[\mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})], \mathrm{b}[\ell_{a+sgn(i-a)}(\delta)]]\cdot t \neq 0.$$

We get a contradiction, since $\widetilde{\ell}$ is one of the concatenations (corresponding to the decomposition $\alpha = (\delta + \alpha_{\overline{a+1}\to\overline{a-1}}) + (\delta)$) in the right-hand side of (3.10) for α.
∘ If $\ell_{\overline{a+1}\to\overline{a-1}}(\delta) > \ell_{a+sgn(i-a)}(\delta)$, then the costandard factorization (2.4) of $\mathrm{SL}(\alpha)$ in (4.71) must be of the form $\mathrm{SL}(\alpha) = \ell_1'\ell_2'$ with $\ell_2' = \ell_{a+sgn(i-a)}(\delta)$ and $\ell_1' = \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{\overline{a+1}\to\overline{a-1}}(\delta)$. We get a contradiction again, since ℓ_1' is an SL-word and so $\ell_1' = \mathrm{SL}(\mathbf{deg}\ \ell_1') = \mathrm{SL}(\delta + \alpha_{\overline{a+1}\to\overline{a-1}}) = \mathrm{SL}(\alpha_{\overline{a+1}\to\overline{a-1}})\ell_{a+sgn(i-a)}(\delta)$.

This completes our proof of (4.70). Assuming $\mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) < \mathrm{SL}(r_1\delta + \alpha_{a\to b})$ and combining all the above, we obtain the following inequalities for the corresponding concatenation $\ell := \mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}})\mathrm{SL}(r_1\delta + \alpha_{a\to b})$:

$$\ell \leq \mathrm{SL}(\alpha_{\overline{d+1}\to\overline{d-1}})\underbrace{\ell_{d+sgn(i-d)}(\delta)}_{r\text{ times}}\, d \qquad \forall\, d \in [a;b]. \tag{4.72}$$

We also note that (4.72) still holds for $r_1 = r$, due to Lemma 4.4.

The standard bracketings of the words from the right-hand side of (4.57) are

$$\mathrm{b}[\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\underbrace{\ell_{c+sgn(i-c)}(\delta)}_{r\text{ times}}\, c] \doteq \begin{cases} (E_{cc} - E_{c+1,c+1})t^{r+1} & \text{if } 1 < c \leq n \\ (E_{n+1,n+1} - E_{11})t^{r+1} & \text{if } c = 0 \end{cases}. \tag{4.73}$$

We shall now compute the standard bracketing of ℓ. We have two possibilities (due to the inequalities of Remark 4.8(c)):

1) The costandard factorization (2.4) of ℓ is of the form

$$\ell = \ell_1'\ell_2' \quad \text{with} \quad \ell_1' = \mathrm{SL}((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}),\ \ell_2' = \mathrm{SL}(r_1\delta + \alpha_{a\to b}).$$

Hence, the standard bracketing of ℓ is

$$\mathrm{b}[\ell] = [\mathrm{b}[\ell_1'], \mathrm{b}[\ell_2']] \doteq (E_{aa} - E_{b+1,b+1})t^{r+1} \doteq (E_{aa} - E_{a+1,a+1})t^{r+1} + (E_{a+1,a+1} - E_{a+2,a+2})t^{r+1} + \cdots + (E_{bb} - E_{b+1,b+1})t^{r+1}.$$

Thus, if ℓ is not a word from the right-hand side of (4.57) for $k = r+1$, then $\mathrm{b}[\ell]$ is a linear combination of the standard bracketings of the larger words $\{\ell_d(\delta) \mid d \in [a;b]\}$, cf. (4.72, 4.73). Hence, the word ℓ cannot be standard.

2) The costandard factorization (2.4) of ℓ is of the form

$$\ell = \ell_1'\ell_2' \quad \text{with} \quad \ell_1' = \ell_1 \underbrace{\ell_{\overline{b+1}\to\overline{a-1}}(\delta)}_{(r-r_1-1)\text{ times}}, \ \ell_2' = \ell_2 \, \mathrm{SL}(r_1\delta + \alpha_{a\to b}).$$

Hence, the standard bracketing of ℓ is either $\mathrm{b}[\ell] \doteq (E_{cc} - E_{b+1,b+1})t^{r+1}$ for $c \in [a;b]$ or $\mathrm{b}[\ell] \doteq (E_{aa} - E_{cc})t^{r+1}$ for $c \in [a+1;b]$. Thus, analogously to 1), if ℓ is not a word from the right-hand side of (4.57) for $k = r+1$, then $\mathrm{b}[\ell]$ is a linear combination of the standard bracketings of the larger words $\{\ell_d(\delta) \,|\, d \in [a;b]\}$, cf. (4.72, 4.73). Therefore, the word ℓ cannot be standard.

Finally, if $\mathrm{SL}((r - r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) > \mathrm{SL}(r_1\delta + \alpha_{a\to b})$, then the concatenation $\widetilde{\ell}$ arising from the decomposition $(r+1)\delta = (r_1\delta + \alpha_{a\to b}) + ((r-r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}})$ is

$$\widetilde{\ell} = \mathrm{SL}(r_1\delta + \alpha_{a\to b})\mathrm{SL}((r - r_1)\delta + \alpha_{\overline{b+1}\to\overline{a-1}}) < \ell, \tag{4.74}$$

due to Lemma 2.4. By induction hypothesis, we have $\mathrm{b}[\widetilde{\ell}] \doteq (E_{pp} - E_{qq})t^{r+1}$ for some $p, q \in [a;\overline{b+1}]$. The latter is a linear combination of standard bracketings of the larger words $\{\ell_d(\delta) \,|\, d \in [a;b]\}$, cf. (4.72)–(4.74), hence $\widetilde{\ell}$ is not standard either.

• Proof of (4.58) for $k = r+1$.

Consider $\alpha = (r+1)\delta + \alpha_{a\to b}$ with $1 \prec a \preceq b \prec i$. Its possible decompositions are $\alpha = (r_1\delta + \alpha_{a\to c}) + (r_2\delta + \alpha_{\overline{c+1}\to b})$ with $r_1 + r_2 = r$ or $r+1$, depending on c.

First, we show that decompositions with $c \notin [a;b]$ give rise to concatenated words that are lexicographically smaller than the word in the right-hand side of (4.58) for $k = r+1$. There are four cases to consider: $1 \in [a;c]$ or $1 \in [\overline{c+1};b]$, treating separately $r_1 = 0, r_1 \geq 1$ in the first case and $r_2 = 0, r_2 \geq 1$ in the second case.

1) If $1 \in [a;c] \neq \widehat{I}$ and $r_1 = 0$, then $1 \in [a;c] \subset [e+1;e-1]$ for any $e \in [c+1;a-1]$, and so $\mathrm{SL}(\alpha_{a\to c}) \leq \mathrm{SL}(\alpha_{(e+1)\to(e-1)})$ by Lemma 4.4. As $1 = \min \widehat{I}$, we get $\mathrm{SL}(\alpha_{a\to c})\,1 < \mathrm{SL}(\alpha_{(e+1)\to(e-1)})\,e = \ell_e(\delta) < \ell_a(\delta) < \ell_{b+1}(\delta)$ with the last two inequalities due to Lemma 4.5. We note that $\mathrm{SL}(\alpha_{a\to c})\,1$ cannot be a proper prefix of $\ell_{b+1}(\delta)$ (as the former word contains the letter 1 twice) and $\mathrm{SL}(r\delta + \alpha_{\overline{c+1}\to b})$ starts with 1. Thus, the concatenation $\mathrm{SL}(\alpha_{a\to c})\mathrm{SL}(r\delta + \alpha_{\overline{c+1}\to b})$ is lexicographically smaller than $\ell_{b+1}(\delta)$, hence, smaller than the right-hand side of (4.58) for $k = r+1$.

2) If $1 \in [\overline{c+1};b]$ and $r_2 = 0$, then $1 \in [\overline{c+1};b] \subset [\overline{b+2};b]$, and so $\mathrm{SL}(\alpha_{\overline{c+1}\to b}) \leq \mathrm{SL}(\alpha_{\overline{b+2}\to b})$ by Lemma 4.4. Thus, $\mathrm{SL}(\alpha_{\overline{c+1}\to b})\,1 < \mathrm{SL}(\alpha_{\overline{b+2}\to b})(b+1) = \ell_{b+1}(\delta)$. The rest of the argument proceeds exactly as in 1) above.

3) If $1 \in [a;c] \neq \widehat{I}$ and $r_1 \geq 1$, then $\mathrm{SL}(r_1\delta + \alpha_{a\to c}) = \ell_1 \underbrace{\ell_{a\to c}(\delta)}_{(r_1-1)\text{ times}} \ell_2$ with ℓ_1 and ℓ_2 defined through the costandard factorization $\mathrm{SL}(\delta + \alpha_{a\to c}) = \ell_1\ell_2$. We claim that $\ell_1 < \ell_{b+1}(\delta)$, from which the argument proceeds exactly as in 1) above. Indeed, according to Lemma 4.11, ℓ_1 is given by one of the following two formulas:

(A) $\ell_1 = \mathrm{SL}(\alpha_{a\to d})$ for $d \in [c \to (a-1))$;
(B) $\ell_1 = \mathrm{SL}(\alpha_{d\to c})$ for $d \in [(c+2) \to a)$.

According to Lemmas 4.4, 4.5, we thus get: $\ell_1 \leq \mathrm{SL}(\alpha_{a\to(a-2)}) < \ell_{a-1}(\delta) < \ell_{b+1}(\delta)$ in case (A) and $\ell_1 \leq \mathrm{SL}(\alpha_{(c+2)\to c}) < \ell_{c+1}(\delta) < \ell_{b+1}(\delta)$ in case (B), as stated above.

4) If $1 \in [\overline{c+1};b] \neq \widehat{I}$ and $r_2 \geq 1$, then $\mathrm{SL}(r_2\delta + \alpha_{\overline{c+1}\to b}) = \ell_1 \underbrace{\ell_{\overline{c+1}\to b}(\delta)}_{(r_2-1)\text{ times}} \ell_2$ with ℓ_1 and ℓ_2 defined through the costandard factorization $\mathrm{SL}(\delta + \alpha_{\overline{c+1}\to b}) = \ell_1\ell_2$. We claim that $\ell_1 < \ell_{b+1}(\delta)$, from which the argument proceeds exactly as in 1) above. Indeed, according to Lemma 4.11, ℓ_1 is given by one of the following two formulas:

(A) $\ell_1 = \mathrm{SL}(\alpha_{d\to b})$ for $d \in [\overline{b+2};\overline{c+1}]$;
(B) $\ell_1 = \mathrm{SL}(\alpha_{\overline{c+1}\to d})$ for $d \in [(b+1) \to c)$.

According to Lemmas 4.4, 4.5, we thus get: $\ell_1 \leq \mathrm{SL}(\alpha_{\overline{b+2}\to b}) < \ell_{b+1}(\delta)$ in case (A) and $\ell_1 < \ell_2 = \mathrm{SL}(\alpha_{\overline{d+1}\to b}) \leq \mathrm{SL}(\alpha_{\overline{b+2}\to b}) < \ell_{b+1}(\delta)$ in case (B), as claimed above.

