ORTHOGONAL BASES FOR TWO-PARAMETER QUANTUM GROUPS

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Abstract. In this note, we construct dual PBW bases of the positive and negative subalgebras of the two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ in classical types, as used in [MT]. Following the ideas of Leclerc [L] and Clark-Hill-Wang [CHW], we introduce the two-parameter shuffle algebra and relate it to the subalgebras above. We then use the combinatorics of dominant Lyndon words to establish the main results.

Contents

1. Introduction

1.1. Summary.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Corresponding to any polarization $\Phi = \Phi^+ \sqcup (-\Phi^+)$ of the root system of $\mathfrak{g}$, there is a root space decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \text{with} \qquad \mathfrak{n}^\pm = \oplus_{\alpha\in\Phi^+}\mathbb{C}\cdot e_{\pm\alpha}. \tag{1.1}$$

The elements $e_{\pm\alpha}$ are called *root vectors*. This induces a decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$, and the ordered products in $\{e_{\pm\alpha}\}_{\alpha\in\Phi^+}$ form a basis of $U(\mathfrak{n}^\pm)$ for any total order on Φ^+. In fact, the root vectors can be normalized so that

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha \in (\mathbb{Z}\setminus\{0\})\cdot e_{\alpha+\beta} \qquad \text{for all} \quad \alpha,\beta\in\Phi^+ \ \text{ such that } \ \alpha+\beta\in\Phi^+.$$

For each such $\mathfrak{g}$, Drinfeld and Jimbo simultaneously defined the quantum group $U_q(\mathfrak{g})$, which is a quantization of the universal enveloping algebra of $\mathfrak{g}$. As with $U(\mathfrak{g})$, the quantum groups possess a triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-)\otimes U_q(\mathfrak{h})\otimes U_q(\mathfrak{n}^+)$. Furthermore, $U_q(\mathfrak{n}^\pm)$ admit PBW-type bases

$$U_q(\mathfrak{n}^\pm) \ = \bigoplus_{\gamma_1\geq\cdots\geq\gamma_k\in\Phi^+} \mathbb{C}(q)\cdot e_{\pm\gamma_1}\dots e_{\pm\gamma_k}$$

formed by the ordered products of *q-deformed root vectors* $e_{\pm\alpha}\in U_q(\mathfrak{n}^\pm)$. The latter are defined via Lusztig's braid group action, which requires one to choose a reduced decomposition of the longest element w_0 in the Weyl group of $\mathfrak{g}$, and the order $\geq$ on Φ^+ used above is induced by this reduced decomposition.

However, there is also a purely combinatorial approach to the construction of PBW-type bases of $U_q(\mathfrak{n}^\pm)$ that goes back to the works of Kharchev, Leclerc, and Rosso. To this end, recall Lalonde-Ram's bijection [LR]:

$$\ell\colon \Phi^+ \xrightarrow{\sim} \Big\{\text{standard Lyndon words in } I\Big\}. \tag{1.2}$$

Here, the notion of standard Lyndon words intrinsically depends on a fixed total order of the indexing set I of simple roots of $\mathfrak{g}$. Furthermore, this bijection ℓ gives rise to a total order on Φ^+ via:

$$\alpha<\beta \quad \Longleftrightarrow \quad \ell(\alpha)<\ell(\beta) \text{ lexicographically.}$$

It was shown in [L, Proposition 26] that this order is *convex* and thus corresponds to a reduced decomposition of w_0, which allows one to define $e_{\pm\alpha}$ as iterated q-commutators, eliminating Lusztig's braid group action.

Although theory of multiparameter quantum groups goes back to the early 1990s (see e.g. [AST, Re, T]), much of the current interest in the subject stems from the papers [BW1, BW2, BW3], which study the two-parameter quantum group $U_{r,s}(\mathfrak{gl}_n)$ and subsequently give an application to pointed finite-dimensional Hopf algebras. In [BW2], they developed the theory of finite-dimensional representations in a complete analogy with the one-parameter case, computed the two-parameter R-matrix for the first fundamental $U_{r,s}(\mathfrak{gl}_n)$-representation, and used it to establish the Schur-Weyl duality between $U_{r,s}(\mathfrak{gl}_n)$ and a two-parameter Hecke algebra. These works of Benkart and Witherspoon stimulated an increased interest in the theory of two-parameter quantum groups. In particular, the definitions of $U_{r,s}(\mathfrak{g})$ for other classical simple Lie algebras $\mathfrak{g}$ were first given in [BGH1, BGH2], where basic results on the structure and representation theory of $U_{r,s}(\mathfrak{g})$ were also established. Subsequently, these algebras have been treated case-by-case in multiple papers. For a more uniform treatment and complete references, we refer the reader to [HP].

In the companion paper [MT], we evaluated the finite and affine R-matrices for two-parameter quantum groups of classical types, associated with the first fundamental representation and the corresponding evaluation modules. We further presented a factorization of the finite R-matrices into "local factors", parametrized by the set Φ^+. The latter relied on the construction of dual PBW-type bases of the positive and negative subalgebras, which was announced in [MT, Theorem 5.12] without a proof. The main objective of the present note is to provide a proof of this result. Due to the absence of Lusztig's braid group action on $U_{r,s}(\mathfrak{g})$ (noted first in [BGH1]), we use the aforementioned combinatorial construction of orthogonal dual bases of the positive and the negative subalgebras of $U_{r,s}(\mathfrak{g})$ through dominant[1] Lyndon words, which goes back to [L, Ro] in the one-parameter setup, to [CHW] in the super case, and to [BKL] in the two-parameter A-type case.

The main results of our paper are Theorem 7.1 and Theorem 7.2, which provide PBW-type bases of the positive subalgebra $U^+_{r,s}(\mathfrak{g})$ and the negative subalgebra $U^-_{r,s}(\mathfrak{g})$, show that they are orthogonal with respect to the Hopf pairing $(\cdot,\cdot)_H$, and evaluate explicitly all the nonzero pairing constants. The latter is actually the hardest part, and it does play the key role in our factorization of R in [MT]. In particular, while the construction of such bases for type A was already presented (through a slightly different, but equivalent, perspective of Gröbner bases) in [BKL], the results on their pairing already seem to be new in A-type.

Let us briefly outline the strategy of our proof. First, we translate the problem solely into the construction of dual bases of $U^+_{r,s}(\mathfrak{g})$, endowed with a twisted coproduct and a twisted product on its tensor powers, with respect to a different symmetric form $(\cdot,\cdot)$. We then introduce a two-parameter shuffle algebra $\mathcal{F}$ and embed

[1]We use the terminology of [CHW]; they are also called *good* in [L], and coincide with *standard* from [LR].

$U^+_{r,s}(\mathfrak{g})$ into $\mathcal{F}$. Interpreting the coproduct and the pairing on the shuffle side allows us to construct dual bases. The explicit computation of the nonzero pairing constants for a special order of simple roots is then accomplished by brute force. Finally, pulling back these results to the original setup yields Theorems 7.1-7.2.

1.2. **Outline.**

The structure of the present paper is as follows:

- In Section 2, we recall two-parametric quantum groups $U_{r,s}(\mathfrak{g})$ for simple finite-dimensional Lie algebras $\mathfrak{g}$, see Definition 2.1, as well as the Hopf pairing for those, see Proposition 2.2. We also construct the analogues of the Cartan involution and the bar involution, see Proposition 2.6. Finally, we compute the graded dimensions and establish the non-degeneracy of the Hopf pairing, see Propositions 2.8 and 2.11.
- In Section 3, we study several other pairings and ultimately relate them to the Hopf pairing of Proposition 2.2, see Theorem 3.17. This is needed to replace the Hopf subalgebra $U^{\geq}_{r,s}(\mathfrak{g})$ with $U^+_{r,s}(\mathfrak{g})$ (eliminating Cartan elements) at the cost of changing the product structure on the tensor powers, and also modifying the coproduct and the pairing. The latter is essential for the combinatorial constructions in Sections 4–6.
- In Section 4, we introduce the two-parameter shuffle algebra $(\mathcal{F}, *)$ and relate it to the positive subalgebra $U^+_{r,s}(\mathfrak{g})$, see (4.1)–(4.3) and Propositions 4.4, 4.6, 4.7. We also provide a shuffle interpretation of the bar involution and twisted coproduct, see Proposition 4.8, Corollary 4.9, and Proposition 4.10.
- In Section 5, following [CHW, Sections 4-5] (which in turn is largely based on [L]), we recall the notions of dominant and Lyndon words as well as introduce the bracketing of words. We use the latter to introduce the *Lyndon basis* $\{R_w\}_{w\in\mathcal{W}^+}$ and its closely related versions $\{\widetilde{R}_w\}_{w\in\mathcal{W}^+}$, $\{\bar{R}_w\}_{w\in\mathcal{W}^+}$, all parametrized by the set $\mathcal{W}^+$ of dominant words. Our main result is Theorem 5.19, which establishes the orthogonality of the last two bases and expresses all nonzero pairings through the pairings $\{(R_\ell, \bar{R}_\ell)\}_{\ell\in\mathcal{L}^+}$, parametrized by the set $\mathcal{L}^+$ of dominant Lyndon words (which is in bijection (1.2) with the set Φ^+ of positive roots).
- In Section 6, we explicitly compute R_ℓ (which are shuffle incarnations of the quantum root vectors in $U^+_{r,s}(\mathfrak{g})$) and the corresponding pairing $(R_\ell, \bar{R}_\ell)$ for each dominant Lyndon word $\ell \in \mathcal{L}^+$, for the special order $1 < \cdots < n$ on the alphabet $I = \{1, \ldots, n\}$. We treat each type separately, similarly to [CHW, §6].
- In Section 7, we combine Theorem 3.17, Theorem 5.19, and the explicit calculations of Section 6 to prove the main results of this paper: Theorems 7.1 and 7.2. This establishes PBW-type bases of $U^+_{r,s}(\mathfrak{g})$ and $U^-_{r,s}(\mathfrak{g})$ dual with respect to the Hopf pairing of Proposition 2.2, which was announced in our earlier work [MT, Theorem 5.12] without a proof, and used there to factorize the corresponding finite R-matrices.
- In Appendix A, we evaluate $(R_\ell, \bar{R}_\ell)$ for any order on the alphabet I given that the first letter of ℓ occurs exactly once, see Theorem A.9, which is crucially based on an interesting combinatorial Lemma A.5.

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2. Notations and Definitions

In this Section, we recall the notion of two-parametric quantum groups $U_{r,s}(\mathfrak{g})$ for simple finite-dimensional Lie algebras $\mathfrak{g}$, the Hopf algebra structure and the Hopf pairing on those, and finally construct several important (anti)automorpshims. We refer the interested reader to [MT, §1.1, §2] for a full list of references.

2.1. **Two-parameter quantum groups.**

Let E be a Euclidean space with a positive-definite symmetric bilinear form $(\cdot,\cdot)$. Let $\Phi \subset E$ be an irreducible reduced root system with an ordered set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, and let $\mathfrak{g}$ be the corresponding complex simple Lie algebra. We set $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{\pm\alpha}$, where $\mathfrak{g}_\alpha$ denotes the root space of $\mathfrak{g}$ corresponding to $\alpha \in \Phi$, and Φ^+ denotes the set of positive roots of Φ, see (1.1). Let $C = (a_{ij})^n_{i,j=1}$ be the Cartan matrix of $\mathfrak{g}$, with entries given explicitly by $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, and set $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$, where $(\cdot,\cdot)$ is normalized so that the short roots have square length 2. The root and weight lattices of $\mathfrak{g}$ will be denoted by Q and P, respectively:

$$\bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i = Q \subset P = \bigoplus_{i=1}^{n} \mathbb{Z}\varpi_i \qquad \text{with} \quad (\alpha_i, \varpi_j) = d_i\delta_{ij}.$$

Having fixed above the order on the set of simple roots Π, we consider the (modified) Ringel bilinear form $\langle\cdot,\cdot\rangle$ on Q, such that (unless $\{i,j\} = \{n-1,n\}$ in type D_n) we have:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i a_{ij} & \text{if} \quad i<j \\ d_i & \text{if} \quad i=j \\ 0 & \text{if} \quad i>j \end{cases},$$

while in the remaining case of D_n-type system, we set:

$$\langle \alpha_{n-1}, \alpha_n \rangle = \langle \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n \rangle = -1, \qquad \langle \alpha_n, \alpha_{n-1} \rangle = \langle \varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n \rangle = 1.$$

We note that $(\mu,\nu) = \langle \mu,\nu\rangle + \langle \nu,\mu\rangle$ for any $\mu,\nu \in Q$.

We also need the following two-parameter analogues of q-integers, q-factorials, and q-binomial coefficients:

$$[m]_{r,s} = \frac{r^m - s^m}{r-s} = r^{m-1} + r^{m-2}s + \ldots + rs^{m-2} + s^{m-1} \qquad \text{for all} \quad m \in \mathbb{Z}_{\geq 0},$$
$$[m]_{r,s}! = [m]_{r,s}[m-1]_{r,s}\cdots[1]_{r,s} \qquad \text{for} \quad m>0, \qquad [0]_{r,s}! = 1,$$

and

$$\begin{bmatrix} m \\ k \end{bmatrix}_{r,s} = \frac{[m]_{r,s}!}{[m-k]_{r,s}![k]_{r,s}!} \qquad \text{for all} \quad 0 \leq k \leq m.$$

Finally, we also define

$$\begin{aligned} r_\gamma &= r^{(\gamma,\gamma)/2}, \qquad s_\gamma = s^{(\gamma,\gamma)/2} \qquad \text{for all} \quad \gamma \in \Phi, \\ r_i &= r_{\alpha_i} = r^{d_i}, \qquad s_i = s_{\alpha_i} = s^{d_i} \qquad \text{for all} \quad 1 \leq i \leq n. \end{aligned} \tag{2.1}$$

We now recall the definition of the *two-parameter quantum group* of $\mathfrak{g}$:

Definition 2.1. *The two-parameter quantum group $U_{r,s}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ is the associative $\mathbb{C}(r,s)$-algebra generated by $\{e_i, f_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=1}^n$ with the following defining relations (for all $1 \leq i,j \leq n$):*

$$[\omega_i, \omega_j] = [\omega_i, \omega_j'] = [\omega_i', \omega_j'] = 0, \qquad \omega_i^{\pm 1}\omega_i^{\mp 1} = 1 = (\omega_i')^{\pm 1}(\omega_i')^{\mp 1}, \tag{2.2}$$

$$\omega_i e_j = r^{\langle \alpha_j, \alpha_i \rangle} s^{-\langle \alpha_i, \alpha_j \rangle} e_j \omega_i, \qquad \omega_i f_j = r^{-\langle \alpha_j, \alpha_i \rangle} s^{\langle \alpha_i, \alpha_j \rangle} f_j \omega_i, \tag{2.3}$$

$$\omega_i' e_j = r^{-\langle \alpha_i, \alpha_j \rangle} s^{\langle \alpha_j, \alpha_i \rangle} e_j \omega_i', \qquad \omega_i' f_j = r^{\langle \alpha_i, \alpha_j \rangle} s^{-\langle \alpha_j, \alpha_i \rangle} f_j \omega_i', \tag{2.4}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}, \tag{2.5}$$

and quantum (r,s)-Serre relations

$$\begin{aligned} &\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \qquad \text{for } i \neq j, \\ &\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} f_i^k f_j f_i^{1-a_{ij}-k} = 0 \qquad \text{for } i \neq j. \end{aligned} \tag{2.6}$$

We note that the algebra $U_{r,s}(\mathfrak{g})$ admits a Q-grading, defined on the generators via:

$$\deg(e_i) = \alpha_i, \quad \deg(f_i) = -\alpha_i, \quad \deg(\omega_i) = 0, \quad \deg(\omega_i') = 0 \qquad \text{for all} \quad 1 \leq i \leq n.$$

For $\mu \in Q$, let $U_{r,s}(\mathfrak{g})_\mu$ (or simply $(U_{r,s})_\mu$) denote the degree μ component of $U_{r,s}(\mathfrak{g})$ under this Q-grading. We shall also need several subalgebras of $U_{r,s}(\mathfrak{g})$:

- the "positive" subalgebra $U^+_{r,s} = U^+_{r,s}(\mathfrak{g})$, generated by $\{e_i\}_{i=1}^n$,
- the "negative" subalgebra $U^-_{r,s} = U^-_{r,s}(\mathfrak{g})$, generated by $\{f_i\}_{i=1}^n$,
- the "Cartan" subalgebra $U^0_{r,s} = U^0_{r,s}(\mathfrak{g})$, generated by $\{\omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=1}^n$,
- the "non-negative subalgebra" $U^{\geq}_{r,s} = U^{\geq}_{r,s}(\mathfrak{g})$, generated by $\{e_i, \omega_i^{\pm 1}\}_{i=1}^n$,

- the "non-positive subalgebra" $U^{\leq}_{r,s} = U^{\leq}_{r,s}(\mathfrak{g})$, generated by $\{f_i, (\omega'_i)^{\pm 1}\}_{i=1}^n$.