Therefore, it suffices to consider only the following decompositions in (3.10):

$$\alpha = (r_1\delta + \alpha_{a\to c}) + ((r+1-r_1)\delta + \alpha_{(c+1)\to b})\,, \quad a \preceq c \prec b\,,\, 0 \le r_1 \le r+1, \tag{4.75}$$

$$\alpha = (r_1\delta) + ((r+1-r_1)\delta + \alpha_{a\to b})\,, \quad 1 \le r_1 \le r+1. \tag{4.76}$$

∘ Case 1: Concatenations arising through (4.75).

1) If $0 < r_1 < r+1$, then the corresponding concatenated word starts with $\ell_{c+1}(\delta)$, due to the induction hypothesis and the inequality $\ell_{c+1}(\delta) < \ell_{b+1}(\delta)$ of Lemma 4.5. Thus, this concatenation is $<$ the right-hand side of (4.58) for $k = r+1$.

2) If $r_1 = r+1$, then the corresponding concatenated word again starts with $\ell_{c+1}(\delta)$, but now because the first letter of $\ell_{c+1}(\delta)$ is smaller than any of $c+1,\ldots,b$. Therefore, this concatenation is $<$ the right-hand side of (4.58) for $k = r+1$.

3) If $r_1 = 0$, then the concatenation equals $\underbrace{\ell_{b+1}(\delta)}_{(r+1)\text{ times}}\, b(b-1)\ldots(c+1)\,\mathrm{SL}(\alpha_{a\to c})$. But $\mathrm{SL}(\alpha_{a\to c}) \le c(c-1)\ldots a$ (either they differ in the first letters, or Claim 4.6 applies), hence, this concatenation is $\le$ the right-hand side of (4.58) for $k = r+1$.

∘ Case 2: concatenations arising through (4.76).

First, we record the standard bracketing $\mathrm{b}[\mathrm{SL}((r+1-r_1)\delta + \alpha_{a\to b})] \doteq E_{a,b+1}t^{r+1-r_1}$.

1) If $r_1 > 1$, then according to (4.73) the only words from the right-hand side of (4.57) with $k = r_1$ whose standard bracketing does not commute with the above $\mathrm{b}[\mathrm{SL}((r+1-r_1)\delta + \alpha_{a\to b})]$ start with $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})1$ for $c = a-1, a, b, b+1$. Each of these words is lexicographically smaller than $\ell_{b+1}(\delta)$. Hence, the corresponding concatenation is $<$ the right-hand side of (4.58) for $k = r+1$.

2) If $r_1 = 1$, then we should rather use formula (4.67) for the bracketings.

∘ If $b \prec (\mathrm{i}-1)$, then the only $\ell_?(\delta)$ whose standard bracketing does not commute with $\mathrm{b}[\mathrm{SL}(r\delta + \alpha_{a\to b})]$ are $\ell_a(\delta)$ and $\ell_{b+1}(\delta)$. As $\ell_a(\delta) < \ell_{b+1}(\delta)$ by Lemma 4.5, the resulting concatenation is $\le$ the right-hand side of (4.58) for $k = r+1$.

∘ If $b = \mathrm{i}-1$, then the only $\ell_?(\delta)$ whose standard bracketing does not commute with $\mathrm{b}[\mathrm{SL}(r\delta + \alpha_{a\to b})]$ are $\ell_a(\delta)$ and $\{\ell_c(\delta) | c \ge \mathrm{i}\}$. As $\ell_\mathrm{i}(\delta)$ is the maximal of these words (Lemma 4.5), the concatenation is still $\le$ the right-hand side of (4.58) for $k = r+1$.

We note that in both cases above the equality is possible (when $\ell_{b+1}(\delta)$ is used).

This completes our proof of (4.58) for $k = r+1$.

• Proof of (4.59) for $k = r+1$.

The argument is completely analogous to the one used in the previous case (we leave details to the interested reader).

• Proof of (4.60)–(4.63) for $k = r+1$.

Let us prove the most complicated formula (4.60) for the case $\alpha = (r+1)\delta + \alpha_{a\to b}$ with $1 \prec a \prec \mathrm{i} \prec b$ and $\overline{\mathrm{i}-1} < \overline{\mathrm{i}+1}$ (the proofs for the other cases are analogous).

There exists a degree α Lyndon word with a nonzero bracketing that starts with $\mathrm{SL}_1(\delta) = \ell_\mathrm{i}(\delta)$. Therefore, it suffices to consider in (3.10) only those decompositions $\alpha = (r_1\delta + \beta_1) + (r_2\delta + \beta_2)$ such that each word $\mathrm{SL}(r_1\delta + \beta_1)$, $\mathrm{SL}(r_2\delta + \beta_2)$ is either $> \ell_\mathrm{i}(\delta)$ or is a prefix of $\ell_\mathrm{i}(\delta)$. This excludes the following cases (with $p = 1, 2$):

1) $\beta_p = \alpha_{a\to c}$ with $1 \in [a;c] \ne \widehat{I}$, as in this case we have $\mathrm{SL}(\alpha_{a\to c})\,1 < \ell_\mathrm{i}(\delta)$ and $\ell_1\,1 < \ell_\mathrm{i}(\delta)$ with ℓ_1 arising through the costandard factorization $\mathrm{SL}(\delta + \alpha_{a\to c}) = \ell_1\ell_2$, cf. our verification of (4.58) above;

2) $\beta_p = \alpha_{c\to b}$ with $1 \in [c;b] \ne \widehat{I}$, as in this case we have $\mathrm{SL}(\alpha_{c\to b})\,1 < \ell_\mathrm{i}(\delta)$ and $\ell_1\,1 < \ell_\mathrm{i}(\delta)$ with ℓ_1 arising through the costandard factorization $\mathrm{SL}(\delta + \alpha_{c\to b}) = \ell_1\ell_2$, cf. our verification of (4.58) above;

3) $\beta_p = k\delta$ with $k > 1$, as $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\,1 < \mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\,c = \ell_c(\delta) \le \ell_\mathrm{i}(\delta)\ \forall c$;

4) $\beta_p = \alpha_{a\to c}$ with $c \in [a \to (\mathrm{i}-1))$ and $r_p > 0$, as $\mathrm{SL}(r_p\delta + \beta_p)$ then starts with $\ell_{c+1}(\delta)$, which has the same length but is lexicographically smaller than $\ell_\mathrm{i}(\delta)$;

5) $\beta_p = \alpha_{\overline{c+1}\to b}$ with $c \in [\overline{\mathrm{i}+1} \to b)$ and $r_p > 0$, as $\mathrm{SL}(r_p\delta + \beta_p)$ then starts with $\ell_c(\delta)$, which has the same length but is lexicographically smaller than $\ell_\mathrm{i}(\delta)$.

Furthermore, if $\beta_p = \alpha_{a\to c}$ with $c \in [a \to (\mathrm{i}-1))$ and $r_p = 0$, then the corresponding concatenation $\mathrm{SL}((r+1)\delta + \alpha_{(c+1)\to b})\mathrm{SL}(\alpha_{a\to c})$ is $\le$ the right-hand side of (4.60) for $k = r+1$, due to the inequality $\overline{\mathrm{i}-1}\ldots(c+1)\mathrm{SL}(\alpha_{a\to c}) \le \overline{\mathrm{i}-1}\ldots(c+1)\,c\ldots a$ (implied by Claim 4.6) and the induction hypothesis. Likewise, if $\beta_p = \alpha_{\overline{c+1}\to b}$ with $c \in [\overline{\mathrm{i}+1} \to b)$ and $r_p = 0$, then the corresponding concatenation

$\mathrm{SL}((r+1)\delta+\alpha_{a\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to b})$ is $\leq$ the right-hand side of (4.60) for $k=r+1$, due to the similar inequality $\overline{\mathrm{i}+1}\dots c\,\mathrm{SL}(\alpha_{\overline{c+1}\to b})\leq\overline{\mathrm{i}+1}\dots b$ and the induction hypothesis.

Therefore, it suffices to consider only the following decompositions in (3.10):

$$\begin{aligned}
\alpha &= (r_1\delta+\alpha_{a\to\overline{\mathrm{i}-1}})+((r+1-r_1)\delta+\alpha_{\mathrm{i}\to b})\,, \quad 0\leq r_1\leq r+1\\
\alpha &= (r_1\delta+\alpha_{a\to\mathrm{i}})+((r+1-r_1)\delta+\alpha_{\overline{\mathrm{i}+1}\to b})\,, \quad 0\leq r_1\leq r+1\\
\alpha &= (\delta)+(r\delta+\alpha_{a\to b}).
\end{aligned} \tag{4.77}$$

Clearly, we can choose only $\mathrm{SL}_1(\delta)=\ell_{\mathrm{i}}(\delta)$ in the latter case. By the induction hypothesis, all the corresponding concatenations have the following specific form:

$$\ell=\underbrace{\ell_{\mathrm{i}}(\delta)}_{p\text{ times}}\ \ell_1\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{q\text{ times}}\ \ell_2\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{m\text{ times}}\ \ell_3 \quad\text{with}$$

$$p+q+m=r+1 \quad\text{and}\quad \{\ell_1\,,\,\ell_2\,,\,\ell_3\}=\{\overline{\mathrm{i}-1}\dots a\,,\,\mathrm{i}\,,\,\overline{\mathrm{i}+1}\dots b\}. \tag{4.78}$$

Since the corresponding concatenation ℓ is Lyndon (Lemma 2.4) and $\ell_{\mathrm{i}}(\delta)$ starts with 1, which is smaller than the first letter of the words ℓ_1,ℓ_2,ℓ_3, we must have

$$p\geq q \quad\text{and}\quad p\geq m. \tag{4.79}$$

Let us consider three cases:

∘ Case 1: $3\,|\,(r+1)$. According to (4.79), we have $p\geq\frac{r+1}{3}$. To get the lexicographically largest word, we need to pick p the smallest possible: $p=\frac{r+1}{3}$. As $p\geq q,m$ and $p+q+m=r+1$, we have $p=q=m=\frac{r+1}{3}$. Additionally, ℓ being Lyndon implies $\ell_1<\ell_2$ and $\ell_1<\ell_3$ if $p=q=m$. It thus follows that $\ell_1=\mathrm{i}$. As we assumed $\overline{\mathrm{i}+1}>\overline{\mathrm{i}-1}$, the largest word occurs if $\ell_2=\overline{\mathrm{i}+1}\dots b>\ell_3=\overline{\mathrm{i}-1}\dots a$. Thus, we end up exactly with the word in the right-hand side of (4.60) for $k=r+1$:

$$\ell_{\max}\ =\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a.$$

This word arises from the decomposition $\alpha=(\frac{2(r+1)}{3}\delta+\alpha_{\mathrm{i}\to b})+(\frac{r+1}{3}\delta+\alpha_{a\to\overline{\mathrm{i}-1}})$. The latter provides the costandard factorization of $\ell_{\max}$, in particular, $\mathrm{b}[\ell_{\max}]\neq 0$.

∘ Case 2: $3\,|\,(r+2)$. According to (4.79), we have $p\geq\frac{r+2}{3}$. To get the lexicographically largest word, we need to pick p the smallest possible: $p=\frac{r+2}{3}$. Then, we have $\{q,m\}=\{\frac{r+2}{3},\frac{r-1}{3}\}$. As ℓ is Lyndon and $q=p$ or $m=p$, $\ell_1\leq\ell_2$ or $\ell_1\leq\ell_3$, respectively. Thus, ℓ_1 equals i or $\overline{\mathrm{i}-1}\dots a$, and to get the lexicographically largest word, we need to pick $\ell_1=\overline{\mathrm{i}-1}\dots a$ and $q=\frac{r-1}{3}$. Then $m=\frac{r+2}{3}$, and ℓ being Lyndon implies that $\ell_3=\overline{\mathrm{i}+1}\dots b$, so that $\ell_2=\mathrm{i}$. Thus, we end up exactly with the word in the right-hand side of (4.60) for $k=r+1$:

$$\ell_{\max}\ =\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+2}{3}\text{ times}}\ \overline{\mathrm{i}-1}\dots a\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r-1}{3}\text{ times}}\ \mathrm{i}\ \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+2}{3}\text{ times}}\ \overline{\mathrm{i}+1}\dots b.$$

This word arises from the decomposition $\alpha=(\frac{2r+1}{3}\delta+\alpha_{a\to\mathrm{i}})+(\frac{r+2}{3}\delta+\alpha_{\overline{\mathrm{i}+1}\to b})$. The latter provides the costandard factorization of $\ell_{\max}$, in particular, $\mathrm{b}[\ell_{\max}]\neq 0$.

∘ Case 3: $3\,|\,r$. According to (4.79), we have $p\geq\frac{r}{3}+1$. To get the lexicographically largest word, we need to pick p the smallest possible and then ℓ_1 the maximal possible: $p=\frac{r}{3}+1$ and $\ell_1=\overline{\mathrm{i}+1}\dots b$. As ℓ_1 is then larger than ℓ_2,ℓ_3 and ℓ is Lyndon, we must have $q,m<p=\frac{r}{3}+1$. Evoking $p+q+m=r+1$, we thus get $q=m=\frac{r}{3}$. It is then straightforward to see (using the induction hypothesis) that the only possible concatenation corresponds to $\ell_2=\mathrm{i},\ell_3=\overline{\mathrm{i}-1}\dots a$. Thus, we end up exactly with the word in

the right-hand side of (4.60) for $k = r + 1$:

$$\ell_{\max} = \underbrace{\ell_i(\delta)}_{\frac{r+3}{3}\text{ times}} \ \overline{i+1}\dots b \ \underbrace{\ell_i(\delta)}_{\frac{r}{3}\text{ times}} \ i \ \underbrace{\ell_i(\delta)}_{\frac{r}{3}\text{ times}} \ \overline{i-1}\dots a.$$

This word arises from the decomposition $\alpha = (\frac{2r}{3}\delta + \alpha_{a\to i}) + (\frac{r+3}{3}\delta + \alpha_{\overline{i+1}\to b})$. The latter provides the costandard factorization of $\ell_{\max}$, in particular, $\mathrm{b}[\ell_{\max}] \neq 0$.

• Proof of (4.64) for $k = r + 1$.

The last root to consider is $\alpha_{b\to a} + (r+1)\delta$, where $1 \in [b; a] \neq \widehat{1}$. First, let us prove the aforementioned fact about the order of ℓ_1, $\ell_{b\to a}(\delta)$, and ℓ_2 (see Remark 4.8(c)):

$$\ell_1 < \ell_2 \leq \ell_{b\to a}(\delta). \tag{4.80}$$

To prove this we need to look at the word $\mathrm{SL}(2\delta + \alpha_{b\to a})$. The first inequality is clear. According to Claim 4.10, ℓ_2 is either $\ell_*(\delta)$ or one of the words $\mathrm{SL}(\alpha_{d\to a}), \mathrm{SL}(\alpha_{b\to c})$ with $d \in [\overline{a+2}; \overline{b-1}], c \in [a; \overline{b-2}]$, respectively. Let us consider these three cases:

∘ If $\ell_2 = \ell_*(\delta)$, then one gets $\ell_{b\to a}(\delta) = \ell_2$ exactly as in our proof of (4.70).

∘ If $\ell_2 = \mathrm{SL}(\alpha_{d\to a})$, then in fact $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}}) < \ell_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to a})$, due to Lemma 4.11. Also $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) < \mathrm{SL}(\alpha_{\overline{b-1}\to\overline{b-3}})\,\overline{b-2} = \ell_{\overline{b-2}}(\delta)$ by Lemma 4.4.

1) If $i \in [2; \overline{b-2}]$, then $\mathrm{b}[\ell_{\overline{b-2}}(\delta)] \doteq (E_{i,i} - E_{b-1,b-1})t$ by (4.67), which does not commute with $\mathrm{b}[\ell_1] = \mathrm{b}[\mathrm{SL}(\alpha_{b\to\overline{b-2}})] \doteq E_{b,b-1}t^{1-\delta_{b,1}}$. Thus, the word $\ell_1\ell_{\overline{b-2}}(\delta)\ell_2$ is Lyndon and its bracketing is $\mathrm{b}[\ell_1\ell_{\overline{b-2}}(\delta)\ell_2] = [\mathrm{b}[\ell_1\ell_{\overline{b-2}}(\delta)], \mathrm{b}[\ell_2]] = [[\mathrm{b}[\ell_1], \mathrm{b}[\ell_{\overline{b-2}}(\delta)]], \mathrm{b}[\ell_2]] \doteq [\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]]t \neq 0$. Therefore, $\ell_2 < \ell_{\overline{b-2}}(\delta) \leq \ell_{b\to a}(\delta)$.

2) If $i \in [\overline{b-1}; n]$, then $\ell_2 < \ell_{\overline{b-2}}(\delta) < \ell_{\overline{b-1}}(\delta)$ by Lemma 4.5. Also $\mathrm{b}[\ell_{\overline{b-1}}(\delta)] \doteq (E_{i+1,i+1} - E_{b-1,b-1})t$ by (4.67), which again does not commute with $\mathrm{b}[\ell_1] \doteq E_{b,b-1}t^{1-\delta_{b,1}}$. Thus, the word $\ell_1\ell_{\overline{b-1}}(\delta)\ell_2$ is Lyndon and moreover, arguing as in 1), we also get $\mathrm{b}[\ell_1\ell_{\overline{b-1}}(\delta)\ell_2] \neq 0$. Therefore, $\ell_2 < \ell_{\overline{b-1}}(\delta) \leq \ell_{b\to a}(\delta)$.

∘ If $\ell_2 = \mathrm{SL}(\alpha_{b\to c})$, then in fact $\ell_1 = \mathrm{SL}(\alpha_{\overline{a+2}\to a}) < \ell_2 = \mathrm{SL}(\alpha_{b\to\overline{a+1}})$, due to Lemma 4.11. Also $\mathrm{SL}(\alpha_{b\to\overline{a+1}}) < \mathrm{SL}(\alpha_{\overline{a+3}\to\overline{a+1}})\,\overline{a+2} = \ell_{\overline{a+2}}(\delta)$ by Lemma 4.4.

1) If $i \in [\overline{a+2}; n]$, then $\mathrm{b}[\ell_{\overline{a+2}}(\delta)] \doteq (E_{i+1,i+1} - E_{a+2,a+2})t$ by (4.67), which does not commute with $\mathrm{b}[\ell_1] = \mathrm{b}[\mathrm{SL}(\alpha_{\overline{a+2}\to a})] \doteq E_{a+2,a+1}t$. Thus, the word $\ell_1\ell_{\overline{a+2}}(\delta)\ell_2$ is Lyndon and its bracketing is $\mathrm{b}[\ell_1\ell_{\overline{a+2}}(\delta)\ell_2] = [\mathrm{b}[\ell_1\ell_{\overline{a+2}}(\delta)], \mathrm{b}[\ell_2]] = [[\mathrm{b}[\ell_1], \mathrm{b}[\ell_{\overline{a+2}}(\delta)]], \mathrm{b}[\ell_2]] \doteq [\mathrm{b}[\ell_1], \mathrm{b}[\ell_2]]t \neq 0$. Therefore, $\ell_2 < \ell_{\overline{a+2}}(\delta) \leq \ell_{b\to a}(\delta)$.

2) If $i \in [2; \overline{a+1}]$, then $\ell_{\overline{a+2}}(\delta) < \ell_{\overline{a+1}}(\delta)$ by Lemma 4.5 so that $\ell_2 < \ell_{\overline{a+1}}(\delta)$. Note that $\mathrm{b}[\ell_{\overline{a+1}}(\delta)] \doteq (E_{i,i} - E_{a+2,a+2})t$ by (4.67), which again does not commute with $\mathrm{b}[\ell_1] \doteq E_{a+2,a+1}t$. Thus, the word $\ell_1\ell_{\overline{a+1}}(\delta)\ell_2$ is Lyndon and moreover, arguing as in 1), we also get $\mathrm{b}[\ell_1\ell_{\overline{a+1}}(\delta)\ell_2] \neq 0$. Therefore, $\ell_2 < \ell_{\overline{a+1}}(\delta) \leq \ell_{b\to a}(\delta)$.

This completes our proof of (4.80).

We also note the following inequality:

$$\mathrm{SL}(\alpha_{b\to a}) \leq \ell_1 < \mathrm{SL}(\delta + \alpha_{b\to a}) = \ell_1\ell_2. \tag{4.81}$$

According to Lemma 4.11, ℓ_1 is either $\mathrm{SL}(\alpha_{b\to\overline{b-2}})$ or $\mathrm{SL}(\alpha_{\overline{a+2}\to a})$. Evoking Lemma 4.4, we thus get $\mathrm{SL}(\alpha_{b\to a}) \leq \ell_1 < \ell_1\ell_2$ in both cases, as claimed in (4.81).

To prove our key Lemma 4.11 below, we need an explicit algorithm for computing the words $\mathrm{SL}(\alpha_{b\to a})$. This is essentially a description of Lalonde–Ram's bijection (2.12) for a finite type A, generalizing our former Claim 4.6 to the case when the minimal letter on the arch $[b; a]$ is not b or a, and it utilizes the argument from our proof of (4.54, 4.55). We provide two algorithms: building $\mathrm{SL}(\alpha_{b\to a})$ either from right to left or from left to right by stacking "segmental" words accordingly.

Right-to-Left Algorithm for $\mathrm{SL}(\alpha_{b\to a})$ with $1 \in [b; a]$.

This algorithm (which crucially uses the fact that each letter appears at most once) reads off the word $\mathrm{SL}(\alpha_{b\to a})$ from right to left, stacking "segmental" words accordingly. First, we note that 1 will be the first letter. Then, we choose the second smallest letter $1 \neq c \in [b; a]$. If $c \in [2; a]$, then we place the word $u_1 := c\,\overline{c+1}\dots a$ in the very end of $\mathrm{SL}(\alpha_{b\to a})$, while for $c \in [b; 0]$ we place the word $u_1 := c\,\overline{c-1}\dots b$ in the very end of $\mathrm{SL}(\alpha_{b\to a})$. Next, we apply the same algorithm to the arch $[b; c-1]$ or $[\overline{c+1}; a]$, respectively. In other words, we take the second smallest letter among the remaining ones, and place the resulting word u_2 right before u_1, and so on.