Evoking (2.2), for any $\mu = \sum_{i=1}^n c_i\alpha_i \in Q$, we define $\omega_\mu, \omega'_\mu \in U^0_{r,s}(\mathfrak{g})$ via:

$$\omega_\mu = \omega_1^{c_1}\omega_2^{c_2}\cdots\omega_n^{c_n}, \qquad \omega'_\mu = (\omega'_1)^{c_1}(\omega'_2)^{c_2}\cdots(\omega'_n)^{c_n}.$$

Finally, the algebra $U_{r,s}(\mathfrak{g})$ has a Hopf algebra structure, where the coproduct Δ, counit ϵ, and antipode S are defined on the generators by the following formulas:

$$\begin{array}{lll} \Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1}\otimes\omega_i^{\pm 1} & \epsilon(\omega_i^{\pm 1}) = 1 & S(\omega_i^{\pm 1}) = \omega_i^{\mp 1} \\ \Delta((\omega'_i)^{\pm 1}) = (\omega'_i)^{\pm 1}\otimes(\omega'_i)^{\pm 1} & \epsilon((\omega'_i)^{\pm 1}) = 1 & S((\omega'_i)^{\pm 1}) = (\omega'_i)^{\mp 1} \\ \Delta(e_i) = e_i\otimes 1 + \omega_i\otimes e_i & \epsilon(e_i) = 0 & S(e_i) = -\omega_i^{-1}e_i \\ \Delta(f_i) = 1\otimes f_i + f_i\otimes\omega'_i & \epsilon(f_i) = 0 & S(f_i) = -f_i(\omega'_i)^{-1} \end{array}$$

We note that

$$\Delta(x) \in x\otimes 1 + \bigoplus_{0<\nu<\mu} U^+_{r,s}(\mathfrak{g})_{\mu-\nu}\omega_\nu \otimes U^+_{r,s}(\mathfrak{g})_\nu + \omega_\mu\otimes x, \tag{2.7}$$

$$\Delta(y) \in y\otimes\omega'_\mu + \bigoplus_{0<\nu<\mu} U^-_{r,s}(\mathfrak{g})_{-\nu}\otimes U^-_{r,s}(\mathfrak{g})_{-(\mu-\nu)}\omega'_\nu + 1\otimes y, \tag{2.8}$$

for any $x \in U^+_{r,s}(\mathfrak{g})_\mu$ and $y \in U^-_{r,s}(\mathfrak{g})_{-\mu}$. Here, we use the standard order $\leq$ on the root lattice Q:

$$\nu \leq \mu \iff \mu - \nu \in Q^+,$$

where $Q^+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ is the positive cone of the root lattice Q.

2.2. **Hopf pairing.**

In this Subsection, we recall the Hopf pairing on $U_{r,s}(\mathfrak{g})$, which allows one to realize $U_{r,s}(\mathfrak{g})$ as a Drinfeld double of its Hopf subalgebras $U^{\leq}_{r,s}(\mathfrak{g})$, $U^{\geq}_{r,s}(\mathfrak{g})$. This pairing is also crucial to the main results of this paper.

Proposition 2.2. *There exists a unique non-degenerate bilinear pairing*

$$(\cdot,\cdot)_H \colon U^{\leq}_{r,s}(\mathfrak{g}) \times U^{\geq}_{r,s}(\mathfrak{g}) \longrightarrow \mathbb{C}(r,s) \tag{2.9}$$

satisfying the following structural properties

$$(yy', x)_H = (y\otimes y', \Delta(x))_H, \qquad (y, xx')_H = (\Delta(y), x'\otimes x)_H \qquad \forall\, x, x' \in U^{\geq}_{r,s}(\mathfrak{g}),\ y, y' \in U^{\leq}_{r,s}(\mathfrak{g}),$$

where $(x'\otimes x'', y'\otimes y'')_H = (x', y')_H(x'', y'')_H$*, as well as being given on the generators by:*

$$(f_i, \omega_j)_H = 0, \qquad (\omega'_i, e_i)_H = 0, \qquad (f_i, e_j)_H = \delta_{ij}\frac{1}{s_i - r_i} \qquad \text{for all} \quad 1 \leq i, j \leq n,$$

$$(\omega'_\lambda, \omega_\mu)_H = r^{\langle\lambda,\mu\rangle}s^{-\langle\mu,\lambda\rangle} \qquad \text{for all} \quad \lambda, \mu \in Q. \tag{2.10}$$

The above pairing is clearly homogeneous with respect to the above Q-grading:

$$(y, x)_H = 0 \quad \text{for} \quad x \in U^{\geq}_{r,s}(\mathfrak{g})_\mu,\ y \in U^{\leq}_{r,s}(\mathfrak{g})_{-\nu} \quad \text{with} \quad \mu \neq \nu \in Q^+. \tag{2.11}$$

Remark 2.3. *We shall provide a careful proof of the non-degeneracy of* $(\cdot,\cdot)_H$ *in Proposition 2.11.*

Using (2.7), we may define linear maps $p_i, p'_i \colon (U^+_{r,s})_\mu \to (U^+_{r,s})_{\mu-\alpha_i}$ for any $\mu \in Q^+$ via

$$\Delta(x) = x\otimes 1 + \sum_{i=1}^n p_i(x)\omega_i\otimes e_i + \ldots + \sum_{i=1}^n e_i\omega_{\mu-\alpha_i}\otimes p'_i(x) + \omega_\mu\otimes x, \tag{2.12}$$

which satisfy (and are uniquely determined by) $p_i(1) = p'_i(1) = 0$, $p_i(e_j) = p'_i(e_j) = \delta_{ij}$, and the following analogue of the Leibniz rule:

$$\begin{aligned} p_i(xx') &= xp_i(x') + (\omega'_{\deg(x')}, \omega_i)_H \cdot p_i(x)x', \\ p'_i(xx') &= p'_i(x)x' + (\omega'_i, \omega_{\deg(x)})_H \cdot xp'_i(x'), \end{aligned} \tag{2.13}$$

for all homogeneous $x, x' \in U^+_{r,s}$. Combining (2.11) with Proposition 2.2, we obtain the following result:

Proposition 2.4. *For any $x \in U^+_{r,s}$ and $y \in U^-_{r,s}$, we have*

$$(f_i y, x)_H = \frac{1}{s_i - r_i}(y, p'_i(x))_H \qquad \text{and} \qquad (y f_i, x)_H = \frac{1}{s_i - r_i}(y, p_i(x))_H.$$

Likewise, using (2.8), we define linear maps $p_i, p'_i \colon (U^-_{r,s})_{-\mu} \to (U^-_{r,s})_{-(\mu-\alpha_i)}$ for any $\mu \in Q^+$ via

$$\Delta(y) = y \otimes \omega'_\mu + \sum_{i=1}^n p_i(y) \otimes f_i \omega'_{\mu-\alpha_i} + \ldots + \sum_{i=1}^n f_i \otimes p'_i(y)\omega'_{\alpha_i} + 1 \otimes y,$$

which satisfy (and are uniquely determined by) $p_i(1) = p'_i(1) = 0$, $p_i(f_j) = p'_i(f_j) = \delta_{ij}$, and the following analogue of the Leibniz rule:

$$\begin{aligned} p_i(yy') &= p_i(y)y' + (\omega'_{-\deg(y)}, \omega_i)_H \cdot y p_i(y'), \\ p'_i(yy') &= y p'_i(y') + (\omega'_i, \omega_{-\deg(y')})_H \cdot p'_i(y) y', \end{aligned} \tag{2.14}$$

for all homogeneous $y, y' \in U^-_{r,s}$. As above, they are related to the Hopf pairing via:

Proposition 2.5. *For any $y \in U^-_{r,s}$ and $x \in U^+_{r,s}$, we have*

$$(y, e_i x)_H = \frac{1}{s_i - r_i}(p_i(y), x)_H \qquad \text{and} \qquad (y, x e_i)_H = \frac{1}{s_i - r_i}(p'_i(y), x)_H.$$

Since we will frequently use the restriction of $(\cdot,\cdot)_H$ to the Cartan subalgebra $U^0_{r,s}(\mathfrak{g})$ throughout the paper, we shall denote it simply by $(\cdot,\cdot)$ for brevity:

$$(y, x) = (y, x)_H \qquad \text{for any} \quad y, x \in U^0_{r,s}(\mathfrak{g}). \tag{2.15}$$

Let us present explicit formulas for the latter in each of the classical types. To this end, we use the following standard embeddings of the classical-type root systems in Euclidean space:

- *A_n-type* (corresponding to $\mathfrak{g} \simeq \mathfrak{sl}_{n+1}$).
  Let $\{\varepsilon_i\}_{i=1}^{n+1}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. Then, we have
$$\begin{aligned} \Phi_{A_n} &= \big\{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i \ne j \le n+1\big\} \subset \mathbb{R}^{n+1}, \\ \Pi_{A_n} &= \big\{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\big\}_{i=1}^n. \end{aligned}$$
  We shall use the following notation for the set of positive roots Φ^+ in Φ_{A_n}:
$$\gamma_{ij} = \alpha_i + \cdots + \alpha_j \qquad \text{for} \quad 1 \le i \le j \le n. \tag{2.16}$$
- *B_n-type* (corresponding to $\mathfrak{g} \simeq \mathfrak{so}_{2n+1}$).
  Let $\{\varepsilon_i\}_{i=1}^n$ be an orthogonal basis of $\mathbb{R}^n$ with $(\varepsilon_i, \varepsilon_i) = 2$ for all i. Then, we have
$$\begin{aligned} \Phi_{B_n} &= \big\{\pm\varepsilon_i \pm \varepsilon_j \,\big|\, 1 \le i < j \le n\big\} \cup \big\{\pm\varepsilon_i \,\big|\, 1 \le i \le n\big\} \subset \mathbb{R}^n, \\ \Pi_{B_n} &= \big\{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\big\}_{i=1}^{n-1} \cup \big\{\alpha_n = \varepsilon_n\big\}. \end{aligned}$$
  We shall use the following notation for the set of positive roots Φ^+ in Φ_{B_n}:
$$\begin{aligned} \gamma_{ij} &= \alpha_i + \cdots + \alpha_j \qquad \text{for} \quad 1 \le i \le j \le n, \\ \beta_{ij} &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n \qquad \text{for} \quad 1 \le i < j \le n. \end{aligned} \tag{2.17}$$
- *C_n-type* (corresponding to $\mathfrak{g} \simeq \mathfrak{sp}_{2n}$).
  Let $\{\varepsilon_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$. Then, we have
$$\begin{aligned} \Phi_{C_n} &= \big\{\pm\varepsilon_i \pm \varepsilon_j \,\big|\, 1 \le i < j \le n\big\} \cup \big\{\pm 2\varepsilon_i \,\big|\, 1 \le i \le n\big\} \subset \mathbb{R}^n, \\ \Pi_{C_n} &= \big\{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\big\}_{i=1}^{n-1} \cup \big\{\alpha_n = 2\varepsilon_n\big\}. \end{aligned}$$
  We shall use the following notation for the set of positive roots Φ^+ in Φ_{C_n}:
$$\begin{aligned} \gamma_{ij} &= \alpha_i + \cdots + \alpha_j \qquad \text{for} \quad 1 \le i \le j \le n, \\ \beta_{ij} &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n \qquad \text{for} \quad 1 \le i \le j < n. \end{aligned} \tag{2.18}$$

- D_n-*type* (corresponding to $\mathfrak{g} \simeq \mathfrak{so}_{2n}$).
  Let $\{\varepsilon_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$. Then, we have

$$\Phi_{D_n} = \Big\{ \pm \varepsilon_i \pm \varepsilon_j \,\Big|\, 1 \le i < j \le n \Big\} \subset \mathbb{R}^n,$$
$$\Pi_{D_n} = \big\{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\big\}_{i=1}^{n-1} \cup \big\{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\big\}.$$

We shall use the following notation for the set of positive roots Φ^+ in Φ_{D_n}:

$$\begin{aligned}
&\gamma_{ij} = \alpha_i + \cdots + \alpha_j \qquad \text{for} \quad 1 \le i \le j < n,\\
&\beta_{in} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n \qquad \text{for} \quad 1 \le i < n,\\
&\beta_{i,n-1} = \alpha_i + \cdots + \alpha_n \qquad \text{for} \quad 1 \le i < n-1,\\
&\beta_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \qquad \text{for} \quad 1 \le i < j < n-1.
\end{aligned} \tag{2.19}$$

Then, we have the following explicit formulas for the pairing of Cartan elements, where $\lambda = \sum_{i=1}^n c_i\alpha_i \in Q$:

- A_n-*type*

$$(\omega'_\lambda, \omega_i) = r^{(\varepsilon_i,\lambda)} s^{(\varepsilon_{i+1},\lambda)},$$
$$(\omega'_i, \omega_\lambda) = r^{-(\varepsilon_{i+1},\lambda)} s^{-(\varepsilon_i,\lambda)}.$$

- B_n-*type*

$$(\omega'_\lambda, \omega_i) = \begin{cases} r^{(\varepsilon_i,\lambda)} s^{(\varepsilon_{i+1},\lambda)} & \text{if} \ \ 1 \le i < n \\ r^{(\varepsilon_n,\lambda)} (rs)^{-c_n} & \text{if} \ \ i = n \end{cases},$$
$$(\omega'_i, \omega_\lambda) = \begin{cases} r^{-(\varepsilon_{i+1},\lambda)} s^{-(\varepsilon_i,\lambda)} & \text{if} \ \ 1 \le i < n \\ s^{-(\varepsilon_n,\lambda)} (rs)^{c_n} & \text{if} \ \ i = n \end{cases}.$$

- C_n-*type*

$$(\omega'_\lambda, \omega_i) = \begin{cases} r^{(\varepsilon_i,\lambda)} s^{(\varepsilon_{i+1},\lambda)} & \text{if} \ \ 1 \le i < n \\ r^{2(\varepsilon_n,\lambda)} (rs)^{-2c_n} & \text{if} \ \ i = n \end{cases},$$
$$(\omega'_i, \omega_\lambda) = \begin{cases} r^{-(\varepsilon_{i+1},\lambda)} s^{-(\varepsilon_i,\lambda)} & \text{if} \ \ 1 \le i < n \\ s^{-2(\varepsilon_n,\lambda)} (rs)^{2c_n} & \text{if} \ \ i = n \end{cases}.$$

- D_n-*type*

$$(\omega'_\lambda, \omega_i) = \begin{cases} r^{(\varepsilon_i,\lambda)} s^{(\varepsilon_{i+1},\lambda)} & \text{if} \ \ 1 \le i < n \\ r^{(\varepsilon_{n-1},\lambda)} s^{-(\varepsilon_n,\lambda)} (rs)^{-2c_{n-1}} & \text{if} \ \ i = n \end{cases},$$
$$(\omega'_i, \omega_\lambda) = \begin{cases} r^{-(\varepsilon_{i+1},\lambda)} s^{-(\varepsilon_i,\lambda)} & \text{if} \ \ 1 \le i < n \\ r^{(\varepsilon_n,\lambda)} s^{-(\varepsilon_{n-1},\lambda)} (rs)^{2c_{n-1}} & \text{if} \ \ i = n \end{cases}.$$

2.3. **(Anti)automorphisms.**

Finally, we need to introduce several additional structures on $U_{r,s}(\mathfrak{g})$ that will be used later.