Left-to-Right Algorithm for $\mathrm{SL}(\alpha_{b\to a})$ with $1 \in [b;a]$.

Since the lexicographical order compares words from left to right, it is convenient to restate the above algorithm by rather building $\mathrm{SL}(\alpha_{b\to a})$ from left to right. The first letter is clearly 1, while the second letter is the $\max\{0,2\}$. If it is 0, then either $n \notin [b;a]$ in which case we just place the segment $23\ldots a$ after 0, or $n \in [b;a]$ and we compare n and 2, do the same operation, and proceed further. Let us rephrase the above algorithm. Pick the largest letter among 2 and 0 and add after 1 the longest Lyndon segment $23\ldots c$ with $c \in [2;a]$ (if $2 > 0$) or $0n\ldots d$ with $d \in [b;0]$ (if $2 < 0$). Then, compare $\overline{c+1}$ with 0 or $\overline{d-1}$ with 2 accordingly, and so on. This reconstructs $\mathrm{SL}(\alpha_{b\to a})$ by stacking "segmental" words from left to right after 1.

Let us now describe the costandard factorization of $\mathrm{SL}(\delta + \alpha_{b\to a})$ with $1 \in [b;a]$.

Lemma 4.11. Let $\mathrm{SL}(\delta + \alpha_{b\to a}) = \ell_1\ell_2$ be the costandard factorization, $1 \in [b;a]$.
(a) If $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}})$, then: $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}})$, $\ell_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to a})$.
(b) If $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) < \mathrm{SL}(\alpha_{b\to\overline{a+1}})$, then: $\ell_1 = \mathrm{SL}(\alpha_{\overline{a+2}\to a})$, $\ell_2 = \mathrm{SL}(\alpha_{b\to\overline{a+1}})$.

Remark 4.12. For $a = \overline{b-2}$ (equivalently, $b = \overline{a+2}$), we get $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}})$ while the above formulas for ℓ_2 should be understood as follows:

$$\ell_2 = \ell_{b\to\overline{b-2}}(\delta) = \ell_{b-1+sgn(i-(b-1))}(\delta).$$

Proof of Lemma 4.11. For $a = \overline{b-2}$, the above formulas (cf. Remark 4.12) are obvious, since according to Claim 4.10 there is only one decomposition to consider:

$$\alpha_{b\to\overline{b-2}} + \delta = (\alpha_{b\to\overline{b-2}}) + (\delta)\,,$$

and $\mathrm{SL}(\alpha_{b\to\overline{b-2}}) < \ell_{\overline{b-1}}(\delta) \le \ell_{b-1+sgn(i-(b-1))}(\delta)$, cf. Lemma 4.5, Remark 4.8(b).

If $a \neq \overline{b-2}$ and $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}})$, then we claim that

$$\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{b-2}})\,. \tag{4.82}$$

Indeed, let us construct all three SL-words $\mathrm{SL}(\alpha_{\overline{b-1}\to a})$, $\mathrm{SL}(\alpha_{b\to\overline{a+1}})$, $\mathrm{SL}(\alpha_{b\to\overline{b-2}})$ using the above "Left-to-Right Algorithm". Then, $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}})$ implies that at the leftmost spot where these words differ either the former has $\overline{b-1}$ while the latter has some $c < \overline{b-1}$ or the latter has $\overline{a+1}$ while the former has some $c > \overline{a+1}$. In either of these cases, we clearly have $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{b-2}})$.

According to (4.82) and Lemma 2.4, the word $\mathrm{SL}(\alpha_{b\to\overline{b-2}})\mathrm{SL}(\alpha_{\overline{b-1}\to a})$ is Lyndon. Its costandard factorization (2.4) is precisely given by $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}})$ and $\ell_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to a})$, since both words start with 1 (and have no more 1's). Hence, the standard bracketing $\mathrm{b}[\mathrm{SL}(\alpha_{b\to\overline{b-2}})\mathrm{SL}(\alpha_{\overline{b-1}\to a})] = [\mathrm{b}[\mathrm{SL}(\alpha_{b\to\overline{b-2}})], \mathrm{b}[\mathrm{SL}(\alpha_{\overline{b-1}\to a})]] \neq 0$. We thus conclude that $\mathrm{SL}(\delta + \alpha_{b\to a}) \geq \mathrm{SL}(\alpha_{b\to\overline{b-2}})\mathrm{SL}(\alpha_{\overline{b-1}\to a})$. We also note that combining (4.82) with Lemma 4.4, we obtain

$$\mathrm{SL}(\alpha_{b\to c}) \le \mathrm{SL}(\alpha_{b\to\overline{b-2}}) < \mathrm{SL}(\alpha_{\overline{b-1}\to a}) \qquad \forall\, c \in [a;\overline{b-2}]\,. \tag{4.83}$$

Combining Claim 4.10 with (4.83), we get $\mathrm{SL}(\delta + \alpha_{b\to a}) \le \mathrm{SL}(\alpha_{b\to\overline{b-2}})\mathrm{SL}(\alpha_{\overline{b-1}\to a})$. Therefore, we actually have the equality

$$\mathrm{SL}(\delta + \alpha_{b\to a}) = \mathrm{SL}(\alpha_{b\to\overline{b-2}})\mathrm{SL}(\alpha_{\overline{b-1}\to a})$$

and the two words in the right-hand side determine the costandard factorization of $\mathrm{SL}(\delta + \alpha_{b\to a})$, as shown above. This completes our proof of part (a).

The proof of part (b) is analogous and is left to the interested reader. ■

Corollary 4.13. In the setup of Lemma 4.11, we have

$$\ell_1 = \min\left\{\mathrm{SL}(\alpha_{b\to\overline{b-2}}), \mathrm{SL}(\alpha_{\overline{a+2}\to a})\right\}. \tag{4.84}$$

Proof. For $a = \overline{b-2}$, the claim is vacuous by Lemma 4.11. If $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{b\to\overline{a+1}})$, then $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}}) < \mathrm{SL}(\alpha_{\overline{b-1}\to a})$ by (4.82) and Lemma 4.11. But $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) \leq \mathrm{SL}(\alpha_{\overline{a+2}\to a})$ by Lemma 4.4 as $1 \in [\overline{b-1}; a] \subseteq [\overline{a+2}; a]$ for $a \prec \overline{b-2}$. Combining the above, we obtain $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}}) < \mathrm{SL}(\alpha_{\overline{a+2}\to a})$.

The case $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) < \mathrm{SL}(\alpha_{b\to\overline{a+1}})$ is completely analogous. ■

With the inequalities (4.80, 4.81) and Lemma 4.11 at hand, we shall finally proceed to the proof of (4.64) for $k = r+1$. To this end, we consider all possible decompositions of $\alpha = (r+1)\delta + \alpha_{b\to a}$ with $1 \in [b;a]$ case-by-case:

1) $\alpha = (r_1\delta + \alpha_{b\to c}) + ((r+1-r_1)\delta + \alpha_{\overline{c+1}\to a})$, with $c \in [b \to a)$.

Let us assume that $1 \in [b;c]$ (the case $1 \in [\overline{c+1}; a]$ is analogous). The corresponding concatenation ℓ is $\leq \ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{(r_1-1)\text{ times}} \ell'_2\, \mathrm{SL}((r+1-r_1)\delta + \alpha_{\overline{c+1}\to a})$ if $r_1 > 0$, or $\leq \mathrm{SL}(\alpha_{b\to c})\mathrm{SL}((r+1)\delta + \alpha_{\overline{c+1}\to a})$ if $r_1 = 0$. Here, $\mathrm{SL}(\delta + \alpha_{b\to c}) = \ell'_1\ell'_2$ is the costandard factorization. According to (4.80, 4.81), we have $\mathrm{SL}(\alpha_{b\to c}) \leq \ell'_1 < \ell'_2 \leq \ell_{b\to c}(\delta)$, where both equalities hold iff either of them holds. As $c \in [a \to b)$ and $b \neq \overline{a-1}$, we have $\mathrm{SL}(\alpha_{b\to c}) \neq \ell'_1$, due to Lemma 4.11. Thus $\mathrm{SL}(\alpha_{b\to c}) < \ell'_1$, so that $\mathrm{SL}(k_1\delta + \alpha_{b\to c}) < \mathrm{SL}(k_2\delta + \alpha_{b\to c})$, hence $\mathrm{SL}(k_1\delta + \alpha_{b\to c})1 < \mathrm{SL}(k_2\delta + \alpha_{b\to c})$ and the former is not a prefix of the latter for any $0 \leq k_1 < k_2$. Therefore, $\ell \leq \ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{r\text{ times}} \ell'_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to a}) = \mathrm{SL}((r+1)\delta + \alpha_{b\to c}))\mathrm{SL}(\alpha_{\overline{c+1}\to a})$. By Lemma 4.11, ℓ'_2 is either $\mathrm{SL}(\alpha_{\overline{b-1}\to c})$ or $\mathrm{SL}(\alpha_{b\to\overline{c+1}})$. We consider these cases:

∘ If $\ell'_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to c})$ and $a \neq \overline{b-2}$, then $\ell'_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to a}) \leq \mathrm{SL}(\alpha_{\overline{b-1}\to a})$ by Proposition 3.4. Moreover, by Lemma 4.11 and its proof, we also have $\mathrm{SL}(\alpha_{\overline{b-1}\to c}) > \mathrm{SL}(\alpha_{b\to\overline{c+1}})$ and $\mathrm{SL}(\alpha_{\overline{b-1}\to c}) > \mathrm{SL}(\alpha_{b\to\overline{b-2}}) = \ell'_1$. We thus obtain a sequence of inequalities: $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) > \mathrm{SL}(\alpha_{\overline{b-1}\to c}) > \mathrm{SL}(\alpha_{b\to\overline{b-2}}) \geq \mathrm{SL}(\alpha_{b\to\overline{a+1}})$. Hence, applying Lemma 4.11 once again to $\mathrm{SL}(\delta + \alpha_{b\to a})$, we see that its costandard factorization has prefix $\ell_1 = \ell'_1$, suffix $\ell_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to a})$, and therefore $\ell_{b\to a}(\delta) = \ell_{b\to c}(\delta)$. Thus, we derive the desired inequality:

$$\ell \leq \ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{r\text{ times}} \mathrm{SL}(\alpha_{\overline{b-1}\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to a}) \leq \ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{r\text{ times}} \mathrm{SL}(\alpha_{\overline{b-1}\to a}) = \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r\text{ times}} \ell_2\,.$$

Moreover, the equality is possible for $r_1 = r+1$ and a specific $c \in [b \to a)$ such that $\mathrm{SL}(\alpha_{\overline{b-1}\to a}) = \mathrm{SL}(\alpha_{\overline{b-1}\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to a})$ is the costandard factorization.

Let us now consider the case $\ell'_2 = \mathrm{SL}(\alpha_{\overline{b-1}\to c})$ and $a = \overline{b-2}$. If $\mathrm{SL}(\alpha_{\overline{b-1}\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to a}) \leq \ell_{b\to c}(\delta)$, then $\ell \leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r+1\text{ times}} = \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r\text{ times}} \ell_2$ still holds. Otherwise, if $\mathrm{SL}(\alpha) = \ell$, we would have $\mathrm{SL}(\alpha) = \mathrm{SL}((r+1)\delta + \alpha_{b\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to a}) = \ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{r\text{ times}} \mathrm{SL}(\alpha_{\overline{b-1}\to c})\mathrm{SL}(\alpha_{\overline{c+1}\to a})$, due to Proposition 3.4(a). However, the costandard factorization of the above word passes to the right of ℓ'_1, and the costandard factorization of the resulting suffix passes to the right of the first $\ell_{b\to c}(\delta)$, a contradiction with Remark 3.5. Hence, ℓ cannot be standard Lyndon.

∘ If $\ell'_2 = \mathrm{SL}(\alpha_{b\to\overline{c+1}})$, then $\mathbf{deg}\, \ell'_2 + \alpha_{\overline{c+1}\to a} \notin \widehat{\Delta}^+$ and so $\mathrm{b}[\ell'_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to a})] = 0$. Likewise, by the degree reasons and evoking inequalities (4.80), we find

$$\mathrm{b}[\mathrm{SL}((r+1)\delta + \alpha_{b\to c}))\mathrm{SL}(\alpha_{\overline{c+1}\to a})] = \mathrm{b}[\ell'_1 \underbrace{\ell_{b\to c}(\delta)}_{r\text{ times}} \ell'_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to a})] = 0$$

as the costandard factorization of this concatenation passes on the left of ℓ'_2 or some $\ell_{b\to c}(\delta)$. Since $\ell \leq \mathrm{SL}((r+1)\delta + \alpha_{b\to c}))\mathrm{SL}(\alpha_{\overline{c+1}\to a})$, we see that if $\mathrm{SL}(\alpha) = \ell$, then we would have $\mathrm{SL}(\alpha) = \mathrm{SL}((r+1)\delta + \alpha_{b\to c}))\mathrm{SL}(\alpha_{\overline{c+1}\to a})$, due to Proposition 3.4(a). However, the rightmost word cannot be standard Lyndon as its standard bracketing was shown above to be 0. Hence, a contradiction with $\mathrm{SL}(\alpha) = \ell$.

2) $\alpha = (r_1\delta + \alpha_{b\to c}) + ((r-r_1)\delta + \alpha_{\overline{c+1}\to a})$, where $1 \in [b;c]$ and $1 \in [\overline{c+1}; a]$.

Let $\mathrm{SL}(\delta + \alpha_{b\to a}) = \ell_1\ell_2$ be the costandard factorization. We claim that one of length n prefixes of the words $\mathrm{SL}(\delta + \alpha_{b\to c})$, $\mathrm{SL}(\delta + \alpha_{\overline{c+1}\to a})$ is $\leq \ell_1$. Indeed, assume that $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}})$ (the case $\ell_1 = \mathrm{SL}(\alpha_{\overline{a+2}\to a})$ is treated similarly). Then, the length n prefix ℓ'_1 of $\mathrm{SL}(\delta + \alpha_{b\to c})$ is ℓ_1 or $\mathrm{SL}(\alpha_{\overline{c+2}\to c})$, due to Lemma 4.11. Note that $\mathrm{SL}(\alpha_{\overline{c+2}\to c}) = \ell_1$ if $c = \overline{b-2}$, while the inequality $\mathrm{SL}(\alpha_{\overline{c+2}\to c}) < \mathrm{SL}(\alpha_{b\to\overline{c+1}})$ for $c \neq \overline{b-2}$ is proved similarly to (4.82). Combining the latter inequality with $\mathrm{SL}(\alpha_{b\to\overline{c+1}}) \leq \mathrm{SL}(\alpha_{b\to\overline{b-2}}) = \ell_1$ due to Lemma 4.4, we obtain $\ell_1' < \ell_1$ as claimed. Henceforth, we assume that $\ell_1 = \mathrm{SL}(\alpha_{b\to\overline{b-2}})$, leaving the other case to the reader.

First consider $0 < r_1 < r$. If the length n prefix ℓ_1' of $\mathrm{SL}(\delta + \alpha_{b\to c})$ is $< \ell_1$, then the corresponding concatenation ℓ is lexicographically smaller than the right-hand side of (4.64) for $k = r+1$. If $\ell_1' = \ell_1$, then we get a costandard factorization $\mathrm{SL}(\delta + \alpha_{b\to c}) = \ell_1\ell_3$ and so $\ell_{b\to c}(\delta) = \ell_{b\to a}(\delta)$. If $c \neq \overline{b-2}$, then $\ell_3 < \ell_{b\to c}(\delta)$ by (4.80) and so $\ell_3\, 1 \leq \ell_{b\to c}(\delta) = \ell_{b\to a}(\delta)$. Thus, we get the desired inequality:

$$\ell \leq \mathrm{SL}(r_1\delta + \alpha_{b\to c})\mathrm{SL}((r-r_1)\delta + \alpha_{(c+1)\to a}) < \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2 \,.$$

If $c = \overline{b-2}$, then $\ell_3 = \ell_{b\to\overline{b-2}}(\delta) = \ell_{b-1+sgn(i-(b-1))}(\delta) \geq \ell_{\overline{b-2}}(\delta)$, with the last inequality due to Lemma 4.5. Let $\mathrm{SL}(\delta + \alpha_{\overline{b-1}\to a}) = \ell_4\ell_5$ be the costandard factorization. Then, $\ell_4 \leq \mathrm{SL}(\alpha_{\overline{b-1}\to\overline{b-3}}) < \mathrm{SL}(\alpha_{\overline{b-1}\to\overline{b-3}})\,\overline{b-2} = \ell_{\overline{b-2}}(\delta)$, due to (4.84). Hence, the corresponding concatenation ℓ satisfies the desired inequality:

$$\ell \leq \ell_1 \underbrace{\ell_{b\to\overline{b-2}}(\delta)}_{r_1 \text{ times}} \ell_4 \underbrace{\ell_{\overline{b-1}\to a}(\delta)}_{(r-r_1-1) \text{ times}} \ell_5 < \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2 \,.$$

For $r_1 = r$, it suffices to consider the case when $\mathrm{SL}(\delta + \alpha_{b\to c})$ starts with ℓ_1. The case $c \neq \overline{b-2}$ is treated as above. If $c = \overline{b-2}$, then $\ell_2 = \mathrm{SL}(\alpha_{\overline{c+1}\to a})$ and $\ell_3 = \ell_{b\to c}(\delta) = \ell_{b\to a}(\delta)$, and thus the resulting concatenation $\ell \leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2$.

Finally, if $r_1 = 0$ and $\mathrm{SL}(\delta + \alpha_{\overline{c+1}\to a}) = \ell_4\ell_5$ is the costandard factorization, then $\mathrm{SL}(\alpha_{b\to c}) \leq \ell_1' \leq \ell_1$. If $c \neq \overline{b-2}$, then $\mathrm{SL}(\alpha_{b\to c}) < \ell_1$, so that $\mathrm{SL}(\alpha_{b\to c})1 < \ell_1$ and the former is not a prefix of the latter, implying $\ell < \ell_1$. If $c = \overline{b-2}$, then $\mathrm{SL}(\alpha_{b\to c}) = \ell_1$ and $\ell_4 \leq \mathrm{SL}(\alpha_{\overline{b-1}\to\overline{b-3}}) < \ell_{\overline{b-2}}(\delta) \leq \ell_{b\to a}(\delta)$ by above, so that

$$\ell \leq \mathrm{SL}(\alpha_{b\to c})\mathrm{SL}(r\delta + \alpha_{\overline{c+1}\to a}) \leq \ell_1\ell_4 \underbrace{\ell_{\overline{c+1}\to a}(\delta)}_{(r-1) \text{ times}} \ell_5 < \ell_1\ell_{b\to a}(\delta) < \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2 \,.$$

3) $\alpha = (r_1\delta) + ((r+1-r_1)\delta + \alpha_{b\to a})$.

If $a \neq \overline{b-2}$, then (using the induction hypothesis) the corresponding concatenated word ℓ is $\leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{(r-r_1) \text{ times}} \ell_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}}) \underbrace{\ell_{c+sgn(i-c)}(\delta)}_{(r_1-1) \text{ times}} c$ if $r_1 \leq r$, or $\leq \mathrm{SL}(\alpha_{b\to a})\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}}) \underbrace{\ell_{c+sgn(i-c)}(\delta)}_{r \text{ times}} c$ if $r_1 = r+1$, for some $c \neq 1$. Due to the inequalities $\mathrm{SL}(\alpha_{b\to a}) < \ell_1 < \ell_2 < \ell_{b\to a}(\delta)$, cf. (4.80, 4.81), we obtain ($\forall\, c \neq 1$)

$$\ell \leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{(r-1) \text{ times}} \ell_2\, \mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})c < \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2 \,.$$

Let us now treat the case $a = \overline{b-2}$, for which we utilize the non-commutativity of the corresponding bracketings. We consider the cases $r_1 = 1$ and $r_1 > 1$ separately.

If $r_1 = 1$, then the corresponding concatenation ℓ is $\leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_c(\delta)$, where $\ell_2 = \ell_{b\to a}(\delta) = \ell_{b-1+sgn(i-(b-1))}(\delta)$ by Remark 4.12. Here, $\mathrm{b}[\ell_c(\delta)]$ does not commute with $\mathrm{b}[\mathrm{SL}(r\delta + \alpha_{b\to\overline{b-2}})]$, which is equivalent to $[\mathrm{b}[\ell_c(\delta)], E_{b,b-1}] \neq 0$. The latter guarantees that $\ell_c(\delta) \leq \ell_{b\to a}(\delta)$, due to (4.67) and Lemma 4.5:

- if $b \prec i$ then $c = b-1, b$ and $\ell_c(\delta) \leq \ell_b(\delta) = \ell_{b-1+sgn(i-(b-1))}(\delta)$;
- if $b = i, i+1, i+2$, then $\ell_{b-1+sgn(i-(b-1))}(\delta) = \ell_i(\delta) \geq \ell_c(\delta)$;
- if $b \succ i+2$, then $c = b-1, b-2$ and $\ell_c(\delta) \leq \ell_{b-2}(\delta) = \ell_{b-1+sgn(i-(b-1))}(\delta)$. Hence, we derive the desired inequality:

$$\ell \leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_c(\delta) \leq \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{(r+1) \text{ times}} = \ell_1 \underbrace{\ell_{b\to a}(\delta)}_{r \text{ times}} \ell_2 \,.$$

For $r_1 > 1$, the argument is precisely the same and is based on the inequalities $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}}) < \ell_c(\delta) \leq \ell_{b\to a}(\delta)$. Here, the second inequality is proved as above, but using (4.73) instead of (4.67).

This completes the proof of (4.64). In the particular case $r = 1$, this proves the formula $\mathrm{SL}(2\delta + \alpha_{b\to a}) = \ell_1 \ell_{b\to a}(\delta)\ell_2$ implicitly used in the statement of (4.64).