Proposition 2.6. *(1) There is a unique $\mathbb{C}(r,s)$-algebra anti-automorphism $\varphi\colon U_{r,s}(\mathfrak{g}) \to U_{r,s}(\mathfrak{g})$ (called the Cartan involution) satisfying*

$$\varphi(e_i) = f_i, \qquad \varphi(f_i) = e_i, \qquad \varphi(\omega_i) = \omega_i, \qquad \varphi(\omega'_i) = \omega'_i \qquad \textit{for all} \quad 1 \le i \le n.$$

(2) There is a unique $\mathbb{C}$-algebra anti-automorphism $\tau\colon U_{r,s}(\mathfrak{g}) \to U_{r,s}(\mathfrak{g})$ satisfying $\tau(r) = s^{-1}$, $\tau(s) = r^{-1}$, and

$$\tau(e_i) = e_i, \qquad \tau(f_i) = f_i, \qquad \tau(\omega_i) = (r_is_i)^{-1}\omega'_i, \qquad \tau(\omega'_i) = (r_is_i)^{-1}\omega_i \qquad \textit{for all} \quad 1 \le i \le n.$$

(3) There is a unique $\mathbb{C}$-algebra automorphism $x \mapsto \bar{x}$ of $U_{r,s}(\mathfrak{g})$ (called the bar involution) satisfying $\bar{r} = s$, $\bar{s} = r$, and

$$\bar{e}_i = e_i, \qquad \bar{f}_i = f_i, \qquad \bar{\omega}_i = \omega'_i, \qquad \bar{\omega}'_i = \omega_i \qquad \textit{for all} \quad 1 \le i \le n.$$

Proof. For each of these, we need to check that defining relations (2.2)–(2.6) are preserved. For parts (1) and (3), this is straightforward, and we leave details to the reader. For part (2), it is easy to check that (2.2)–(2.4) and (2.5) with $i \neq j$ are preserved under τ. For (2.5) with $i = j$, we have:

$$\frac{\tau(\omega_i) - \tau(\omega'_i)}{\tau(r_i) - \tau(s_i)} = \frac{(r_i s_i)^{-1}(\omega'_i - \omega_i)}{s_i^{-1} - r_i^{-1}} = \frac{\omega'_i - \omega_i}{r_i - s_i} = f_i e_i - e_i f_i = \tau(f_i)\tau(e_i) - \tau(e_i)\tau(f_i).$$

For the quantum Serre relations (2.6), we shall only carry out the verification for the e_i's, since the calculations for the f_i's is analogous. First, we note that

$$\tau([m]_{r,s}) = \frac{s^{-m} - r^{-m}}{s^{-1} - r^{-1}} = (rs)^{1-m}[m]_{r,s},$$

and therefore

$$\tau\left(\begin{bmatrix} m \\ k \end{bmatrix}_{r,s}\right) = \frac{\tau([m]_{r,s}!)}{\tau([k]_{r,s}!)\tau([m-k]_{r,s}!)} = \frac{(rs)^{-\frac{1}{2}m(m-1)}}{(rs)^{-\frac{1}{2}k(k-1)}(rs)^{-\frac{1}{2}(m-k)(m-k-1)}} \frac{[m]_{r,s}!}{[k]_{r,s}![m-k]_{r,s}!}$$
$$= (rs)^{k(k-m)} \begin{bmatrix} m \\ k \end{bmatrix}_{r,s}.$$

Hence, we have:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \tau\left(\begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i\rangle}\right) \tau(e_i)^k \tau(e_j) \tau(e_i)^{1-a_{ij}-k} =$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k (r_i s_i)^{k(k-1+a_{ij})} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{-\frac{1}{2}k(k-1)} (rs)^{-k\langle \alpha_j, \alpha_i\rangle} e_i^k e_j e_i^{1-a_{ij}-k} =$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^{1-a_{ij}-k} (r_i s_i)^{-(1-a_{ij}-k)k} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{-\frac{1}{2}(1-a_{ij}-k)(-a_{ij}-k)} (rs)^{-(1-a_{ij}-k)\langle \alpha_j, \alpha_i\rangle} e_i^{1-a_{ij}-k} e_j e_i^k.$$

Since

$$-(1-a_{ij}-k)k - \tfrac{1}{2}(1-a_{ij}-k)(-a_{ij}-k) = \tfrac{1}{2}k(k-1) - \tfrac{1}{2}a_{ij}(a_{ij}-1),$$

we thus find that applying τ to the first relation in (2.6), we get:

$$0 = \sum_{k=0}^{1-a_{ij}} (-1)^k \tau\left(\begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i\rangle}\right) \tau(e_i)^k \tau(e_j) \tau(e_i)^{1-a_{ij}-k} =$$
$$(-1)^{1-a_{ij}} (r_i s_i)^{-\frac{1}{2}a_{ij}(a_{ij}-1)} (rs)^{-(1-a_{ij})\langle \alpha_j, \alpha_i\rangle} \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{r_i,s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i\rangle} e_i^{1-a_{ij}-k} e_j e_i^k.$$

The above shows that the first relation in (2.6) is indeed preserved by τ. This completes the proof. ■

2.4. **Non-degeneracy of pairing and weight space dimensions.**

Let $\lambda = \sum_{i=1}^n l_i \varpi_i \in P \cap Q$ with all $l_i \geq 0$ be a dominant weight (we only consider such weights that lie in the root lattice just to avoid extending the base field $\mathbb{C}(r,s)$). Recall that a vector v in a $U_{r,s}(\mathfrak{g})$-module V is said to have *weight* λ if

$$\omega_i \cdot v = (\omega'_\lambda, \omega_i) v \qquad \text{and} \qquad \omega'_i \cdot v = (\omega'_i, \omega_\lambda)^{-1} v \quad \text{for all} \quad 1 \leq i \leq n.$$

We denote the subspace of all vectors of weight λ in V by V_λ.

Let $M(\lambda) = U_{r,s}(\mathfrak{g}) \otimes_{U^{\leq}_{r,s}(\mathfrak{g})} \mathbb{C}(r,s)$ be the $U_{r,s}(\mathfrak{g})$-*Verma module* with highest weight λ, where the action of $U^{\leq}_{r,s}(\mathfrak{g})$ on $\mathbb{C}(r,s)$ is defined by

$$e_i \cdot 1 = 0, \quad f_i \cdot 1 = 0, \quad \omega_i \cdot 1 = (\omega'_\lambda, \omega_i), \quad \omega'_i \cdot 1 = (\omega'_i, \omega_\lambda)^{-1} \quad \text{for all} \quad 1 \leq i \leq n.$$

Let $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$. If $v_\lambda \in M(\lambda)$ is a nonzero highest weight vector, set

$$\widetilde{L}(\lambda) = M(\lambda) \Big/ \sum_{i=1}^n U^-_{r,s} f_i^{l_i+1} v_\lambda.$$

For classical $\mathfrak{g}$, the $U_{r,s}(\mathfrak{g})$-module $\widetilde{L}(\lambda)$ is known to be finite-dimensional of highest weight λ (see [BW2, proof of Lemma 2.12] for A-type and [BGH2, Proposition 2.16] for BCD-types). Since $\widetilde{L}(\lambda)$ surjects onto $L(\lambda)$, the module $L(\lambda)$ is also finite-dimensional. Now, the argument of [MT, Proposition 3.8] can be carried out for both $\widetilde{L}(\lambda)$ and $L(\lambda)$, as they are both finite-dimensional highest weight $U_{r,s}(\mathfrak{g})$-modules of weight λ. This shows that $\widetilde{L}(\lambda)$ and $L(\lambda)$ have the same dimension, and since $L(\lambda)$ is a quotient of $\widetilde{L}(\lambda)$, we conclude that $L(\lambda) \simeq \widetilde{L}(\lambda)$. This observation allows us to prove the following result:

Proposition 2.7. *Let $\lambda = \sum_{i=1}^n l_i \varpi_i \in P \cap Q$ be a dominant weight, and let $v_\lambda \in L(\lambda)$ be a nonzero vector of weight λ. For any $\mu = \sum_{i=1}^n m_i \alpha_i \in Q^+$ satisfying $m_i \le l_i$ for all $1 \le i \le n$, the map $(U^-_{r,s})_{-\mu} \to L(\lambda)_{\lambda-\mu}$ defined by $y \mapsto y v_\lambda$ is bijective.*

Proof. Let $J^- \subset U^-_{r,s}$ be the left ideal generated by the elements $\{f_i^{l_i+1}\}_{i=1}^n$. Then the map $U^-_{r,s} \to M(\lambda)$ defined by $y \mapsto y v_\lambda$ induces a $U^-_{r,s}$-module isomorphism $U^-_{r,s}/J^- \xrightarrow{\sim} \widetilde{L}(\lambda) \simeq L(\lambda)$. But since $m_i \le l_i$ for all i, we have $J^- \cap (U^-_{r,s})_{-\mu} = 0$, so that the restriction of $U^-_{r,s} \twoheadrightarrow U^-_{r,s}/J^- \xrightarrow{\sim} L(\lambda)$ to $(U^-_{r,s})_{-\mu}$ gives rise to the claimed isomorphism $(U^-_{r,s})_{-\mu} \xrightarrow{\sim} L(\lambda)_{\lambda-\mu}$. ■

As our first application of the proposition above, let us prove the following result on the dimensions of the weight spaces $(U^\pm_{r,s})_\mu$, which will be needed later for Theorem 5.5:

Proposition 2.8. *For all $\mu \in Q^+$, we have*

$$\dim_{\mathbb{C}(r,s)} (U^+_{r,s})_\mu = \dim_{\mathbb{C}(r,s)} (U^-_{r,s})_{-\mu} = \dim_{\mathbb{C}} U(\mathfrak{n}^\pm)_{\pm\mu}.$$

Proof. Let $\mu = \sum_{i=1}^n m_i \alpha_i \in Q^+$, and let $\lambda = \sum_{i=1}^n l_i \varpi_i \in P$ be a dominant weight with $l_i \ge m_i$. Because P is contained in the $\mathbb{Q}$-span of $\alpha_1, \ldots, \alpha_n$, we may replace λ by a suitable positive integer multiple to ensure that $\lambda \in P \cap Q$. Then $\dim_{\mathbb{C}(r,s)} (U^-_{r,s})_{-\mu} = \dim_{\mathbb{C}(r,s)} L(\lambda)_{\lambda-\mu}$, due to Proposition 2.7. On the other hand, by [MT, Proposition 3.8] and the choice of λ, we have $\dim_{\mathbb{C}(r,s)} L(\lambda)_{\lambda-\mu} = \dim_{\mathbb{C}} U(\mathfrak{n}^-)_{-\mu}$, which proves the claim for $U^-_{r,s}$. The result for $U^+_{r,s}$ follows by applying the anti-automorphism φ of Proposition 2.6(1). ■

We shall now prove the non-degeneracy of the Hopf pairing $(\cdot,\cdot)_H$ introduced in Proposition 2.2. For this, we require the following result, which is another application of Proposition 2.7:

Proposition 2.9. *Let $\mu \in Q^+ \setminus \{0\}$. If $y \in (U^-_{r,s})_{-\mu}$ satisfies $e_i y - y e_i = 0$ for all $1 \le i \le n$, then $y = 0$. Similarly, if $x \in (U^+_{r,s})_\mu$ satisfies $f_i x - x f_i = 0$ for all $1 \le i \le n$, then $x = 0$.*

Proof. Given μ, choose λ as in Proposition 2.7. Then, if $y \ne 0$, we have $y v_\lambda \ne 0$, where $v_\lambda \in L(\lambda)$ is a nonzero vector of weight λ. The assumption that $e_i y - y e_i = 0$ for all i then implies that $e_i y v_\lambda = y e_i v_\lambda = 0$ for all i, and hence $y v_\lambda \in L(\lambda)$ is a highest weight vector of weight $\lambda - \mu < \lambda$. This contradicts the fact that $L(\lambda)$ is irreducible, so we must have $y = 0$. The result for $U^+_{r,s}$ is obtained by applying φ of Proposition 2.6(1). ■

We also need the following lemma:

Lemma 2.10. *(1) For all homogeneous $y \in U^-_{r,s}$ and all $1 \le i \le n$, we have*

$$e_i y - y e_i = \frac{1}{r_i - s_i}\big(\omega_i p_i(y) - p'_i(y)\omega'_i\big). \tag{2.20}$$

(2) For all homogeneous $x \in U^+_{r,s}$ and all $1 \le i \le n$, we have

$$x f_i - f_i x = \frac{1}{r_i - s_i}\big(p_i(x)\omega_i - \omega'_i p'_i(x)\big).$$

Proof. For part (1), the equality is clear when $y = 1$, and for $y = f_j$ it follows from (2.5). Since both sides of (2.20) are linear in y, it is enough to show that if (2.20) holds for y' and y'', then it also holds for $y = y' y''$.

Using the identities $y'\omega_i = (\omega'_{-\deg(y')}, \omega_i)\omega_i y'$ and $\omega'_i y'' = (\omega'_i, \omega_{-\deg(y'')})y''\omega'_i$, we obtain:

$$\begin{aligned} e_i(y'y'') - (y'y'')e_i &= (e_iy' - y'e_i)y'' + y'(e_iy'' - y''e_i) \\ &= \frac{1}{r_i - s_i}\left((\omega_i p_i(y') - p'_i(y')\omega'_i)y'' + y'(\omega_i p_i(y'') - p'_i(y'')\omega'_i)\right) \\ &= \frac{1}{r_i - s_i}\Big(\omega_i(p_i(y')y'' + (\omega'_{-\deg(y')}, \omega_i)y'p_i(y'')) - (y'p'_i(y'') + (\omega'_i, \omega_{-\deg(y'')})p'_i(y')y'')\omega'_i\Big) \\ &= \frac{1}{r_i - s_i}(\omega_i p_i(y'y'') - p'_i(y'y'')\omega'_i), \end{aligned}$$

where the last equality follows from (2.14).

As for part (2), one can either use a similar argument, or rather note that it follows by applying φ to (2.20) since $\varphi \circ p_i = p_i \circ \varphi$ (the latter can be established by comparing the first formulas of (2.13) and (2.14)). ■

Proposition 2.11. *The restriction of the Hopf pairing $(\cdot,\cdot)_H$ of Proposition 2.2 to $(U^-_{r,s})_{-\mu} \times (U^+_{r,s})_\mu$ is non-degenerate for all $\mu \in Q^+$.*

Proof. The claim is obvious for $\mu = 0$. Now, suppose that the claim holds for all $\nu \in Q^+$ with $0 \leq \nu < \mu$ and let $y \in (U^-_{r,s})_{-\mu}$ be an element such that $(y, x)_H = 0$ for all $x \in (U^+_{r,s})_\mu$. Then for all $1 \leq i \leq n$ and $x \in (U^+_{r,s})_{\mu-\alpha_i}$, we have $0 = (y, e_i x)_H = (y, xe_i)_H$, which implies $(p_i(y), x)_H = (p'_i(y), x)_H = 0$, due to Proposition 2.5. Since $p_i(y), p'_i(y) \in (U^-_{r,s})_{-(\mu-\alpha_i)}$, we must have $p_i(y) = p'_i(y) = 0$ for all $1 \leq i \leq n$ by the induction hypothesis. Then $e_i y - y e_i = 0$ for all $1 \leq i \leq n$ by Lemma 2.10(1), and therefore Proposition 2.9 implies that $y = 0$, as claimed. If $x \in (U^+_{r,s})_\mu$ satisfies $(y, x)_H = 0$ for all $y \in (U^-_{r,s})_{-\mu}$, then applying similar arguments one shows that $x = 0$. This completes the proof. ■

3. Bilinear Forms

In this Section, we discuss several other pairings and their relation to the Hopf pairing $(\cdot,\cdot)_H$ of (2.9). This allows us to translate the problem solely into the construction of dual bases of $U^+_{r,s}$, endowed with a twisted coproduct and a twisted product on its tensor powers, with respect to a different symmetric form $(\cdot,\cdot)$. This technical part is essential to the rest of the paper, as it eliminates Cartan elements from consideration.