5 Properties of Orders

To account for $\mathbf{dim}(\widehat{\mathfrak{g}}_{k\delta}) = |I|$ in (3.9), let us extend $\widehat{\Delta}^+$ to $\widehat{\Delta}^{+,\mathrm{ext}}$:

$$\widehat{\Delta}^{+,\mathrm{ext}} := \widehat{\Delta}^{+,\mathrm{re}} \sqcup \big\{(k\delta, r) \,\big|\, k \geq 1, 1 \leq r \leq |I|\big\}. \tag{5.1}$$

We define $\mathrm{SL}((k\delta, r)) := \mathrm{SL}_r(k\delta)$ accordingly. Consider the order on $\widehat{\Delta}^{+,\mathrm{ext}}$ induced from the lexicographical order on affine standard Lyndon words, cf. (2.15):

$$\alpha < \beta \quad \Longleftrightarrow \quad \mathrm{SL}(\alpha) < \mathrm{SL}(\beta) \ \text{ lexicographically.} \tag{5.2}$$

In this section, we investigate some properties of this order using Theorem 4.7.

Example 5.1. The only case when $\widehat{\Delta}^{+,\mathrm{ext}} = \widehat{\Delta}^+$ is the case of $\widehat{\mathfrak{sl}}_2$. Using the formulas of Proposition 3.7 (with the order $1 < 0$), we see that (5.2) recovers the usual order:

$$\alpha_1 < \alpha_1 + \delta < \alpha_1 + 2\delta < \cdots < \cdots < 3\delta < 2\delta < \delta < \cdots < 2\delta + \alpha_0 < \delta + \alpha_0 < \alpha_0\,.$$

5.2 Important counterexample

Unlike the orders on $\widehat{\Delta}^{+,\mathrm{ext}}$ in the theory of affine quantum groups ([1, 4]), arising through the affine braid group action, the order (5.2) does separate imaginary roots. Explicitly, for type $A_n^{(1)}$ $(n > 1)$ and any order on $\widehat{I}$, one always has

$$(k_1\delta, n) < \alpha < (k_2\delta, 1) \quad \text{for some } \alpha \in \widehat{\Delta}^{+,\mathrm{re}}\,,\ k_1, k_2 \geq 1.$$

It is thus natural to ask (motivated by Levendorsky–Soibelman convexity property):

Question: Is it true that we cannot have a pattern

$$(k_2\delta, n) < \beta_2 < \beta_1 < (k_1\delta, 1) \quad \text{with} \quad \beta_1, \beta_2 \in \widehat{\Delta}^{+,\mathrm{re}}\,,\ \beta_1 + \beta_2 = (k_1 + k_2)\delta.$$

The answer is actually negative, as shown by the following simplest counterexample.

Counterexample: Consider the affine Lie algebra $\widehat{\mathfrak{sl}}_5$ with the standard order $1 < 2 < 3 < 4 < 0$ on $\widehat{I}$. For $k, m > 0$, set $\beta_1 = k\delta + \alpha_4$, $\beta_2 = m\delta + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)$ and $k_1 = 1, k_2 = k + m$. According to Theorem 4.2, we have

$$\mathrm{SL}_1(\delta) = 10432\,, \qquad \mathrm{SL}_4((k+m)\delta) = 1234\ \underbrace{10234}_{(k+m-1)\text{ times}}\ 0,$$

$$\mathrm{SL}(\beta_1) = \underbrace{10423}_{k\text{ times}}4\,, \qquad \mathrm{SL}(\beta_2) = 1023\,\underbrace{10423}_{m\text{ times}}.$$

Thus, indeed $(k_2\delta, 4) < \beta_2 < \beta_1 < (\delta, 1)$ with respect to the order (5.2) on $\widehat{\Delta}^{+,\mathrm{ext}}$.

5.3 Chain monotonicity in type $A_n^{(1)}$

For $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, define the *chain* Ch_α as the sequence $\alpha, \alpha + \delta, \alpha + 2\delta, \ldots \in \widehat{\Delta}^{+,\mathrm{re}}$.

Proposition 5.4. For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, the chain Ch_α is monotonous:

$$\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\delta) < \mathrm{SL}(\alpha+2\delta) < \cdots \quad \text{or} \quad \mathrm{SL}(\alpha) > \mathrm{SL}(\alpha+\delta) > \mathrm{SL}(\alpha+2\delta) > \cdots$$

Proof. Without loss of generality, we can assume that (4.52) holds, so that the formulas of Theorem 4.7 apply. The proof follows by a simple case-by-case analysis:

- $\alpha = \alpha_{a\to b}$ with $i \prec a \preceq b \preceq 0$.

According to (4.59), we have $\mathrm{SL}(k\delta+\alpha_{a\to b}) = \underbrace{\ell_{\overline{a-1}}(\delta)}_{k\text{ times}}\, a\,\overline{a+1}\ldots b$ for all $k \geq 1$. As $a\,\overline{a+1}\ldots b$ starts with a letter a that is larger than 1, the first letter of $\ell_{\overline{a-1}}(\delta)$, we obtain $\mathrm{SL}(k\delta+\alpha_{a\to b}) > \mathrm{SL}((k+1)\delta+\alpha_{a\to b})$ for any $k \geq 1$. In the remaining case $k = 0$, we also have $\mathrm{SL}(\alpha_{a\to b}) > \mathrm{SL}(\delta+\alpha_{a\to b})$, as $\mathrm{SL}(\alpha_{a\to b})$ starts with a letter $\mathbf{min}\{a,\ldots,b\}$, which is larger than 1, the first letter of $\mathrm{SL}(\delta+\alpha_{a\to b})$.

- $\alpha = \alpha_{a\to b}$ with $1 \prec a \preceq b \prec i$.

The proof of $\mathrm{SL}(k\delta+\alpha_{a\to b}) > \mathrm{SL}((k+1)\delta+\alpha_{a\to b})$ for any $k \geq 0$ is exactly the same as above, with $\ell_{b+1}(\delta)$ used instead of $\ell_{\overline{a-1}}(\delta)$.

- $\alpha = \alpha_{a\to b}$ with $1 \prec a \prec i \prec b$.

Combining formula (4.60) with the inequalities $\overline{i\pm 1} > i > 1 =$ first letter of $\ell_i(\delta)$, we obtain $\mathrm{SL}(k\delta+\alpha_{a\to b}) > \mathrm{SL}((k+1)\delta+\alpha_{a\to b})$ for any $k \geq 1$. In the remaining case $k = 0$, we also have $\mathrm{SL}(\alpha_{a\to b}) > \mathrm{SL}(\delta+\alpha_{a\to b})$, as $1 \notin [a;b]$.

- $\alpha = \alpha_{a\to b}$ with $a = i$ or $b = i$ and $1 \notin [a;b]$.

The proof of $\mathrm{SL}(k\delta+\alpha_{a\to b}) > \mathrm{SL}((k+1)\delta+\alpha_{a\to b})$ for any $k \geq 0$ is exactly the same as above, where we now use one of (4.61)–(4.63) instead of (4.60).

- $\alpha = \alpha_{b\to a}$ with $1 \in [b;a]$.

According to (4.64), we have $\mathrm{SL}(k\delta+\alpha_{b\to a}) = \ell_1\,\underbrace{\ell_{b\to a}(\delta)}_{(k-1)\text{ times}}\,\ell_2$ for all $k \geq 1$. Here, we have $\ell_2 \leq \ell_{b\to a}(\delta)$, due to (4.80), so that $\ell_2 < \ell_{b\to a}(\delta)\ell_2$. Thus, we obtain $\mathrm{SL}(k\delta+\alpha_{b\to a}) < \mathrm{SL}((k+1)\delta+\alpha_{b\to a})$ for any $k \geq 1$. In the remaining case $k = 0$, we also have $\mathrm{SL}(\alpha_{b\to a}) < \mathrm{SL}(\delta+\alpha_{b\to a})$, due to (4.81). ■

Remark 5.5. It follows from the proof that the chain Ch_α monotonously increases if $\alpha = k\delta+\alpha_{a\to b}$ with $\mathbf{min}\{\widehat{I}\} \in [a;b]$, and monotonously decreases otherwise.

Remark 5.6. For any $k \geq 1$ and $c \neq 1$, we also have $\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\,\underbrace{\ell_{c+sgn(i-c)}(\delta)}_{(k-1)\text{ times}}\,c > \mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\,\underbrace{\ell_{c+sgn(i-c)}(\delta)}_{k\text{ times}}\,c$, cf. (4.57). Since the order among length n words $\{\mathrm{SL}(\alpha_{\overline{c+1}\to\overline{c-1}})\,|\,c \neq 1\}$ determines the order among the n words in the right-hand side of (4.57) for any k, we also see that $\{\mathrm{SL}(k\delta, r)\}_{k\geq 1}$ monotonously decreases:

$$\mathrm{SL}(\delta, r) > \mathrm{SL}(2\delta, r) > \mathrm{SL}(3\delta, r) > \cdots \qquad \forall\, 1 \leq r \leq n.$$

5.7 Pre-convexity in type $A_n^{(1)}$

Motivated by Definition 2.18, we shall call an order $<$ on $\widehat{\Delta}^{+,\mathrm{re}}$ *pre-convex* if

$$\alpha < \alpha+\beta < \beta \quad \text{or} \quad \beta < \alpha+\beta < \alpha \qquad \forall\, \alpha,\beta,\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}. \tag{5.3}$$

Proposition 5.8. The restriction of (5.2) to $\widehat{\Delta}^{+,\mathrm{re}}$ is pre-convex.

Proof. Without loss of generality, we can assume that (4.52) holds, so that the formulas of Theorem 4.7 apply. The proof follows by a direct case-by-case analysis:

- $\alpha = \alpha_{a\to b}+k\delta$, $\beta = \alpha_{(b+1)\to c}+r\delta$ for $1 \prec a \preceq b \prec c \prec i$.
  - Case 1: $k, r > 0$. In this case, we have $\mathrm{SL}(\alpha) = \underbrace{\ell_{b+1}(\delta)}_{k\text{ times}}\,b(b-1)\ldots a$, $\mathrm{SL}(\beta) = \underbrace{\ell_{c+1}(\delta)}_{r\text{ times}}\,c(c-1)\ldots(b+1)$,

$\mathrm{SL}(\alpha+\beta) = \underbrace{\ell_{c+1}(\delta)}_{(k+r)\text{ times}}\,c(c-1)\ldots a$. The inequality $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta)$ is a consequence of $\ell_{c+1}(\delta) > \ell_{b+1}(\delta)$ (due to Lemma 4.5), while the inequality $\mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$ is obvious as $\ell_{c+1}(\delta)$ starts with $1 < c$.

∘ Case 2: $k=0, r>0$. In this case, we have $\mathrm{SL}(\beta)=\underbrace{\ell_{c+1}(\delta)}_{r\text{ times}}c(c-1)\dots(b+1)$, $\mathrm{SL}(\alpha+\beta)=\underbrace{\ell_{c+1}(\delta)}_{r\text{ times}}c(c-1)\dots a$, while $\mathrm{SL}(\alpha)$ starts with a letter >1. Therefore, we immediately get $\mathrm{SL}(\alpha)>\mathrm{SL}(\alpha+\beta)>\mathrm{SL}(\beta)$.

∘ Case 3: $k>0, r=0$. In this case, we have $\mathrm{SL}(\alpha)=\underbrace{\ell_{b+1}(\delta)}_{k\text{ times}}b(b-1)\dots a$, $\mathrm{SL}(\alpha+\beta)=\underbrace{\ell_{c+1}(\delta)}_{k\text{ times}}c(c-1)\dots a$, while $\mathrm{SL}(\beta)$ starts with a letter >1. Evoking the inequality $\ell_{c+1}(\delta)>\ell_{b+1}(\delta)$, we immediately get $\mathrm{SL}(\alpha)<\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\beta)$.

∘ Case 4: $k=r=0$. In this case, $\alpha,\beta,\alpha+\beta\in\Delta^+$, hence the claim follows from Proposition 2.20 (a priori we do not know which of the two possible orders holds).

- $\alpha=\alpha_{a\to b}+k\delta$, $\beta=\alpha_{\overline{b+1}\to c}+r\delta$ for $i\prec a\preceq b\prec c\preceq 0$.

∘ Case 1: $k,r>0$. In this case, we have $\mathrm{SL}(\alpha)=\underbrace{\ell_{a-1}(\delta)}_{k\text{ times}}a\,\overline{a+1}\dots b$, $\mathrm{SL}(\beta)=\underbrace{\ell_b(\delta)}_{r\text{ times}}\overline{b+1}\,\overline{b+2}\dots c$, $\mathrm{SL}(\alpha+\beta)=\underbrace{\ell_{a-1}(\delta)}_{(k+r)\text{ times}}a\,\overline{a+1}\dots c$. The inequality $\mathrm{SL}(\beta)<\mathrm{SL}(\alpha+\beta)$ is a consequence of $\ell_{a-1}(\delta)>\ell_b(\delta)$ (due to Lemma 4.5), while the inequality $\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\alpha)$ is obvious as $\ell_{a-1}(\delta)$ starts with 1, which is $<a$.

∘ Case 2: $k=0, r>0$. In this case, we have $\mathrm{SL}(\beta)=\underbrace{\ell_b(\delta)}_{r\text{ times}}\overline{b+1}\,\overline{b+2}\dots c$, $\mathrm{SL}(\alpha+\beta)=\underbrace{\ell_{a-1}(\delta)}_{r\text{ times}}a\,\overline{a+1}\dots c$, while $\mathrm{SL}(\alpha)$ starts with a letter >1. Evoking the inequality $\ell_{a-1}(\delta)>\ell_b(\delta)$, we immediately get $\mathrm{SL}(\beta)<\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\alpha)$.

∘ Case 3: $k>0, r=0$. In this case, we have $\mathrm{SL}(\alpha)=\underbrace{\ell_{a-1}(\delta)}_{k\text{ times}}a\,\overline{a+1}\dots b$, $\mathrm{SL}(\alpha+\beta)=\underbrace{\ell_{a-1}(\delta)}_{k\text{ times}}a\,\overline{a+1}\dots c$, while $\mathrm{SL}(\beta)$ starts with a letter >1. Therefore, we get

$$\mathrm{SL}(\alpha)<\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\beta).$$

∘ Case 4: $k=r=0$. In this case, the claim follows from Proposition 2.20 again.

- $\alpha=\alpha_{a\to(i-1)}+k\delta$, $\beta=\alpha_i+r\delta$ for $1\prec a\prec i$.

∘ Case 1: $k>0, r\geq 0$. In this case, we have $\mathrm{SL}(\alpha)=\underbrace{\ell_i(\delta)}_{k\text{ times}}\overline{i-1}\,\overline{i-2}\dots a$, $\mathrm{SL}(\alpha+\beta)=$
$$\begin{cases}\underbrace{\ell_i(\delta)}_{\frac{k+r}{2}\text{ times}}\;i\;\underbrace{\ell_i(\delta)}_{\frac{k+r}{2}\text{ times}}\;\overline{i-1}\dots a & \text{if } 2\mid(k+r)\\ \underbrace{\ell_i(\delta)}_{\frac{k+r+1}{2}\text{ times}}\;\overline{i-1}\dots a\;\underbrace{\ell_i(\delta)}_{\frac{k+r-1}{2}\text{ times}}\;i & \text{if } 2\nmid(k+r)\end{cases}$$
, and $\mathrm{SL}(\beta)=\underbrace{\ell_i(\delta)}_{r\text{ times}}i$.

If $2\mid(k+r)$ and $k>\frac{k+r}{2}>r$, then clearly $\mathrm{SL}(\alpha)<\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\beta)$. If $2\mid(k+r)$ and $k\leq\frac{k+r}{2}\leq r$, then clearly $\mathrm{SL}(\alpha)>\mathrm{SL}(\alpha+\beta)>\mathrm{SL}(\beta)$.

If $2\nmid(k+r)$ and $k\geq\frac{k+r+1}{2}>r$, then clearly $\mathrm{SL}(\alpha)<\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\beta)$. If $2\nmid(k+r)$ and $k<\frac{k+r+1}{2}\leq r$, then clearly $\mathrm{SL}(\alpha)>\mathrm{SL}(\alpha+\beta)>\mathrm{SL}(\beta)$.

∘ Case 2: $k=0, r>0$. In this case, $\mathrm{SL}(\alpha)$ starts with a letter >1, $\mathrm{SL}(\beta)=\underbrace{\ell_i(\delta)}_{r\text{ times}}i$, $\mathrm{SL}(\alpha+\beta)=$
$$\begin{cases}\underbrace{\ell_i(\delta)}_{\frac{r}{2}\text{ times}}\;i\;\underbrace{\ell_i(\delta)}_{\frac{r}{2}\text{ times}}\;\overline{i-1}\dots a & \text{if } 2\mid r\\ \underbrace{\ell_i(\delta)}_{\frac{r+1}{2}\text{ times}}\;\overline{i-1}\dots a\;\underbrace{\ell_i(\delta)}_{\frac{r-1}{2}\text{ times}}\;i & \text{if } 2\nmid r\end{cases}$$
. Therefore, we immediately get $\mathrm{SL}(\alpha)>\mathrm{SL}(\alpha+\beta)>\mathrm{SL}(\beta)$.

∘ Case 3: $k=r=0$. In this case, the claim follows from Proposition 2.20 again. In fact, we get $\mathrm{SL}(\alpha)>\mathrm{SL}(\alpha+\beta)>\mathrm{SL}(\beta)$ since $\mathrm{SL}(\alpha)>\mathrm{SL}(\beta)$ (as $i<a,\dots,i-1$).

- $\alpha=\alpha_{a\to b}+k\delta$, $\beta=\alpha_{(b+1)\to i}+r\delta$ for $1\prec a\preceq b\prec i-1$.

∘ Case 1: $k,r>0$. Combining (4.58, 4.62) and Lemma 4.5, we obtain

$$\mathrm{SL}(\alpha)=\underbrace{\ell_{b+1}(\delta)}_{k\text{ times}}b\,\overline{b-1}\dots a<\ell_i(\delta)<\mathrm{SL}(\beta),\ \mathrm{SL}(\alpha+\beta)\,.$$

It thus remains to prove that $\mathrm{SL}(\alpha+\beta)<\mathrm{SL}(\beta)$. This is obvious unless $k=1$ and $2\nmid r$, as $\mathrm{SL}(\alpha+\beta)$ contains more copies of $\ell_i(\delta)$'s in the beginning than $\mathrm{SL}(\beta)$, due to (4.62) and $\lceil\frac{k+r}{2}\rceil>\lceil\frac{r}{2}\rceil$.

Meanwhile, for $k = 1$ and $2 \nmid r$ we have

$$\mathrm{SL}(\alpha+\beta) = \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{2}}\, \mathrm{i}\, \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{2}}\, \overline{\mathrm{i}-1} \ldots a < \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r+1}{2}}\, \overline{\mathrm{i}-1} \ldots \overline{b+1}\, \underbrace{\ell_{\mathrm{i}}(\delta)}_{\frac{r-1}{2}}\, \mathrm{i} = \mathrm{SL}(\beta)\,.$$

∘ Case 2: $k = 0, r > 0$. We have $\mathrm{SL}(\alpha_{(b+1)\to \mathrm{i}}+r\delta)\mathrm{SL}(\alpha_{a\to b}) \le \mathrm{SL}(\alpha_{a\to \mathrm{i}}+r\delta)$, due to Proposition 3.4. Therefore: $\mathrm{SL}(\beta) < \mathrm{SL}(\beta)\mathrm{SL}(\alpha) \le \mathrm{SL}(\alpha_{a\to \mathrm{i}}+r\delta) = \mathrm{SL}(\alpha+\beta)$. On the other hand, $\mathrm{SL}(\alpha)$ starts with $\mathbf{min}\{a,\ldots,b\}$, which is $> 1 =$ the first letter of $\mathrm{SL}(\alpha+\beta)$. Hence, $\mathrm{SL}(\beta) < \mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\alpha)$.

∘ Case 3: $r{=}0, k{>}0$. In this case, we have $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha{+}\beta) < \mathrm{SL}(\beta)$, due to $\ell_{b+1}(\delta) < \ell_{\mathrm{i}}(\delta)$ (by Lemma 4.5) and $1 < \mathrm{i}$.

∘ Case 4: $k = r = 0$. In this case, the claim follows from Proposition 2.20 again. In fact, we get $\mathrm{SL}(\alpha) > \mathrm{SL}(\alpha+\beta) > \mathrm{SL}(\beta)$ since $\mathrm{SL}(\alpha) > \mathrm{SL}(\beta)$ (as $\mathrm{i} < a,\ldots,\mathrm{i}-1$).

• $\alpha = \alpha_{a\to b} + k\delta$, $\beta = \alpha_{(b+1)\to c} + r\delta$ for $1 \prec a \preceq b \prec \mathrm{i}-1$ and $\mathrm{i} \prec c \preceq 0$.

The proof is absolutely analogous to the previous case, but we should now look at $r \bmod 3$ (rather than $r \bmod 2$) and use formula (4.60) instead of (4.62).

• $\alpha = \alpha_{a\to(\mathrm{i}-1)} + k\delta$, $\beta = \alpha_{\mathrm{i}\to b} + r\delta$ for $1 \prec a \prec \mathrm{i} \prec b \preceq 0$.

∘ Case 1: $k, r > 0$. Let us compare the multiplicity of the word $\ell_{\mathrm{i}}(\delta)$ in the beginning of our words: it is k for $\mathrm{SL}(\alpha)$, $\lceil \frac{r}{2} \rceil$ for $\mathrm{SL}(\beta)$, and $\lceil \frac{k+r}{3} \rceil$ for $\mathrm{SL}(\alpha+\beta)$. If $r = 2k+3$ or $r > 2k+4$, then $k < \lceil \frac{k+r}{3} \rceil < \lceil \frac{r}{2} \rceil$ (as $\lceil \frac{k+r}{3} \rceil \le \frac{k+r+2}{3} < \frac{r}{2} \le \lceil \frac{r}{2} \rceil$ for $r > 2k+4$), and so $\mathrm{SL}(\beta) < \mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\alpha)$. If $r < 2k-3$, then likewise $k > \lceil \frac{k+r}{3} \rceil > \lceil \frac{r}{2} \rceil$ (as $\lceil \frac{k+r}{3} \rceil \ge \frac{k+r}{3} > \frac{r+1}{2} \ge \lceil \frac{r}{2} \rceil$), and so $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$. Thus, it remains to consider $r \in \{2k-3, 2k-2, 2k-1, 2k, 2k+1, 2k+2, 2k+4\}$. Let us illustrate the argument for $r = 2k-2$, while the other six cases are treated completely analogously. For $r = 2k-2$, $\lceil \frac{r}{2} \rceil < k = \lceil \frac{k+r}{3} \rceil$, and so it suffices to prove that $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta)$. Comparing formulas (4.58, 4.60), we see that either $\mathrm{SL}(\alpha)$ is a proper prefix of $\mathrm{SL}(\alpha+\beta)$ if $\overline{\mathrm{i}-1} > \overline{\mathrm{i}+1}$, or its first letter after k copies of $\ell_{\mathrm{i}}(\delta)$ is smaller than that of $\mathrm{SL}(\alpha+\beta)$ if $\overline{\mathrm{i}-1} < \overline{\mathrm{i}+1}$. Thus $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta)$.