3.1. Twisted product and compatible pairing.

Let $\mathcal{F}$ be the free associative $\mathbb{C}(r,s)$-algebra generated by the finite alphabet $I = \{1, 2, \ldots, n\}$. Let $\mathcal{W}$ be the set of words in I, i.e. the monoid generated by I. We shall often use the notation $[i_1 \ldots i_d] = i_1 i_2 \ldots i_d$ for the elements in $\mathcal{W}$, where $i_1, \ldots, i_d \in I$. The algebra $\mathcal{F}$ has a natural grading by the positive cone Q^+ of the root lattice Q, defined by $|i| = \alpha_i$. For any $a, b \in \mathbb{C}(r,s)$, we define the twisted product $\odot_{a,b}$ on $\mathcal{F}^{\otimes n}$ via

$$(x_1 \otimes \cdots \otimes x_n) \odot_{a,b} (x'_1 \otimes x'_2 \otimes \cdots \otimes x'_n) = a^{-\sum_{1\leq i<j\leq n}\langle |x_j|, |x'_i|\rangle} b^{\sum_{1\leq i<j\leq n}\langle |x'_i|, |x_j|\rangle} x_1 x'_1 \otimes \cdots \otimes x_n x'_n \tag{3.1}$$

for all homogeneous $x_i, x'_i \in \mathcal{F}$. In particular, evoking (2.10), for all homogeneous $x, x', y, y' \in \mathcal{F}$, we have:

$$(x \otimes y) \odot_{r,s} (x' \otimes y') = (\omega'_{|y|}, \omega_{|x'|})^{-1} xx' \otimes yy',$$
$$(x \otimes y) \odot_{s^{-1},r^{-1}} (x' \otimes y') = (\omega'_{|x'|}, \omega_{|y|})^{-1} xx' \otimes yy',$$

cf. (2.15). For a fixed choice of a, b as above, we define the algebra homomorphism

$$\Delta_{a,b} \colon \mathcal{F} \longrightarrow (\mathcal{F} \otimes \mathcal{F}, \odot_{a,b}) \qquad \text{via} \qquad \Delta(i) = i \otimes \emptyset + \emptyset \otimes i. \tag{3.2}$$

For any element $x \in \mathcal{F}$, we shall use the notation

$$\Delta_{a,b}(x) = \sum_{(x)} x_{1;a,b} \otimes x_{2;a,b}.$$

If x is homogeneous, then we have $|x_{1;a,b}| + |x_{2;a,b}| = |x|$, by the definition of $\Delta_{a,b}$. If the values of a and b are clear from context, we will omit the subscript a, b, and write simply $\Delta_{a,b}(x) = \sum_{(x)} x_1 \otimes x_2$ instead. The following result shows that (3.2) is in fact coassociative:

Proposition 3.1. *For any $a, b \in \mathbb{C}(r,s)$, we have $(\Delta_{a,b} \otimes 1) \circ \Delta_{a,b} = (1 \otimes \Delta_{a,b}) \circ \Delta_{a,b}$.*

Proof. This is clearly true on the generators, so it suffices to show that $\Delta_{a,b}\otimes 1$ and $1\otimes\Delta_{a,b}$ are both algebra homomorphisms $(\mathcal{F}^{\otimes 2},\odot_{a,b})\to(\mathcal{F}^{\otimes 3},\odot_{a,b})$.

To this end, let $x,x',y,y'\in\mathcal{F}$ be any homogeneous elements. Then, we have:

$$\begin{aligned}(\Delta_{a,b}\otimes 1)((x\otimes y)\odot_{a,b}(x'\otimes y')) &= a^{-\langle|y|,|x'|\rangle}b^{\langle|x'|,|y|\rangle}(\Delta_{a,b}(x)\odot_{a,b}\Delta_{a,b}(x'))\otimes yy' \\ &= a^{-\langle|y|,|x'|\rangle}b^{\langle|x'|,|y|\rangle}\sum_{(x)(x')}a^{-\langle|x_2|,|x_1'|\rangle}b^{\langle|x_1'|,|x_2|\rangle}x_1x_1'\otimes x_2x_2'\otimes yy'\end{aligned}$$

as well as

$$\begin{aligned}(\Delta_{a,b}(x)\otimes y)\odot_{a,b}(\Delta_{a,b}(x')\otimes y') &= \sum_{(x)(x')}(x_1\otimes x_2\otimes y)\odot_{a,b}(x_1'\otimes x_2'\otimes y') \\ &= \sum_{(x)(x')}a^{-\langle|x_2|,|x_1'|\rangle-\langle|y|,|x_1'|+|x_2'|\rangle}b^{\langle|x_1'|,|x_2|\rangle+\langle|x_1'|+|x_2'|,|y|\rangle}x_1x_1'\otimes x_2x_2'\otimes yy' \\ &= a^{-\langle|y|,|x'|\rangle}b^{\langle|x'|,|y|\rangle}\sum_{(x)(x')}a^{-\langle|x_2|,|x_1'|\rangle}b^{\langle|x_1'|,|x_2|\rangle}x_1x_1'\otimes x_2x_2'\otimes yy',\end{aligned}$$

where we used $|x_1'|+|x_2'|=|x'|$ in the last equality. Comparison of the above two formulas shows that $\Delta_{a,b}\otimes 1$ is an algebra homomorphism. The proof that $1\otimes\Delta_{a,b}$ is an algebra homomorphism is similar. ■

Consequently, we can define a product, also denoted by $\odot_{a,b}$, on $\mathcal{F}^*$ via

$$(f\odot_{a,b}g)(x)=(f\otimes g)(\Delta_{a,b}(x)).$$

The coassociativity property of Proposition 3.1 guarantees that this product is associative. The identity element on $\mathcal{F}^*$ is the map $\varepsilon\colon\mathcal{F}\to\mathbb{C}(r,s)$ given by $\varepsilon(\emptyset)=1$ and $\varepsilon(i)=0$ for $i\in I$. We can now use this to prove the following theorem:

Theorem 3.2. *There is a unique bilinear form* $\{\cdot,\cdot\}\colon\mathcal{F}\times\mathcal{F}\to\mathbb{C}(r,s)$ *satisfying*

(1) $\{1,1\}=1$,
(2) $\{i,j\}=\delta_{ij}$ *for all* $i,j\in I$,
(3) $\{xy,z\}=\{x\otimes y,\Delta_{s^{-1},r^{-1}}(z)\}$ *for all* $x,y,z\in\mathcal{F}$,
(4) $\{x,yz\}=\{\Delta_{r,s}(x),y\otimes z\}$ *for all* $x,y,z\in\mathcal{F}$,

where $\{x'\otimes x'',y'\otimes y''\}=\{x',y'\}\{x'',y''\}$ *for any* $x',x'',y',y''\in\mathcal{F}$.

Proof. For each $i\in I$, define a linear map $i^*\colon\mathcal{F}\to\mathbb{C}(r,s)$ given by $i^*(i)=1$ and $i^*(x)=0$ if x is any word other than i. We now define an algebra homomorphism $\psi\colon\mathcal{F}\to(\mathcal{F}^*,\odot_{s^{-1},r^{-1}})$ via $\psi(i)=i^*$. Then we may define a bilinear form on $\mathcal{F}$ by

$$\{x,y\}=\psi(x)(y)\qquad\text{for all}\quad x,y\in\mathcal{F}.$$

First, it is easy to show that if $|x|=\mu$, then $\psi(x)\in\mathcal{F}_\mu^*$, i.e. $\psi(x)(y)=0$ unless $y\in\mathcal{F}_\mu$. This translates to $\{x,y\}=0$ unless $|x|=|y|$. Moreover, this bilinear form clearly satisfies (1) and (2), while property (3) follows from the fact that ψ is an algebra homomorphism. Indeed, for any $x,y,z\in\mathcal{F}$, we have

$$\{xy,z\}=\psi(xy)(z)=(\psi(x)\odot_{s^{-1},r^{-1}}\psi(y))(z)=(\psi(x)\otimes\psi(y))(\Delta_{s^{-1},r^{-1}}(z))=\{x\otimes y,\Delta_{s^{-1},r^{-1}}(z)\}.$$

It is sufficient to prove (4) for any word $x\in\mathcal{W}$, and we shall do so by induction on $\mathrm{ht}(|x|)$. First, we note that it is clearly true whenever $|x|$ has height 0 or 1, for any $y,z\in\mathcal{F}$. If $\mathrm{ht}(|x|)>1$, we can write $x=x'x''$ for $\mathrm{ht}(|x'|),\mathrm{ht}(|x''|)<\mathrm{ht}(|x|)$. Then, by the induction assumption, we obtain:

$$\begin{aligned}&\{x,yz\}\\ &=\{x'\otimes x'',\Delta_{s^{-1},r^{-1}}(yz)\}\\ &=\sum_{(y)(z)}(\omega'_{|z_{1;s^{-1},r^{-1}}|},\omega_{|y_{2;s^{-1},r^{-1}}|})^{-1}\{x',y_{1;s^{-1},r^{-1}}z_{1;s^{-1},r^{-1}}\}\{x'',y_{2;s^{-1},r^{-1}},z_{2;s^{-1},r^{-1}}\}\\ &=\sum_{(y)(z)}(\omega'_{|z_{1;s^{-1},r^{-1}}|},\omega_{|y_{2;s^{-1},r^{-1}}|})^{-1}\{\Delta_{r,s}(x'),y_{1;s^{-1},r^{-1}}\otimes z_{1;s^{-1},r^{-1}}\}\{\Delta_{r,s}(x''),y_{2;s^{-1},r^{-1}}\otimes z_{2;s^{-1},r^{-1}}\}\\ &=\sum_{\substack{(y)(z)\\(x')(x'')}}(\omega'_{|z_{1;s^{-1},r^{-1}}|},\omega_{|y_{2;s^{-1},r^{-1}}|})^{-1}\{x'_{1;r,s},y_{1;s^{-1},r^{-1}}\}\{x'_{2;r,s},z_{1;s^{-1},r^{-1}}\}\{x''_{1;r,s},y_{2;s^{-1},r^{-1}}\}\{x''_{2;r,s},z_{2;s^{-1},r^{-1}}\}\end{aligned}$$

while

$$\begin{aligned}\{\Delta_{r,s}(x), y\otimes z\} &= \{\Delta_{r,s}(x'x''), y\otimes z\}\\ &= \sum_{(x')(x'')} (\omega'_{|x'_{2;r,s}|}, \omega_{|x''_{1;r,s}|})^{-1}\{x'_{1;r,s}x''_{1;r,s}, y\}\{x'_{2,r,s}x''_{2;r,s}, z\}\\ &= \sum_{\substack{(y)(z)\\(x')(x'')}} (\omega'_{|x'_{2;r,s}|}, \omega_{|x''_{1;r,s}|})^{-1}\{x'_{1;r,s}, y_{1;s^{-1},r^{-1}}\}\{x'_{2;r,s}, z_{1;s^{-1},r^{-1}}\}\{x''_{1;r,s}, y_{2;s^{-1},r^{-1}}\}\{x''_{2;r,s}, z_{2;s^{-1},r^{-1}}\}.\end{aligned}$$

But since $\{u,v\}=0$ unless $|u|=|v|$, it follows that $(\omega'_{|x'_{2;r,s}|}, \omega_{|x''_{1;r,s}|})^{-1} = (\omega'_{|z_{1;s^{-1},r^{-1}}|}, \omega_{|y_{2;s^{-1},r^{-1}}|})^{-1}$ for all nonzero terms in the last sum above. Thus, we have $\{x, yz\} = \{\Delta_{r,s}(x), y\otimes z\}$, as desired. ■

Let us now introduce two linear maps ∂_i, ∂'_i that we will use frequently throughout the remainder of this Section. For any $x\in\mathcal{F}$, we have

$$\Delta_{r,s}(x) = x\otimes 1 + \sum_{i=1}^n \partial_i(x)\otimes i + \ldots + \sum_{i=1}^n i\otimes\partial'_i(x) + 1\otimes x. \tag{3.3}$$

The resulting linear maps $\partial_i, \partial'_i\colon \mathcal{F}\to\mathcal{F}$ satisfy (and are uniquely determined by) $\partial_i(1)=\partial'_i(1)=0$, $\partial_i(j)=\partial'_i(j)=\delta_{ij}$, and the following analogue of the Leibniz rule:

$$\begin{aligned}\partial_i(xx') &= x\partial_i(x') + (\omega'_i, \omega_{|x'|})^{-1}\partial_i(x)x',\\ \partial'_i(xx') &= \partial'_i(x)x' + (\omega'_{|x|}, \omega_i)^{-1}x\partial'_i(x'),\end{aligned} \tag{3.4}$$

for all homogeneous elements $x, x'\in\mathcal{F}$.

Lemma 3.3. *The bilinear form of Theorem 3.2 has the following properties:*

$$\{x, iy\} = \{\partial'_i(x), y\} \qquad\text{and}\qquad \{x, yi\} = \{\partial_i(x), y\} \qquad \text{for any } x,y\in\mathcal{F},\ i\in I.$$

Proof. This follows immediately by combining (3.3) with (2.11) and Theorem 3.2(2,4). ■

Likewise, we introduce linear maps $\tilde{\partial}_i, \tilde{\partial}'_i$ on $\mathcal{F}$ via:

$$\Delta_{s^{-1},r^{-1}}(x) = x\otimes 1 + \sum_{i=1}^n \tilde{\partial}_i(x)\otimes i + \ldots + \sum_{i=1}^n i\otimes\tilde{\partial}'_i(x) + 1\otimes x. \tag{3.5}$$

Similarly to above, the resulting linear maps satisfy (and are uniquely determined by) $\tilde{\partial}_i(1)=\tilde{\partial}'_i(1)=0$, $\tilde{\partial}_i(j)=\tilde{\partial}'_i(j)=\delta_{ij}$, and the following analogue of the Leibniz rule:

$$\begin{aligned}\tilde{\partial}_i(xx') &= x\tilde{\partial}_i(x') + (\omega'_{|x'|}, \omega_i)^{-1}\tilde{\partial}_i(x)x',\\ \tilde{\partial}'_i(xx') &= \tilde{\partial}'_i(x)x' + (\omega'_i, \omega_{|x|})^{-1}x\tilde{\partial}'_i(x'),\end{aligned}$$

for all homogeneous elements $x, x'\in\mathcal{F}$. The following is analogous to Lemma 3.3:

Lemma 3.4. *The bilinear form of Theorem 3.2 has the following properties:*

$$\{ix, y\} = \{x, \tilde{\partial}'_i(y)\} \qquad\text{and}\qquad \{xi, y\} = \{x, \tilde{\partial}_i(y)\} \qquad \text{for any } x,y\in\mathcal{F},\ i\in I.$$

3.2. **Reduction modulo radicals.**

Let $\mathcal{I}$ be the left radical of the form $\{\cdot,\cdot\}$ introduced in Theorem 3.2:

$$\mathcal{I} = \big\{x\in\mathcal{F}\,\big|\,\{x,y\}=0 \text{ for any } y\in\mathcal{F}\big\}. \tag{3.6}$$

We define $\bar{\mathcal{F}} = \mathcal{F}/\mathcal{I}$, and denote the images of i in $\mathcal{F}/\mathcal{I}$ by e_i (we shall see later in (3.13) that $\mathcal{F}/\mathcal{I}\simeq U^+_{r,s}$, justifying this notation). It is easy to see that $\mathcal{I}$ is a homogeneous ideal, so that $\bar{\mathcal{F}}$ inherits the Q^+-grading from $\mathcal{F}$. Moreover, $\mathcal{I}$ is actually a two-sided ideal of $\mathcal{F}$, according to Theorem 3.2(3), so that $\bar{\mathcal{F}}$ is an algebra. Let us now define the divided powers

$$e_i^{(k)} = \frac{e_i^k}{[k]_{r_i,s_i}!} \qquad \text{for all} \quad k\in\mathbb{Z}_{\geq 0},\ i\in I.$$

Our next goal is to prove the following theorem:

Theorem 3.5. *For any $i \neq j$, the following relation holds in the algebra $\bar{\mathcal{F}}$:*

$$\sum_{k=0}^{1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0.$$

Before proceeding to the proof of this result, we observe the following easy consequence of Lemma 3.3:

Lemma 3.6. *The ideal $\mathcal{I}$ of (3.6) is stable under the maps ∂_i, ∂'_i of (3.3) for all $i \in I$.*

Therefore, we obtain the same-named linear maps $\partial_i, \partial'_i \colon \bar{\mathcal{F}} \to \bar{\mathcal{F}}$ on the quotient algebra $\bar{\mathcal{F}}$. Moreover, we have the following result:

Proposition 3.7. *(1) If $x \in \bar{\mathcal{F}}$ satisfies $\partial_i(x) = 0$ for all $i \in I$, then $x = 0$.*

(2) If $x \in \bar{\mathcal{F}}$ satisfies $\partial'_i(x) = 0$ for all $i \in I$, then $x = 0$.

Proof. It is enough to prove (1) and (2) for a homogeneous $x \in \bar{\mathcal{F}}$. If $x \neq 0$ and $\hat{x} \in \mathcal{F}$ is its lift of the same degree, then by the very definition of $\bar{\mathcal{F}}$, there must be some $\hat{y} \in \mathcal{F}$ such that $\{\hat{x}, \hat{y}\} \neq 0$, and moreover we must have $|\hat{y}| = |\hat{x}|$. Furthermore, since $\mathcal{F}_\mu$ is spanned by all $[i_1 \dots i_k]$ for which $\alpha_{i_1} + \dots + \alpha_{i_k} = \mu$, it follows that in fact there is some sequence $(i_1, \dots, i_k)$ such that $\{\hat{x}, [i_1 \dots i_k]\} \neq 0$. But then we have

$$\{\partial'_{i_1}(\hat{x}), [i_2 \dots i_k]\} = \{\hat{x}, [i_1 i_2 \dots i_k]\} \neq 0$$

and

$$\{\partial_{i_k}(\hat{x}), [i_1 \dots i_{k-1}]\} = \{\hat{x}, [i_1 \dots i_{k-1} i_k]\} \neq 0,$$

so it follows that $\partial'_{i_1}(x) \neq 0$ and $\partial_{i_k}(x) \neq 0$ in $\bar{\mathcal{F}}$. ∎

We shall now use this proposition to prove Theorem 3.5.

Proof of Theorem 3.5. For $i \neq j$, set

$$S_{ij} = \sum_{k=0}^{1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} e_i^{(1-a_{ij}-k)} e_j e_i^{(k)}. \tag{3.7}$$

According to Proposition 3.7, it suffices to verify $\partial'_p(S_{ij}) = 0$ for all $p \in I$. This vanishing is clear for $p \neq i, j$, so it remains to verify $\partial'_i(S_{ij}) = \partial'_j(S_{ij}) = 0$.

First, a simple inductive argument shows that

$$\partial'_i(e_i^{(k)}) = r_i^{1-k} e_i^{(k-1)},$$

so that

$$\partial'_i(e_i^{(k)} e_j e_i^{(l)}) = r_i^{1-k} e_i^{(k-1)} e_j e_i^{(l)} + (\omega'_j, \omega_i)^{-1} (r_i^{-1} s_i)^k r_i^{1-l} e_i^{(k)} e_j e_i^{(l-1)}$$

for all $k, l \geq 0$, with the convention that $e_i^{(m)} = 0$ for $m < 0$. Thus, we obtain:

$$\begin{aligned} \partial'_i(S_{ij}) = &\sum_{k=0}^{1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}k(k-1)} r_i^{k+a_{ij}} (rs)^{k\langle \alpha_j, \alpha_i \rangle} e_i^{(-a_{ij}-k)} e_j e_i^{(k)} \\ &+ \sum_{k=0}^{1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle + \langle \alpha_i, \alpha_j \rangle} s_i^{1-a_{ij}-k} e_i^{(1-a_{ij}-k)} e_j e_i^{(k-1)}. \end{aligned}$$

Combining the k-th term of the first sum with the $(k+1)$-st term of the second sum yields

$$\partial'_i(S_{ij}) = \sum_{k=0}^{-a_{ij}} (-1)^k \left((r_i s_i)^{\frac{1}{2}k(k-1)} r_i^{k+a_{ij}} (rs)^{k\langle \alpha_j, \alpha_i \rangle} - (r_i s_i)^{\frac{1}{2}(k+1)k} (rs)^{(k+1)\langle \alpha_j, \alpha_i \rangle + \langle \alpha_i, \alpha_j \rangle} s_i^{-a_{ij}-k} \right) e_i^{(-a_{ij}-k)} e_j e_i^{(k)},$$

and since

$$\begin{aligned} (r_i s_i)^{\frac{1}{2}(k+1)k} (rs)^{(k+1)\langle \alpha_j, \alpha_i \rangle + \langle \alpha_i, \alpha_j \rangle} s_i^{-a_{ij}-k} &= (r_i s_i)^{\frac{1}{2}k(k+1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} (r_i s_i)^{a_{ij}} s_i^{-a_{ij}-k} \\ &= (r_i s_i)^{\frac{1}{2}k(k-1)} r_i^{k+a_{ij}} (rs)^{k\langle \alpha_j, \alpha_i \rangle}, \end{aligned}$$

we finally obtain $\partial'_i(S_{ij}) = 0$, as desired.

For ∂'_j, we first note that the one-parameter identity (cf. [J, §0.2(4)])

$$\sum_{t=0}^{a}(-1)^t q^{t(1-a)}\begin{bmatrix}a\\t\end{bmatrix}_q = 0$$

implies, by setting $q=(r_is_i^{-1})^{1/2}$, the identity

$$\sum_{t=0}^{a}(-1)^t (r_is_i)^{\frac{1}{2}t(t-1)} r_i^{t(1-a)}\begin{bmatrix}a\\t\end{bmatrix}_{r_i,s_i} = 0. \tag{3.8}$$

According to (3.4), we have

$$\partial'_j(e_i^{(k)}e_je_i^{(l)}) = \partial'_j(e_i^{(k)}e_j)e_i^{(l)} = (\omega'_i,\omega_j)^{-k}e_i^{(k)}e_i^{(l)}$$

for any $k,l\geq 0$. We thus obtain:

$$\begin{aligned}\partial'_j(S_{ij}) &= \sum_{k=0}^{1-a_{ij}}(-1)^k(r_is_i)^{\frac{1}{2}k(k-1)}(rs)^{k\langle\alpha_j,\alpha_i\rangle}(\omega'_i,\omega_j)^{-(1-a_{ij}-k)}e_i^{(1-a_{ij}-k)}e_i^{(k)}\\ &= \sum_{k=0}^{1-a_{ij}}(-1)^k(r_is_i)^{\frac{1}{2}k(k-1)}r^{k\langle\alpha_j,\alpha_i\rangle-(1-a_{ij}-k)\langle\alpha_i,\alpha_j\rangle}s^{(1-a_{ij})\langle\alpha_j,\alpha_i\rangle}\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}_{r_i,s_i}e_i^{(1-a_{ij})}\\ &= r^{(a_{ij}-1)\langle\alpha_i,\alpha_j\rangle}s^{(1-a_{ij})\langle\alpha_j,\alpha_i\rangle}\left(\sum_{k=0}^{1-a_{ij}}(-1)^k(r_is_i)^{\frac{1}{2}k(k-1)}r_i^{ka_{ij}}\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}_{r_i,s_i}\right)e_i^{(1-a_{ij})}=0,\end{aligned}$$

where we used (3.8) in the last equality. This completes our proof of the theorem. ■

Likewise, let $\mathcal{I}'$ be the right radical of the bilinear form $\{\cdot,\cdot\}$ from Theorem 3.2:

$$\mathcal{I}' = \big\{y\in\mathcal{F}\,\big|\,\{x,y\}=0 \text{ for any } x\in\mathcal{F}\big\}. \tag{3.9}$$

We set $\bar{\mathcal{F}}'=\mathcal{F}/\mathcal{I}'$, and denote the images of i in $\bar{\mathcal{F}}'$ by f_i for all $i\in I$ (this notation is justified by (3.13), where we show that $\bar{\mathcal{F}}'\simeq U^-_{r,s}$). We note that $\mathcal{I}'$ is actually a two-sided ideal of $\mathcal{F}$, due to Theorem 3.2(4), so that $\bar{\mathcal{F}}'$ is an algebra.

As above, we have the following easy consequence of Lemma 3.4:

Lemma 3.8. *The ideal $\mathcal{I}'$ of (3.9) is stable under the maps $\tilde{\partial}_i,\tilde{\partial}'_i$ of (3.5) for all $i\in I$.*

We thus obtain the same-named linear maps $\tilde{\partial}_i,\tilde{\partial}'_i\colon\bar{\mathcal{F}}'\to\bar{\mathcal{F}}'$, satisfying the analogue of Proposition 3.7:

Proposition 3.9. *(1) If $x\in\bar{\mathcal{F}}'$ satisfies $\tilde{\partial}_i(x)=0$ for all $i\in I$, then $x=0$.*

(2) If $x\in\bar{\mathcal{F}}'$ satisfies $\tilde{\partial}'_i(x)=0$ for all $i\in I$, then $x=0$.

Using this proposition, we also obtain the following counterpart of Theorem 3.5:

Theorem 3.10. *For any $i\neq j$, the following relation holds in the algebra $\bar{\mathcal{F}}'$:*

$$\sum_{k=0}^{1-a_{ij}}(-1)^k(r_is_i)^{\frac{1}{2}k(k-1)}(rs)^{k\langle\alpha_j,\alpha_i\rangle}f_i^{(k)}f_jf_i^{(1-a_{ij}-k)}=0.$$

We now introduce a $\mathbb{C}(r,s)$-algebra anti-isomorphism $\varphi\colon\bar{\mathcal{F}}\to\bar{\mathcal{F}}'$, which will ultimately be matched with the corresponding map on $U^{\pm}_{r,s}$ introduced in Proposition 2.6(1). To do so, we first consider the $\mathbb{C}(r,s)$-algebra anti-involution $\varphi\colon\mathcal{F}\to\mathcal{F}$ defined by $\varphi(i)=i$ for all $i\in I$. We then have:

Lemma 3.11. *For all $x\in\mathcal{F}$, we have $\partial_i(\varphi(x))=\varphi(\tilde{\partial}'_i(x))$ and $\varphi(\partial_i(x))=\tilde{\partial}'_i(\varphi(x))$.*

Proof. For the first equality, it is enough to consider $x\in\mathcal{W}$, for which we proceed by induction on $\mathrm{ht}(|x|)$. If $\mathrm{ht}(|x|)\leq 1$, the claim is obvious. If $\mathrm{ht}(|x|)=m>1$, then we may write $x=x'x''$ for some $x',x''\in\mathcal{W}$ with $\mathrm{ht}(|x'|),\mathrm{ht}(|x''|)<m$. Then by the induction assumption, we have:

$$\begin{aligned}\partial_i(\varphi(x)) &= \partial_i(\varphi(x'')\varphi(x')) = \varphi(x'')\partial_i(\varphi(x'))+(\omega'_i,\omega_{|x'|})^{-1}\partial_i(\varphi(x''))\varphi(x')\\ &= \varphi(x'')\varphi(\tilde{\partial}'_i(x'))+(\omega'_i,\omega_{|x'|})^{-1}\varphi(\tilde{\partial}_i(x''))\varphi(x') = \varphi\left(\tilde{\partial}'_i(x')x''+(\omega'_i,\omega_{|x'|})^{-1}x'\tilde{\partial}'_i(x'')\right)\\ &= \varphi(\tilde{\partial}'_i(x'x''))=\varphi(\tilde{\partial}'_i(x)).\end{aligned}$$

This proves that $\partial_i \circ \varphi = \varphi \circ \tilde{\partial}'_i$ for any $i \in I$. Since φ is an anti-involution, we also get $\varphi \circ \partial_i = \tilde{\partial}'_i \circ \varphi$. ■

We can now easily derive the desired result:

Proposition 3.12. *There is a unique $\mathbb{C}(r,s)$-algebra anti-isomorphism $\varphi\colon \bar{\mathcal{F}} \to \bar{\mathcal{F}}'$ such that*

$$\varphi(e_i) = f_i \qquad \text{for all} \quad i \in I.$$

Proof. To prove this, it suffices to show that $\varphi(\mathcal{I}) \subseteq \mathcal{I}'$ and $\varphi(\mathcal{I}') \subseteq \mathcal{I}$. To this end, suppose that $x \in \mathcal{F}$ satisfies $\varphi(x) \notin \mathcal{I}'$. By Proposition 3.9, we then have $0 \neq \tilde{\partial}'_i(\varphi(x)) = \varphi(\partial_i(x))$ for some $i \in I$. The latter implies $\partial_i(x) \neq 0$, and so $x \notin \mathcal{I}$ by Proposition 3.7. This proves the first inclusion $\varphi(\mathcal{I}) \subseteq \mathcal{I}'$. Similarly, if $y \in \mathcal{F}$ satisfies $\varphi(y) \notin \mathcal{I}$, then by Proposition 3.7, there is some $j \in I$ such that $0 \neq \partial_j(\varphi(y)) = \varphi(\tilde{\partial}'_j(y))$. This means that $\tilde{\partial}'_j(y) \neq 0$ and so $y \notin \mathcal{I}'$, by Proposition 3.9, establishing the second inclusion $\varphi(\mathcal{I}') \subseteq \mathcal{I}$. ■

We note that $\mathcal{I}$ is a Hopf ideal with respect to $\Delta_{r,s}$, i.e. $\Delta_{r,s}(\mathcal{I}) \subseteq \mathcal{F}\otimes\mathcal{I}+\mathcal{I}\otimes\mathcal{F}$, due to Theorem 3.2(4). This implies that $\Delta_{r,s}$ descends from $\mathcal{F}$ to the same-named algebra homomorphism $\Delta_{r,s}\colon \bar{\mathcal{F}} \to \bar{\mathcal{F}} \otimes \bar{\mathcal{F}}$. Likewise, we have $\Delta_{s^{-1},r^{-1}}(\mathcal{I}') \subseteq \mathcal{F}\otimes\mathcal{I}'+\mathcal{I}'\otimes\mathcal{F}$ by Theorem 3.2(3), so that $\Delta_{s^{-1},r^{-1}}$ descends from $\mathcal{F}$ to the same-named algebra homomorphism $\Delta_{s^{-1},r^{-1}}\colon \bar{\mathcal{F}}' \to \bar{\mathcal{F}}' \otimes \bar{\mathcal{F}}'$. Finally, we define the linear map

$$T\colon \bar{\mathcal{F}}' \otimes \bar{\mathcal{F}}' \longrightarrow \bar{\mathcal{F}}' \otimes \bar{\mathcal{F}}' \qquad \textit{via} \qquad T(x\otimes y) = y \otimes x.$$

Then, we have the following result:

Proposition 3.13. *For all $z \in \bar{\mathcal{F}}$, we have*

$$(T \circ (\varphi\otimes\varphi) \circ \Delta_{r,s})(z) = (\Delta_{s^{-1},r^{-1}} \circ \varphi)(z).$$

Proof. As usual, it is enough to prove the claim for elements of the form $z = e_{i_1}e_{i_2}\dots e_{i_m}$. We proceed by induction on $\mathrm{ht}(|z|)$. The claim is obvious for $\mathrm{ht}(|z|) \leq 1$, so let us suppose that $\mathrm{ht}(|z|) = m > 1$ and the claim holds for all z' with $\mathrm{ht}(|z'|) < m$. By assumption, there is some i such that $z = z'e_i$, where $\mathrm{ht}(|z'|) < \mathrm{ht}(|z|)$. Thus, by the induction assumption, we have:

$$\begin{aligned}\Delta_{s^{-1},r^{-1}}(\varphi(z)) &= \Delta_{s^{-1},r^{-1}}(f_i\varphi(z')) = (f_i\otimes 1 + 1\otimes f_i) \odot_{s^{-1},r^{-1}} \sum_{(z')} \varphi(z'_{2;r,s})\otimes\varphi(z'_{1;r,s}) \\ &= \sum_{(z')} f_i\varphi(z'_{2;r,s})\otimes\varphi(z'_{1;r,s}) + \sum_{(z')} (\omega'_{|z'_{2;r,s}|},\omega_i)^{-1}\varphi(z'_{2;r,s})\otimes f_i\varphi(z'_{1;r,s}).\end{aligned}$$

On the other hand, we also have:

$$\begin{aligned}T(\varphi\otimes\varphi)\Delta_{r,s}(z'e_i) &= T(\varphi\otimes\varphi)\left(\left(\sum_{(z')} z'_{1;r,s}\otimes z'_{2;r,s}\right) \odot_{r,s} (e_i\otimes 1 + 1\otimes e_i)\right) \\ &= T(\varphi\otimes\varphi)\left(\sum_{(z')} (\omega'_{|z'_{2;r,s}|},\omega_i)^{-1} z'_{1;r,s}e_i\otimes z'_{2;r,s} + \sum_{(z')} z'_{1;r,s}\otimes z'_{2;r,s}e_i\right) \\ &= \sum_{(z')} (\omega'_{|z'_{2;r,s}|},\omega_i)^{-1}\varphi(z'_{2;r,s})\otimes f_i\varphi(z'_{1;r,s}) + \sum_{(z')} f_i\varphi(z'_{2;r,s})\otimes\varphi(z'_{1;r,s}).\end{aligned}$$

This completes the proof, since the right-hand sides of the above two equations coincide. ■

3.3. Symmetric version of pairing.

The pairing $\{\cdot,\cdot\}\colon \mathcal{F}\times\mathcal{F} \to \mathbb{C}(r,s)$ of Theorem 3.2 induces the same-named non-degenerate pairing $\{\cdot,\cdot\}$ on $\bar{\mathcal{F}}\times\bar{\mathcal{F}}'$, by the definition of (3.6, 3.9). The latter can be turned into the pairing $(\cdot,\cdot)\colon \bar{\mathcal{F}}\times\bar{\mathcal{F}} \to \mathbb{C}(r,s)$ through the use of the anti-isomorphism φ from Proposition 3.12. More specifically, we set

$$(x,x') = \{x,\varphi(x')\} \qquad \text{for any} \quad x,x' \in \bar{\mathcal{F}}. \tag{3.10}$$

Lemma 3.14. *The resulting bilinear form $(\cdot,\cdot)$ satisfies the following five properties:*

(1) $(1,1) = 1$,
(2) $(e_i,e_j) = \delta_{ij}$,
(3) $(x,y) = 0$ *if* $|x| \neq |y|$,
(4) $(x,yz) = (\Delta_{r,s}(x), z\otimes y)$,

(5) $(xy,z)=(y\otimes x,\Delta_{r,s}(z))$,
where $(x'\otimes x'',y'\otimes y'')=(x',y')(x'',y'')$ *for any* $x',x'',y',y''\in\bar{\mathcal{F}}$.