∘ Case 2: $k = 0, r > 0$. Comparing the first letters, we get $\mathrm{SL}(\alpha) > \mathrm{SL}(\alpha+\beta)$. It thus remains to prove $\mathrm{SL}(\alpha+\beta) > \mathrm{SL}(\beta)$. For $r = 3$ or $r > 4$, this follows from $\lceil \frac{r}{2} \rceil > \lceil \frac{r}{3} \rceil$. The cases $r \in \{1,2,4\}$ are treated similarly to $r = 2k-2$ in Case 1.

∘ Case 3: $k > 0, r = 0$. Comparing the first letters, we get $\mathrm{SL}(\beta) > \mathrm{SL}(\alpha+\beta)$, while $\mathrm{SL}(\alpha+\beta) > \mathrm{SL}(\alpha)$ is verified alike $\mathrm{SL}(\alpha+\beta) > \mathrm{SL}(\beta)$ in Case 2.

∘ Case 4: $k = r = 0$. In this case, the claim follows from Proposition 2.20 again. In fact, we get $\mathrm{SL}(\alpha) > \mathrm{SL}(\alpha+\beta) > \mathrm{SL}(\beta)$ since $\mathrm{SL}(\alpha) > \mathrm{SL}(\beta)$ (as $\mathrm{i} < a,\ldots,\mathrm{i}-1$).

The next four cases are absolutely similar to the previous four:

• $\alpha = \alpha_{\mathrm{i}} + k\delta$, $\beta = \alpha_{\overline{\mathrm{i}+1}\to b} + r\delta$ for $\mathrm{i} \prec b \preceq 0$.

• $\alpha = \alpha_{\mathrm{i}\to b} + k\delta$, $\beta = \alpha_{\overline{b+1}\to c} + r\delta$ for $\mathrm{i} \prec b \prec c \preceq 0$.

• $\alpha = \alpha_{a\to b} + k\delta$, $\beta = \alpha_{\overline{b+1}\to c} + r\delta$ for $1 \prec a \prec \mathrm{i} \prec b \prec c \preceq 0$.

• $\alpha = \alpha_{a\to \mathrm{i}} + k\delta$, $\beta = \alpha_{\overline{\mathrm{i}+1}\to b} + r\delta$ for $1 \prec a \prec \mathrm{i} \prec b \preceq 0$.

Finally, let us treat the remaining three cases that utilize (4.64) and its proof.

• $\alpha = (\alpha_{a\to b} + k\delta)$, $\beta = (\alpha_{\overline{b+1}\to c} + r\delta)$ for $1 \in [a;b]$ and $1 \notin [\overline{b+1};c]$.

∘ Case 1: $c \in [\overline{b+1}; \overline{a-2}]$.

If $k > 0, r > 0$, then $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha)\mathrm{SL}(\beta) \le \mathrm{SL}(\alpha+\beta)$ with the second inequality proved in case 1) of our proof of (4.64). Hence, it remains to show that $\mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$. By Corollary 4.13, $\mathrm{SL}(\alpha+\beta)$ starts with $\mathbf{min}\{\mathrm{SL}(\alpha_{a\to\overline{a-2}})\,1, \mathrm{SL}(\alpha_{\overline{c+2}\to c})\,1\} < \mathrm{SL}(\alpha_{\overline{c+2}\to c})\overline{c+1} = \ell_{\overline{c+1}}(\delta)$. On the other hand, $\mathrm{SL}(\beta)$ starts with $\ell_{\mathrm{i}}(\delta) \ge \ell_{\overline{c+1}}(\delta)$ if $\mathrm{i} \in [(b+1) \to c)$, with $\ell_{\overline{c+1}}(\delta)$ if $1 \prec b+1 \preceq c \prec \mathrm{i}$, with $\ell_b(\delta) > \ell_{\overline{c+1}}(\delta)$ for $\mathrm{i} \prec b+1 \preceq c$ (by Lemma 4.5). This completes the proof of $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$ for $k, r > 0$. The inequalities are similar when $k \ne r = 0$ or $r \ne k = 0$. Finally, for $k = r = 0$ the claim follows from Proposition 2.20. In fact, since 1 is the minimal element of $\widehat{I}$, we get $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$.

∘ Case 2: $c \in [a;0]$.

Note that $\mathrm{SL}(\alpha_{\overline{b+1}\to c}{+}r\delta) > \mathrm{SL}(\alpha_{\overline{b+1}\to c}{+}(r{+}k{+}1)\delta)$ by Remark 5.5. We also have $\mathrm{SL}(\alpha_{\overline{b+1}\to c}{+}(r{+}k{+}1)\delta) > \mathrm{SL}(\alpha_{a\to c} + (r+k+1)\delta)$, due to an already established pre-convexity for roots $(\alpha_{a\to c} + (r+k+1)\delta) + \alpha_{\overline{b+1}\to\overline{a-1}} = \alpha_{\overline{b+1}\to c} + (r+k+1)\delta$. Combining the two inequalities above, we obtain $\mathrm{SL}(\alpha+\beta) < \mathrm{SL}(\beta)$. Evoking Corollary 4.13, we see that $\mathrm{SL}(\alpha)$ starts with $\ell_1 1 \le \mathrm{SL}(\alpha_{\overline{a-2}\to a})1 < \ell_{\overline{a-1}}(\delta)$, so that $\mathrm{SL}(\alpha) < \ell_{\overline{a-1}}(\delta)$. On the other hand, $\mathrm{SL}(\alpha+\beta)$ is lexicographically larger than $\ell_{\overline{a-1}}(\delta)$, due to explicit formulas of Theorem 4.7 and Lemma 4.5. Combining these inequalities, we obtain $\mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\beta)$.

• $\alpha = (\alpha_{a\to b} + k\delta)$, $\beta = (\alpha_{\overline{b+1}\to c} + r\delta)$ for $1 \notin [a;b]$ and $1 \in [\overline{b+1};c]$.

The proof in this case is completely analogous to the previous one.

• $\alpha = \alpha_{a\to b} + k\delta$, $\beta = \alpha_{\overline{b+1}\to c} + r\delta$ for $1 \in [a;b]$ and $1 \in [\overline{b+1};c]$.

According to (4.81), we have $\mathrm{SL}(\alpha_{a\to b}+k\delta) \geq \mathrm{SL}(\alpha_{a\to b})$ and $\mathrm{SL}(\alpha_{\overline{b+1}\to c}+r\delta) \geq \mathrm{SL}(\alpha_{\overline{b+1}\to c})$ for $k, r \geq 0$. Thus, $\mathrm{SL}(\alpha_{a\to b} + k\delta) \geq \mathrm{SL}(\alpha_{a\to \overline{c+1}})$ and $\mathrm{SL}(\alpha_{\overline{b+1}\to c} + r\delta) \geq \mathrm{SL}(\alpha_{\overline{a-1}\to c})$ by Lemma 4.5 as $1 \in [a; \overline{c+1}] \subset [a;b]$ and $1 \in [\overline{a-1}; c] \subseteq [\overline{b+1}; c]$. Evoking the proof of Lemma 4.11, see (4.82), we conclude that one of the words $\mathrm{SL}(\alpha_{\overline{a-1}\to c})$ and $\mathrm{SL}(\alpha_{a\to\overline{c+1}})$ is $> \mathrm{SL}(\alpha_{a\to c} + (k + r + 1)\delta)$. This implies that $\max\{\mathrm{SL}(\alpha), \mathrm{SL}(\beta)\} > \mathrm{SL}(\alpha + \beta)$. The other inequality is obvious: $\min\{\mathrm{SL}(\alpha), \mathrm{SL}(\beta)\} < \mathrm{SL}(\alpha + \beta)$, cf. our treatment of case 2) in the proof of (4.64). This competes the proof for any $k, r \geq 0$. ■

Funding

This work was partially supported by the NSF [DMS-2037602 and DMS-2302661 to A.T.].

Acknowledgments

This paper is written as a part of the project under the Yulia's Dream initiative, a subdivision of the MIT PRIMES program aimed at Ukrainian high-school students. We are grateful to the referees for their useful suggestions that improved the exposition. A.T. is deeply indebted to Andrei Neguţ for sharing his invaluable insights, teaching the Lyndon word's theory, and stimulating discussions through the entire project. A.T. is grateful to IHES for the hospitality and wonderful working conditions in the Spring 2023, when the final version of this note was prepared.

References

1. Beck, J. "Convex bases of PBW type for quantum affine algebras." *Commun. Math. Phys.* **165**, no. 1 (1994): 193–9. https://doi.org/10.1007/BF02099742.
2. Damiani, I. "A basis of type Poincare–Birkhoff–Witt for the quantum algebra of $\widehat{\mathfrak{sl}}(2)$." *J. Algebra* **161**, no. 2 (1993): 291–310.
3. Green, J. "Quantum groups, Hall algebras and quantized shuffles." *Finite Reductive Groups (Luminy 1994)*. 273–90. Birkhäuser Prog. Math. 141.
4. Khoroshkin, S. and V. Tolstoy. "The universal R-matrix for quantum untwisted affine Lie algebras." *Funct. Anal. Appl.* **26**, no. 1 (1992): 69–71.
5. Lalonde, P. and A. Ram. "Standard Lyndon bases of Lie algebras and enveloping algebras." *Trans. Amer. Math. Soc.* **347**, no. 5 (1995): 1821–30. https://doi.org/10.1090/S0002-9947-1995-1273505-4.
6. Leclerc, B. "Dual canonical bases, quantum shuffles and q-characters." *Math. Z.* **246**, no. 4 (2004): 691–732. https://doi.org/10.1007/s00209-003-0609-9.
7. Lothaire, M. *Combinatorics of Words*. Cambridge: Cambridge University Press, 1997.
8. Neguţ, A. and A. Tsymbaliuk. "Quantum loop groups and shuffle algebras via Lyndon words." *Adv. Math.* **439** (2024): Paper No. 109482.
9. Papi, P. "A characterization of a special ordering in a root system." *Proc. Amer. Math. Soc.* **120**, no. 3 (1994): 661–5. https://doi.org/10.1090/S0002-9939-1994-1169886-7.
10. Rosso, M. "Quantum groups and quantum shuffles." *Invent. Math.* **133**, no. 2 (1998): 399–416. https://doi.org/10.1007/s002220050249.
11. Rosso, M. "Lyndon bases and the multiplicative formula for R-matrices." (2002): Preprint.
12. Schauenburg, P. "A characterization of the Borel-like subalgebras of quantum enveloping algebras." *Comm. Algebra* **24**, no. 9 (1996): 2811–23. https://doi.org/10.1080/00927879608825714.