Proof. Properties (1)–(3) are immediate from the definition and the corresponding properties of $\{\cdot,\cdot\}$. Property (4) follows from $(x,yz)=\{x,\varphi(yz)\}=\{x,\varphi(z)\varphi(y)\}=\{\Delta_{r,s}(x),\varphi(z)\otimes\varphi(y)\}=(\Delta_{r,s}(x),z\otimes y)$. Property (5) follows along the same line by evoking Proposition 3.13:

$$(xy,z)=\{x\otimes y,\Delta_{s^{-1},r^{-1}}(\varphi(z))\}=\{x\otimes y,T(\varphi\otimes\varphi)(\Delta_{r,s}(z))\}=\{y\otimes x,\varphi\otimes\varphi(\Delta_{r,s}(z))\}=(y\otimes x,\Delta_{r,s}(z)).$$

This completes the proof of this lemma. ∎

We also note the following important property of $(\cdot,\cdot)$:

Lemma 3.15. *The pairing* $(\cdot,\cdot)$ *of* (3.10) *is symmetric.*

Proof. The pairing $(\cdot,\cdot)$ not only satisfies the above properties (1)–(5), but is also uniquely determined by them. However, the pairing $(\cdot,\cdot)^\circ$ defined via $(x,y)^\circ=(y,x)$ clearly satisfies the same properties. Therefore, $(\cdot,\cdot)^\circ=(\cdot,\cdot)$, which shows that $(\cdot,\cdot)$ is indeed symmetric. ∎

Finally, as an immediate consequence of Lemma 3.3, we obtain:

Lemma 3.16. *For any* $x,x'\in\bar{\mathcal{F}}$ *and* $i\in I$*, we have the following two equalities:*

$$(x,e_ix')=(\partial_i(x),x')\qquad\text{and}\qquad(x,x'e_i)=(\partial'_i(x),x').$$

3.4. Relation to the Hopf pairing.

Recall the Hopf pairing $(\cdot,\cdot)_H\colon U^{\leq}_{r,s}\times U^{\geq}_{r,s}\to\mathbb{C}(r,s)$ from (2.9). We conclude this Section with an explicit relation between $(\cdot,\cdot)_H$ and $(\cdot,\cdot)$ of (3.10), which will allow us to show that $U^+_{r,s}\simeq\bar{\mathcal{F}}$, $U^-_{r,s}\simeq\bar{\mathcal{F}}'$, see (3.13). This relationship is also instrumental to the proof of our main Theorems 7.1 and 7.2.

Before proceeding, we note first that by Theorems 3.5 and 3.10, there are natural $\mathbb{C}(r,s)$-algebra homomorphisms $\psi^+\colon U^+_{r,s}\to\bar{\mathcal{F}}$ and $\psi^-\colon U^-_{r,s}\to\bar{\mathcal{F}}'$, determined by $\psi^+(e_i)=e_i$ and $\psi^-(f_i)=f_i$ for all $i\in I$. We also recall the bar involution $x\mapsto\bar{x}$ on $U^\pm_{r,s}$ from Proposition 2.6(3). We can now relate $(\cdot,\cdot)_H$ to $(\cdot,\cdot)$:

Theorem 3.17. *For all* $y\in(U^-_{r,s})_{-\mu}$ *and* $x\in(U^+_{r,s})_\mu$*, where* $\mu=\sum_{i=1}^n c_i\alpha_i\in Q^+$*, we have*[2]

$$\overline{(\bar{y},\bar{x})_H}=\left(\prod_{i=1}^n\frac{1}{(r_i-s_i)^{c_i}}\right)\big(\psi^+(\varphi(y)),\psi^+(x)\big).$$

Proof. Evoking the linear maps $p_i\colon U^+_{r,s}\to U^+_{r,s}$ of (2.12), let us define a $\mathbb{C}(r,s)$-linear map

$$\bar{p}_i\colon U^+_{r,s}\to U^+_{r,s}\qquad\text{via}\qquad\bar{p}_i(x)=\overline{p_i(\bar{x})}.$$

Then, $\bar{p}_i(1)=0$, $\bar{p}_i(e_j)=\delta_{ij}$, and we claim that they satisfy the following analogue of the Leibniz rule:

$$\bar{p}_i(xx')=x\bar{p}_i(x')+(\omega'_i,\omega_{\deg(x')})^{-1}\bar{p}_i(x)x' \tag{3.11}$$

for all homogeneous $x,x'\in U^+_{r,s}$. Indeed, this follows from (2.13) and $\overline{(\omega'_\mu,\omega_\nu)}=(\omega'_\nu,\omega_\mu)^{-1}$:

$$\bar{p}_i(xx')=\overline{p_i(\bar{x}\bar{x}')}=\overline{\bar{x}p_i(\bar{x}')+(\omega'_{\deg(x')},\omega_i)p_i(\bar{x})\bar{x}'}=x\bar{p}_i(x')+(\omega'_i,\omega_{\deg(x')})^{-1}\bar{p}_i(x)x'.$$

Let us now record the relation between these $\bar{p}_i$ and the linear maps ∂_i of (3.3):

$$\partial_i(\psi^+(x))=\psi^+(\bar{p}_i(x))\qquad\text{for all}\quad x\in U^+_{r,s}. \tag{3.12}$$

This equality is clear when $x\in\mathbb{C}(r,s)$ or $x=e_j$ with $j\in I$. Thus, it remains to show that if (3.12) holds for x' and x'', then it also holds for $x'x''$. To this end, we have:

$$\begin{aligned}\partial_i(\psi^+(x'x''))=\partial_i(\psi^+(x')\psi^+(x''))&\overset{(3.4)}{=}\psi^+(x')\partial_i(\psi^+(x''))+(\omega'_i,\omega_{\deg(x'')})^{-1}\partial_i(\psi^+(x'))\psi^+(x'')\\&=\psi^+(x')\psi^+(\bar{p}_i(x''))+(\omega'_i,\omega_{\deg(x'')})^{-1}\psi^+(\bar{p}_i(x'))\psi^+(x'')\\&=\psi^+(x'\bar{p}_i(x'')+(\omega'_i,\omega_{\deg(x'')})^{-1}\bar{p}_i(x')x'')\overset{(3.11)}{=}\psi^+(\bar{p}_i(x'x'')).\end{aligned}$$

[2]Our convention (2.15) is not in contradiction with (3.10), as the latter is not defined on the Cartan subalgebra.

To prove the theorem, we proceed by induction on $\mathrm{ht}(\mu)$, the base case $\mathrm{ht}(\mu)=0$ being obvious. Assuming $\mathrm{ht}(\mu)>0$, it is enough to consider $y=y'f_i$ for some i. Then by Proposition 2.4 and Lemma 3.16, we have:

$$\begin{aligned}
\overline{(\bar{y},\bar{x})_H} &= \overline{(\bar{y}'f_i,\bar{x})_H} = \frac{1}{r_i-s_i}\overline{(\bar{y}',p_i(\bar{x}))_H} = \frac{1}{r_i-s_i}\overline{(\bar{y}',\overline{\bar{p}_i(x)})_H} \\
&= \frac{1}{r_i-s_i}\left(\frac{1}{(r_i-s_i)^{c_i-1}}\cdot\prod_{j\neq i}\frac{1}{(r_j-s_j)^{c_j}}\right)\left(\psi^+(\varphi(y')),\psi^+(\bar{p}_i(x))\right) \\
&\overset{(3.12)}{=} \left(\prod_{i=1}^n\frac{1}{(r_i-s_i)^{c_i}}\right)\left(\psi^+(\varphi(y')),\partial_i(\psi^+(x))\right) = \left(\prod_{i=1}^n\frac{1}{(r_i-s_i)^{c_i}}\right)\left(e_i\psi^+(\varphi(y')),\psi^+(x)\right) \\
&= \left(\prod_{i=1}^n\frac{1}{(r_i-s_i)^{c_i}}\right)\left(\psi^+(\varphi(y'f_i)),\psi^+(x)\right) = \left(\prod_{i=1}^n\frac{1}{(r_i-s_i)^{c_i}}\right)\left(\psi^+(\varphi(y)),\psi^+(x)\right),
\end{aligned}$$

where we used the induction hypothesis in the second line. This completes the proof of the theorem. ■

Corollary 3.18. *The above algebra homomorphisms $\psi^\pm$ are actually algebra isomorphisms:*

$$\psi^+\colon U^+_{r,s}\xrightarrow{\sim}\bar{\mathcal{F}} \qquad\text{and}\qquad \psi^-\colon U^-_{r,s}\xrightarrow{\sim}\bar{\mathcal{F}}'. \tag{3.13}$$

Proof. Suppose that $\psi^+(x)=0\in\bar{\mathcal{F}}$ for some $x\in(U^+_{r,s})_\mu$. Then in particular, we have $(\psi^+(\varphi(y)),\psi^+(x))=0$ for all $y\in(U^-_{r,s})_{-\mu}$, and therefore Theorem 3.17 implies that $(\bar{y},\bar{x})_H=0$ for all $y\in(U^-_{r,s})_{-\mu}$. However, since $(\cdot,\cdot)_H$ is non-degenerate (see Proposition 2.11) and $y\mapsto\bar{y}$ is an algebra automorphism, we thus get $x=0$.

Now suppose that $\psi^-(y)=0\in\bar{\mathcal{F}}'$ for some $y\in(U^-_{r,s})_{-\mu}$. As $\varphi\circ\psi^-=\psi^+\circ\varphi$, we have $\psi^+(\varphi(y))=0$, so that $\varphi(y)=0$ by above. Thus $y=0$ as claimed, since $\varphi\colon U^-_{r,s}\to U^+_{r,s}$ is an anti-isomorphism. ■

Combining Proposition 2.11 with Theorem 3.17 and using (3.13) to identify $\bar{\mathcal{F}}$ with $U^+_{r,s}$, we obtain:

Corollary 3.19. *The pairing $(\cdot,\cdot)\colon U^+_{r,s}\times U^+_{r,s}\to\mathbb{C}(r,s)$ is non-degenerate.*

4. Shuffle Algebras

In this Section, we introduce the two-parameter shuffle algebra $(\mathcal{F},*)$, relate it to the positive subalgebra $U^+_{r,s}$, and provide a shuffle interpretation of some of the structures on the latter. Our exposition closely follows that of the one-parameter setup from [L, Section 2] and [CHW, Section 3].

4.1. Two-parameter shuffle algebra.

Recall that $\mathcal{F}$ is the free associative $\mathbb{C}(r,s)$-algebra generated by the finite alphabet $I=\{1,2,\ldots,n\}$, and $\mathcal{W}$ is the set of words in I. Recall also the notation $[i_1\ldots i_d]=i_1i_2\ldots i_d$ for the elements in $\mathcal{W}$, where $i_1,\ldots,i_d\in I$. As before, $\mathcal{F}$ has a natural Q^+-grading induced by declaring the degree of $[i]$ equal to α_i. For a homogeneous element $x\in\mathcal{F}$, we write $|x|$ for the degree of x. For any $a,b\in\mathbb{C}(r,s)$, we now define the quantum shuffle product $*_{a,b}\colon\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ inductively via

$$(xi)*_{a,b}(yj)=(x*_{a,b}(yj))i+a^{-\langle|xi|,\alpha_j\rangle}b^{\langle\alpha_j,|xi|\rangle}((xi)*_{a,b}y)j, \qquad \emptyset*_{a,b}x=x*_{a,b}\emptyset=x, \tag{4.1}$$

for all $i,j\in I$ and all homogeneous $x,y\in\mathcal{F}$. By iterating this definition, we find that

$$[i_1\ldots i_m]*_{a,b}[i_{m+1}\ldots i_{m+d}]=\sum_\sigma e_{a,b}(\sigma)[i_{\sigma^{-1}(1)}\ldots i_{\sigma^{-1}(m+d)}], \tag{4.2}$$

where

$$e_{a,b}(\sigma)=\prod_{\substack{k\leq m<l\\ \sigma(k)<\sigma(l)}}a^{-\langle\alpha_{i_k},\alpha_{i_l}\rangle}b^{\langle\alpha_{i_l},\alpha_{i_k}\rangle} \tag{4.3}$$

and the sum runs over all (m,d)-shuffles of $\{1,2,\ldots,m+d\}$, i.e. the permutations $\sigma\in S_{m+d}$ such that $\sigma(1)<\sigma(2)<\cdots<\sigma(m)$ and $\sigma(m+1)<\cdots<\sigma(m+d)$. There are four choices of a,b that are of interest to us; in these cases, the inductive formula (4.1) takes the following form:

$$(xi) *_{r,s} (yj) = (x *_{r,s} (yj))i + (\omega'_{|xi|}, \omega_j)^{-1}((xi) *_{r,s} y)j,$$
$$(xi) *_{s,r} (yj) = (x *_{s,r} (yj))i + (\omega'_j, \omega_{|xi|})((xi) *_{s,r} y)j,$$
$$(xi) *_{s^{-1},r^{-1}} (yj) = (x *_{s^{-1},r^{-1}} (yj))i + (\omega'_j, \omega_{|xi|})^{-1}((xi) *_{s^{-1},r^{-1}} y)j,$$
$$(xi) *_{r^{-1},s^{-1}} (yj) = (x *_{r^{-1},s^{-1}} (yj))i + (\omega'_{|xi|}, \omega_j)((xi) *_{r^{-1},s^{-1}} y)j,$$

cf. (2.15), and the corresponding expressions for $e_{a,b}(\sigma)$ of (4.3) are:

$$e_{r,s}(\sigma) = \prod_{\substack{k \le m < l \\ \sigma(k) < \sigma(l)}} (\omega'_{i_k}, \omega_{i_l})^{-1}, \qquad e_{s,r}(\sigma) = \prod_{\substack{k \le m < l \\ \sigma(k) < \sigma(l)}} (\omega'_{i_l}, \omega_{i_k}),$$
$$e_{s^{-1},r^{-1}}(\sigma) = \prod_{\substack{k \le m < l \\ \sigma(k) < \sigma(l)}} (\omega'_{i_l}, \omega_{i_k})^{-1}, \qquad e_{r^{-1},s^{-1}}(\sigma) = \prod_{\substack{k \le m < l \\ \sigma(k) < \sigma(l)}} (\omega'_{i_k}, \omega_{i_l}).$$

Since the product structure $*_{r,s}$ will be used most frequently, we shall often omit the subscript for this operation, and just use the notation $*$ instead.

We have the following basic result, the proof of which is a direct computation using (4.1):

Proposition 4.1. *The bilinear map* $*_{a,b} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *is associative.*

For the later use, we note that $*_{r,s}$ and $*_{s,r}$ are related via the following result:

Proposition 4.2. *For all homogeneous* $x, y \in \mathcal{F}$*, we have*

$$x *_{r,s} y = (\omega'_{|x|}, \omega_{|y|})^{-1} y *_{s,r} x.$$

Proof. We proceed by induction on $m = \mathrm{ht}(|x|) + \mathrm{ht}(|y|)$. If $m = 0$ or 1, then the result is obvious, because in these cases one of x and y must be the empty word. Thus, we may assume that $\mathrm{ht}(|x|) \ge 1$ and $\mathrm{ht}(|y|) \ge 1$, and that the result holds for all homogeneous elements x, y with $\mathrm{ht}(|x|) + \mathrm{ht}(|y|) < m$. We may further assume that $x, y \in \mathcal{W}$. Then we may write $x = x'i$ and $y = y'j$ for some $i, j \in I$, and by induction we have:

$$\begin{aligned} x *_{r,s} y &= (x'i) *_{r,s} (y'j) = (x' *_{r,s} (y'j))i + (\omega'_{|x|}, \omega_j)^{-1}((x'i) *_{r,s} y')j \\ &= (\omega'_{|x'|}, \omega_{|y|})^{-1}((y'j) *_{s,r} x')i + (\omega'_{|x|}, \omega_j)^{-1}(\omega'_{|x|}, \omega_{|y'|})^{-1}(y' *_{s,r} (x'i))j \\ &= (\omega'_{|x|}, \omega_{|y|})^{-1} \left((\omega'_i, \omega_{|y|})((y'j) *_{s,r} x')i + (y' *_{s,r} (x'i))j \right) \\ &= (\omega'_{|x|}, \omega_{|y|})^{-1}(y'j) *_{s,r} (x'i) = (\omega'_{|x|}, \omega_{|y|})^{-1} y *_{s,r} x. \end{aligned}$$

This completes the proof. ■

Let $\pi_+ : \mathcal{F} \to U^+_{r,s}$ and $\pi_- : \mathcal{F} \to U^-_{r,s}$ be the canonical algebra homomorphisms determined by $\pi_+(i) = e_i$ and $\pi_-(i) = f_i$ for $i \in I$. We note that by the definition of $\Delta_{r,s}$ on $\bar{\mathcal{F}} \simeq U^+_{r,s}$, cf. (3.13), we have

$$\Delta_{r,s}\pi_+ = (\pi_+ \otimes \pi_+)\Delta_{r,s}.$$

For $w = [i_1 \dots i_d]$ and any $P = \{k_1 < \dots < k_m\} \subseteq \{1, 2, \dots, d\}$, define $w_P = [i_{k_1} \dots i_{k_m}]$. We then have

$$\Delta_{r,s}(w) = \sum_{P \subseteq \{1,2,\dots,d\}} z(P),$$

where $z(P) = z_1 \odot_{r,s} \dots \odot_{r,s} z_d$ with $z_k = i_k \otimes 1$ when $k \in P$ and $z_k = 1 \otimes i_k$ when $k \in P^c = \{1, 2, \dots, d\} \setminus P$. If σ_P denotes the $(d-m, m)$-shuffle determined by $\sigma_P(d - m + i) = k_i$, then we have

$$z(P) = e_{r,s}(\sigma_P) w_P \otimes w_{P^c},$$

which follows immediately from formula (3.1) for $\odot_{r,s}$ and the definition (4.3) of $e_{r,s}(\sigma)$. This implies that

$$\Delta_{r,s}(w) = \sum_{P \subseteq \{1,2,\dots,d\}} e_{r,s}(\sigma_P) w_P \otimes w_{P^c}.$$

Let $\mathcal{F}^*$ be the graded dual of $\mathcal{F}$, and for each word $w \in \mathcal{W}$ we define $w^* \in \mathcal{F}^*$ by

$$w^*(v) = \delta_{w,v} \quad \text{for all} \quad v \in \mathcal{W}. \tag{4.4}$$

Consider the product on $\mathcal{F}^*$ defined by:

$$(fg)(x) = (g \otimes f)(\Delta_{r,s}(x)).$$

Lemma 4.3. *The linear map $\phi\colon \mathcal{F}^* \to (\mathcal{F}, *)$ defined by $w^* \mapsto w$ is a $\mathbb{C}(r,s)$-algebra isomorphism.*

Proof. The map ϕ is clearly a vector space isomorphism, so we only need to show that $\phi(fg) = \phi(f) * \phi(g)$. For this, let $u = [i_1 \dots i_d] \in \mathcal{W}$, $v = [i_{d+1} \dots i_{d+m}] \in \mathcal{W}$, and let w be any word of weight $|u| + |v|$. Then

$$(u^* v^*)(w) = (v^* \otimes u^*) \left(\sum_{P \subseteq \{1,\dots,d+m\}} e_{r,s}(\sigma_P) w_P \otimes w_{P^c} \right).$$

If $\lambda^w_{u,v} = \sum e_{r,s}(\sigma_P)$ with the sum over all $P \subseteq \{1, 2, \dots, d+m\}$ satisfying $w_P = v, w_{P^c} = u$, then we get:

$$u^* v^* = \sum \lambda^w_{u,v} w^*. \tag{4.5}$$

On the other hand, we have

$$u * v = \sum_\sigma e_{r,s}(\sigma)[i_{\sigma^{-1}(1)} \dots i_{\sigma^{-1}(d+m)}],$$

so that the coefficient of any word $w \in \mathcal{W}$ of weight $|u| + |v|$ in the expansion of $u * v$ is precisely equal to $\sum e_{r,s}(\sigma)$, where the sum ranges over all (d, m)-shuffles σ such that, if $P = \{\sigma(d+1), \dots, \sigma(d+m)\}$, then $w_P = v$ and $w_{P^c} = u$. We thus get $u * v = \sum \lambda^w_{u,v} w$, which together with (4.5) completes the proof. ■

Proposition 4.4. *There is a unique $\mathbb{C}(r,s)$-algebra homomorphism $\Psi\colon U^+_{r,s} \to (\mathcal{F}, *)$ such that $\Psi(e_i) = i$. Moreover, Ψ is injective.*

Proof. The quotient map $\pi_+ \colon \mathcal{F} \to U^+_{r,s}$ induces an embedding of graded duals $\pi_+^* \colon (U^+_{r,s})^* \to \mathcal{F}^*$, where multiplication is defined by $(fg)(x) = (g \otimes f)(\Delta_{r,s}(x))$ in both cases. As the pairing $(\cdot,\cdot)$ on $\bar{\mathcal{F}} \simeq U^+_{r,s}$ is non-degenerate by Corollary 3.19, we have a vector space isomorphism $\psi\colon U^+_{r,s} \to (U^+_{r,s})^*$ given by $\psi(x)(y) = (x, y)$ for all $x, y \in U^+_{r,s}$. Evoking that $(fg)(x) = (g \otimes f)(\Delta_{r,s}(x))$ for $f, g \in (U^+_{r,s})^*$, we thus obtain:

$$(xx', y) = (x' \otimes x, \Delta_{r,s}(y)) = (\psi(x') \otimes \psi(x))(\Delta_{r,s}(y)) = (\psi(x)\psi(x'))(y).$$

This shows that the map ψ is actually an algebra isomorphism. Now, define $\Psi = \phi \circ \pi_+^* \circ \psi$. Then Ψ is an algebra embedding, and since $((\pi_+^* \circ \psi)(e_i))(j) = \psi(e_i)(e_j) = (e_i, e_j) = \delta_{ij}$, it also follows that $\Psi(e_i) = i$. ■

We shall now give an alternative description of the above map Ψ, making use of the operators ∂'_i introduced in (3.3). For each word $w = [i_1 \dots i_d] \in \mathcal{W}$, we define

$$\partial'_w = \partial'_{i_1} \partial'_{i_2} \dots \partial'_{i_d}.$$

We then define a $\mathbb{C}(r,s)$-linear map $\Upsilon\colon U^+_{r,s} \to \mathcal{F}$ by

$$\Upsilon(u) = \sum_{w \in \mathcal{W}_\mu} \partial'_w(u) w \qquad \text{for} \quad u \in (U^+_{r,s})_\mu. \tag{4.6}$$

We will show in Proposition 4.6 below that this map coincides with the map Ψ of Proposition 4.4. To do so, we need to introduce analogues of the operators ∂'_i for $(\mathcal{F}, *)$, which is the content of the following lemma:

Lemma 4.5. *For each $i = 1, 2, \dots, n$, define the $\mathbb{C}(r,s)$-linear map $\epsilon'_i \colon \mathcal{F} \to \mathcal{F}$ by*

$$\epsilon'_i([i_1 \dots i_d]) = \delta_{i,i_d}[i_1 \dots i_{d-1}], \qquad \epsilon'_i(\emptyset) = 0.$$

Then $\epsilon'_i(j) = \delta_{ij}$ and we have

$$\epsilon'_i(x * y) = \epsilon'_i(x) * y + (\omega'_{|x|}, \omega_i)^{-1}(x * \epsilon'_i(y))$$

for all homogeneous $x, y \in \mathcal{F}$.

Proof. It suffices to assume that $x, y \in \mathcal{W}$. If one of x, y has length zero, then the formula is obvious. Otherwise, we may write $x = x'j, y = y'k$. Then, we have:

$$\begin{aligned}\epsilon'_i(x * y) &= \epsilon'_i((x' * (y'k))j + (\omega'_{|x|}, \omega_k)^{-1}((x'j) * y')k) \\ &= \delta_{ij}(x' * (y'k)) + \delta_{ik}(\omega'_{|x|}, \omega_k)^{-1}((x'j) * y') = \epsilon'_i(x) * y + (\omega'_{|x|}, \omega_i)^{-1}(x * \epsilon'_i(y)),\end{aligned}$$

as desired. ■

For a word $w=[i_1\dots i_d]$, we also define $\epsilon'_w\colon \mathcal{F}\to\mathcal{F}$ via

$$\epsilon'_w=\epsilon'_{i_1}\epsilon'_{i_2}\dots\epsilon'_{i_d}. \tag{4.7}$$

Then for any word $v\in\mathcal{W}$, we have $\epsilon'_w(v)=\delta_{w,v}$.

Proposition 4.6. *The map $\Upsilon\colon U^+_{r,s}\to\mathcal{F}$ of (4.6) is an injective algebra homomorphism satisfying $\Upsilon(e_i)=i$ for all $i\in I$, and hence it coincides with the map Ψ of Proposition 4.4.*

Proof. First, we note that $\Psi\circ\partial'_i=\epsilon'_i\circ\Psi$, by (3.4) and Lemma 4.5. Therefore, if $u\in(U^+_{r,s})_\mu$, $w\in\mathcal{W}_\mu$, and $\gamma_w(u)$ is the coefficient of w in $\Psi(u)$, then

$$\gamma_w(u)=\epsilon'_w(\Psi(u))=\Psi(\partial'_w(u))=\partial'_w(u)\Psi(1)=\partial'_w(u).$$

This shows that $\Psi=\Upsilon$ and completes the proof. ■

Let $\mathcal{U}$ denote the image of Ψ, that is $\mathcal{U}=\Psi(U^+_{r,s})$, which is the subalgebra of $(\mathcal{F},*_{r,s})$ generated by I.

Proposition 4.7. *The element $x=\sum_{w\in\mathcal{W}}\gamma(w)w\in\mathcal{F}$ lies in $\mathcal{U}$ if and only if*

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{r_i,s_i}(r_is_i)^{\frac{1}{2}k(k-1)}(rs)^{k\langle\alpha_j,\alpha_i\rangle}\gamma(z[i]^k[j][i]^{1-a_{ij}-k}t)=0 \tag{4.8}$$

for all $i\neq j$ and $z,t\in\mathcal{W}$.

Proof. Let K be the $\mathbb{C}(r,s)$-subspace of $\mathcal{F}$ spanned by the set of elements $\sum_{w\in\mathcal{W}}\gamma(w)w$ satisfying (4.8). For any $u\in(U^+_{r,s})_\mu$, consider

$$x=\Psi(u)=\sum_{|w|=\mu}\gamma_w(u)w.$$

Then, for any word $w=[i_1\dots i_d]$ with $|w|=\mu$, Proposition 4.6 and Lemma 3.16 imply that

$$\gamma_w(u)=\partial'_{i_1}\dots\partial'_{i_d}(u)=(e_{i_1}\dots e_{i_d},u).$$

Therefore $x\in K$ according to Theorem 3.5, so that $\mathcal{U}\subseteq K$.

To prove the other inclusion, consider the linear map $L\colon\mathcal{F}\to\mathcal{F}^*$ defined by $w\mapsto w^*$ for $w\in\mathcal{W}$, where w^* was defined in (4.4). Then, $f\in K$ if and only if we have $L(x)(f)=0$ for all $x\in\ker(\pi_+)$, since $\ker(\pi_+)$ is generated by $\{S_{ij}\}_{i,j\in I}$ of (3.7) due to (3.13). Thus, it follows that $\dim(K_\mu)=\dim(U^+_{r,s})_\mu$ for any $\mu\in Q^+$. But since $\Psi\colon U^+_{r,s}\to\mathcal{U}$ is an isomorphism, we also have $\dim(\mathcal{U}_\mu)=\dim(U^+_{r,s})_\mu$ for all $\mu\in Q^+$.

Therefore, we must actually have the equality $K=\mathcal{U}$. ■

4.2. Additional structures.

Proposition 4.8. *(1) Let $\tau\colon\mathcal{F}\to\mathcal{F}$ be the $\mathbb{C}$-linear map defined by $\tau(r)=s^{-1},\tau(s)=r^{-1}$, and*

$$\tau([i_1\dots i_d])=[i_d\dots i_1].$$

Then, $\tau(x*_{r,s}y)=\tau(y)*_{r,s}\tau(x)$ for all $x,y\in\mathcal{F}$.

(2) Let $x\mapsto\bar{x}$ be the $\mathbb{C}$-linear map $\mathcal{F}\to\mathcal{F}$ defined by $\bar{r}=s$, $\bar{s}=r$, and

$$\overline{[i_1\dots i_d]}=\left(\prod_{k<l}(\omega'_{i_l},\omega_{i_k})^{-1}\right)[i_d\dots i_1].$$

Then, $\overline{x*_{r,s}y}=\bar{x}*_{r,s}\bar{y}$ for all $x,y\in\mathcal{F}$.

Proof. To prove part (1), let $x=[i_1\dots i_m]$ and $y=[i_{m+1}\dots i_{m+d}]$. Then

$$\begin{aligned}\tau([i_1\dots i_m]*_{r,s}[i_{m+1}\dots i_{m+d}])&=\tau\left(\sum_\sigma e_{r,s}(\sigma)[i_{\sigma^{-1}(1)}\dots i_{\sigma^{-1}(m+d)}]\right)\\ &=\sum_\sigma\prod_{\substack{k\le m<l\\ \sigma(k)<\sigma(l)}}\tau\Big((\omega'_{i_k},\omega_{i_l})^{-1}\Big)[i_{\sigma^{-1}(m+d)}\dots i_{\sigma^{-1}(1)}]\\ &=\sum_{\sigma'}\prod_{\substack{k\le m<l\\ \sigma'(k)>\sigma'(l)}}(\omega'_{i_l},\omega_{i_k})^{-1}[i_{(\sigma')^{-1}(1)}\dots i_{(\sigma')^{-1}(m+d)}],\end{aligned} \tag{4.9}$$

where the first two sums are taken over all (m,d)-shuffles $\sigma \in S_{m+d}$, the last sum is taken over all $\sigma' \in S_{m+d}$ such that $\sigma'(m+d) < \sigma'(m+d-1) < \cdots < \sigma'(m+1)$ and $\sigma'(m) < \sigma'(m-1) < \cdots < \sigma'(1)$, and the final equality is obtained by matching these $\{\sigma\}$ and $\{\sigma'\}$ via $\sigma \mapsto \sigma' = w_0\sigma$ with the longest element $w_0 \in S_{m+d}$.

On the other hand, we have:

$$\begin{aligned}[] [i_{m+d}\dots i_{m+1}] *_{r,s} [i_m \dots i_1] &= \sum_{\tilde\sigma} \prod_{\substack{\tilde k \le d < \tilde l \\ \tilde\sigma(\tilde k) < \tilde\sigma(\tilde l)}} (\omega'_{i_{m+d+1-\tilde k}}, \omega_{i_{m+d+1-\tilde l}})^{-1} [i_{(\tilde\sigma w_0)^{-1}(1)} \dots i_{(\tilde\sigma w_0)^{-1}(m+d)}] \\ &= \sum_{\tilde\sigma} \prod_{\substack{k \le m < l \\ \tilde\sigma w_0(l) < \tilde\sigma w_0(k)}} (\omega'_{i_l}, \omega_{i_k})^{-1} [i_{(\tilde\sigma w_0)^{-1}(1)} \dots i_{(\tilde\sigma w_0)^{-1}(m+d)}], \end{aligned} \tag{4.10}$$

where both sums are taken over all (d,m)-shuffles $\tilde\sigma \in S_{m+d}$, and the indexes k,l in the latter sum are related to $\tilde k, \tilde l$ in the former via $k = m+d+1-\tilde l = w_0(\tilde l)$, $l = m+d+1-\tilde k = w_0(\tilde k)$. The right-hand sides of (4.9, 4.10) coincide under a natural bijection between the corresponding $\{\tilde\sigma\}$ and $\{\sigma'\}$ given by $\sigma' = \tilde\sigma w_0$. This completes the proof of part (1).

For part (2), we first note that for any homogeneous $x', y' \in \mathcal{F}$ and $i, j \in I$, we have by part (1):

$$\begin{aligned} (ix') *_{r,s} (jy') &= \tau((\tau(y')j) *_{r,s} (\tau(x')i)) \\ &= \tau\Big((\tau(y') *_{r,s} (\tau(x')i))j + (\omega'_{|y'j|}, \omega_i)^{-1}((\tau(y')j) *_{r,s} \tau(x'))i \Big) \\ &= j((ix') *_{r,s} y') + (\omega'_i, \omega_{|jy'|})^{-1} i(x' *_{r,s} (jy')). \end{aligned} \tag{4.11}$$

For $x, y \in \mathcal{W}$, we proceed by induction on $m = \mathrm{ht}(|x|) + \mathrm{ht}(|y|)$. Note first that if either $\mathrm{ht}(|x|) = 0$ or $\mathrm{ht}(|y|) = 0$, then the claim is trivial. In particular, the assertion holds for $m = 0$ and $m = 1$. For the step of induction, suppose that $\mathrm{ht}(|x|), \mathrm{ht}(|y|) \ge 1$, $m = \mathrm{ht}(|x|) + \mathrm{ht}(|y|)$, and the claim holds for all $x', y' \in \mathcal{W}$ with $\mathrm{ht}(|x'|) + \mathrm{ht}(|y'|) < m$. By assumption, there are $i, j \in I$ and $x', y' \in \mathcal{W}$ such that $x = ix'$ and $y = jy'$, and we have $\mathrm{ht}(|x'|) + \mathrm{ht}(|y'|) = m - 2$. Thus, combining (4.11) with the equality $\overline{(\omega'_\mu, \omega_\nu)^{-1}} = (\omega'_\nu, \omega_\mu)$ and the induction hypothesis, we obtain:

$$\begin{aligned} \overline{(ix') *_{r,s} (jy')} &= \overline{j((ix') *_{r,s} y')} + \overline{(\omega'_i, \omega_{|jy'|})^{-1} i(x' *_{r,s} (jy'))} \\ &= (\omega'_{|ix'|}\omega'_{|y'|}, \omega_j)^{-1} \left(\overline{(ix') *_{r,s} y'}\right) j + (\omega'_{|jy'|}, \omega_i)(\omega'_{|x'|}\omega'_{|jy'|}, \omega_i)^{-1} \left(\overline{x' *_{r,s} (jy')}\right) i \\ &= (\omega'_{|ix'|}\omega'_{|y'|}, \omega_j)^{-1} (\overline{ix'} *_{r,s} \bar y')j + (\omega'_{|x'|}, \omega_i)^{-1} (\bar x' *_{r,s} \overline{jy'})i \\ &= (\omega'_{|ix'|}\omega'_{|y'|}, \omega_j)^{-1} (\omega'_{|x'|}, \omega_i)^{-1} ((\bar x' i) *_{r,s} \bar y')j + (\omega'_{|x'|}, \omega_i)^{-1} (\omega'_{|y'|}, \omega_j)^{-1} (\bar x' *_{r,s} (\bar y' j))i \\ &= (\omega'_{|x'|}, \omega_i)^{-1} (\omega'_{|y'|}, \omega_j)^{-1} ((\bar x' i) *_{r,s} (\bar y' j)) = \overline{ix'} *_{r,s} \overline{jy'}, \end{aligned}$$

which completes the proof. ■

Comparing the above result with Proposition 2.6, we obtain:

Corollary 4.9. *(1) For all $u \in U^+_{r,s}$, we have $\tau\Psi(u) = \Psi\tau(u)$, where $\tau\colon U^+_{r,s} \to U^+_{r,s}$ is the $\mathbb{C}$-algebra anti-automorphism defined in Proposition 2.6(2).*

(2) For all $u \in U^+_{r,s}$, we have $\overline{\Psi(u)} = \Psi(\bar u)$, where $^-\colon U^+_{r,s} \to U^+_{r,s}$ is the $\mathbb{C}$-algebra automorphism defined in Proposition 2.6(3).

Finally, we equip $\mathcal{F}$ with a coproduct:

Proposition 4.10. *Let $\Delta\colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ be the linear map defined by*

$$\Delta([i_1 \dots i_d]) = \sum_{0 \le k \le d} [i_{k+1} \dots i_d] \otimes [i_1 \dots i_k].$$

Then, $\Delta(x * y) = \Delta(x) * \Delta(y)$, where we define the shuffle product $*$ on $\mathcal{F} \otimes \mathcal{F}$ via

$$(w \otimes x) * (y \otimes z) = (\omega'_{|x|}, \omega_{|y|})^{-1} (w * y) \otimes (x * z). \tag{4.12}$$

Furthermore, we have $\Delta\Psi = (\Psi \otimes \Psi)\Delta_{r,s}$.

Proof. It is enough to prove the claim when $x, y \in \mathcal{W}$. In this case, we write $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ and $\Delta(y) = \sum_{(y)} y_1 \otimes y_2$. We note that for any $i \in I$, we have:

$$\Delta(xi) = \Delta(x) \cdot (i \otimes 1) + 1 \otimes xi = \sum_{(x)} x_1 i \otimes x_2 + 1 \otimes xi,$$

where $\cdot$ denotes the term-wise multiplication on $\mathcal{F} \otimes \mathcal{F}$:

$$(w \otimes x) \cdot (y \otimes z) = wy \otimes xz. \tag{4.13}$$

We shall now proceed by induction on $\mathrm{ht}(|x|) + \mathrm{ht}(|y|) = m$. The cases $m = 0$, $m = 1$, as well as when one of x, y has length zero are trivial. Suppose now $x, y \in \mathcal{W}$ satisfy $\mathrm{ht}(|x|), \mathrm{ht}(|y|) \geq 1$ and $m = \mathrm{ht}(|x|) + \mathrm{ht}(|y|)$, and the claim holds for all $x', y' \in \mathcal{W}$ with $\mathrm{ht}(|x'|) + \mathrm{ht}(|y'|) < m$. By this assumption, we can write $x = x'i$ and $y = y'j$ for some $i, j \in I$ and $x', y' \in \mathcal{W}$. Then, we have:

$$\begin{aligned}
&\Delta((x'i) * (y'j)) = \Delta((x' * (y'j))i) + \Delta((\omega'_{|x'i|}, \omega_j)^{-1}((x'i) * y')j) \\
&\ = \Delta(x' * (y'j)) \cdot (i \otimes 1) + 1 \otimes (x' * (y'j))i + (\omega'_{|x'i|}, \omega_j)^{-1} \left(\Delta((x'i) * y') \cdot (j \otimes 1) + 1 \otimes ((x'i) * y')j\right),
\end{aligned}$$

so the induction hypothesis implies that

$$\begin{aligned}
\Delta((x'i) * (y'j)) &= (\Delta(x') * \Delta(y'j)) \cdot (i \otimes 1) + (\omega'_{|x'i|}, \omega_j)^{-1} (\Delta(x'i) * \Delta(y')) \cdot (j \otimes 1) + 1 \otimes ((x'i) * (y'j)) \\
&= \left(\left(\sum_{(x')} x'_1 \otimes x'_2 \right) * \left(\sum_{(y')} (y'_1 j) \otimes y'_2 + 1 \otimes (y'j) \right) \right) \cdot (i \otimes 1) \\
&\quad + (\omega'_{|x'i|}, \omega_j)^{-1} \left(\left(\sum_{(x')} (x'_1 i) \otimes x'_2 + 1 \otimes (x'i) \right) * \left(\sum_{(y')} y'_1 \otimes y'_2 \right) \right) \cdot (j \otimes 1) \\
&\quad + 1 \otimes ((x'i) * (y'j)) \\
&= \sum_{(x')(y')} (\omega'_{|x'_2|}, \omega_{|y'_1 j|})^{-1} (x'_1 * (y'_1 j)) i \otimes (x'_2 * y'_2) + \sum_{(x')} (x'_1 i) \otimes (x'_2 * (y'j)) \\
&\quad + (\omega'_{|x'i|}, \omega_j)^{-1} \sum_{(x')(y')} (\omega'_{|x'_2|}, \omega_{|y'_1|})^{-1} ((x'_1 i) * y'_1) j \otimes (x'_2 * y'_2) \\
&\quad + (\omega'_{|x'i|}, \omega_j)^{-1} \sum_{(y')} (\omega'_{|x'i|}, \omega_{|y'_1|})^{-1} (y'_1 j) \otimes ((x'i) * y'_2) + 1 \otimes ((x'i) * (y'j)) \\
&= \sum_{(x')(y')} (\omega'_{|x'_2|}, \omega_{|y'_1 j|})^{-1} ((x'_1 i) * (y'_1 j)) \otimes (x'_2 * y'_2) \\
&\quad + \left(\sum_{(x')} (x'_1 i) \otimes x'_2 \right) * (1 \otimes (y'j)) + (1 \otimes (x'i)) * \left(\sum_{(y')} (y'_1 j) \otimes y'_2 \right) + 1 \otimes ((x'i) * (y'j)) \\
&= \left(\sum_{(x')} (x'_1 i) \otimes x'_2 + 1 \otimes (x'i) \right) * \left(\sum_{(y')} (y'_1 j) \otimes y'_2 + 1 \otimes (y'j) \right) = \Delta(x'i) * \Delta(y'j).
\end{aligned}$$

This completes the proof of $\Delta(x * y) = \Delta(x) * \Delta(y)$.

As per the equality $\Delta\Psi = (\Psi \otimes \Psi)\Delta_{r,s} \colon U^+_{r,s} \to \mathcal{F} \otimes \mathcal{F}$, it suffices to verify its validity on the generators, where it immediately follows from $\Delta(\Psi(e_i)) = [i] \otimes \emptyset + \emptyset \otimes [i] = (\Psi \otimes \Psi)(\Delta_{r,s}(e_i))$ for each $i \in I$. ■

5. Orthogonal Bases

This Section closely follows [CHW, Sections 4-5], which in turn is largely based on [L]. So we shall only highlight the key changes in the present setup.

5.1. **Dominant and Lyndon words.**

From now on, we fix an order $\leq$ on the alphabet I, which induces a lexicographical order on the monoid $\mathcal{W}$. For a nonzero $x \in \mathcal{F}$, its *leading term* $\max(x)$, is a word $w \in \mathcal{W}$ such that $x = \sum_{u \leq w} t_u \cdot u$ with $t_u \in \mathbb{C}(r,s)$ and $t_w \neq 0$. Following the terminology of [CHW, §4.1], we call a word $w \in \mathcal{W}$ *dominant* if it appears as a leading term of some element from $\mathcal{U}$. We use $\mathcal{W}^+$ to denote the subset of $\mathcal{W}$ consisting of all dominant words. Then we have the following basic result, proved exactly as in [L, Proposition 12], cf. [CHW, Proposition 4.1]:

Proposition 5.1. *(1) There is a unique basis of homogeneous vectors $\{m_w \,|\, w \in \mathcal{W}^+\}$ in $\mathcal{U}$ such that for all $w_1, w_2 \in \mathcal{W}^+$ with $|w_1| = |w_2|$, we have (cf. (4.7)):*

$$\epsilon'_{w_1}(m_{w_2}) = \delta_{w_1,w_2}.$$

(2) The set $\left\{e_w = e_{i_1} \dots e_{i_d} \,\middle|\, w = [i_1 \dots i_d] \in \mathcal{W}^+\right\}$ is a basis of $U^+_{r,s}$.

A word $w = [i_1 \dots i_d]$ is called *Lyndon* if it is smaller than any of its proper right factors:

$$w < [i_k \dots i_d] \qquad \forall\, 1 < k \leq d.$$

We use $\mathcal{L}$ to denote the set of all Lyndon words. It is well-known that any word w admits a unique factorization as a product of non-increasing Lyndon words:

$$w = \ell_1 \ell_2 \dots \ell_k, \quad \ell_1 \geq \ell_2 \geq \dots \geq \ell_k, \quad \ell_1, \dots, \ell_k \in \mathcal{L}. \tag{5.1}$$

The following Lemma is an extension of [L, Lemma 15] (which covers the case $k = 1$):

Lemma 5.2. *For any $\ell \in \mathcal{L}$ and $w \in \mathcal{W}$ with $\ell \geq w$, we have $\max(\ell^k * w) = \ell^k w$ for all $k \geq 1$.*

Proof. We proceed by induction on k. The base case $k = 1$ is analogous to [CHW, Lemma 4.5]. Suppose that $k > 1$, and the result holds for all smaller values of k. Since all terms in $\ell^k * w$ appear with coefficients in $\mathbb{Z}_{\geq 0}[r^{\pm 1}, s^{\pm 1}]$, it follows that the coefficient of $\ell^k w$ in $\ell^k * w$ is nonzero. Thus, if u is a word in $\ell^k * w$, it suffices to show that $u \leq \ell^k w$. Given such a word u, there is some factorization $w = w_1 w_2$ (with w_1 or w_2 possibly empty) such that u occurs in $(\ell^{k-1} * w_1)(\ell * w_2)$. Then since $\ell \geq w \geq w_1$, the induction hypothesis implies that $\max(\ell^{k-1} * w_1) = \ell^{k-1} w_1$, and therefore u is less than or equal to all words appearing in $\ell^{k-1} w_1 (\ell * w_2)$. Then since every word that appears in $w_1(\ell * w_2)$ also appears in $\ell * w$, and the case $k = 1$ implies that $\max(\ell * w) = \ell w$, all words that appear in $\ell^{k-1} w_1 (\ell * w_2)$ are less than or equal to $\ell^k w$. This implies that $u \leq \ell^k w$, which completes the proof. ∎

Let $\mathcal{L}^+ = \mathcal{W}^+ \cap \mathcal{L}$ be the set of all dominant Lyndon words. Then Lemma 5.2 implies the following result, analogous to [L, Proposition 16] (cf. [CHW, Proposition 4.7]):

Proposition 5.3. *If $\ell \in \mathcal{L}^+$ and $w \in \mathcal{W}^+$ satisfy $\ell \geq w$, then $\ell w \in \mathcal{W}^+$.*

Completely analogously to [L, Proposition 17], we also have:

Proposition 5.4. *A word $w \in \mathcal{W}$ is dominant if and only if it has the form*

$$w = \ell_1 \ell_2 \dots \ell_k, \qquad \ell_1 \geq \ell_2 \geq \dots \geq \ell_k,$$

where $\ell_1, \dots, \ell_k$ are dominant Lyndon words (such a decomposition is unique and coincides with (5.1)).

Finally, due to our earlier dimension count from Proposition 2.8, we also have the following analogue of [L, Proposition 18] (cf. [CHW, Theorem 4.8]):

Theorem 5.5. *The map $\ell \mapsto |\ell|$ defines a bijection from $\mathcal{L}^+$ to Φ^+.*

We shall denote the inverse of this bijection by $\ell \colon \Phi^+ \to \mathcal{L}^+$.

5.2. **Bracketing.**

For homogeneous elements $x, y \in \mathcal{F}$, define their (r,s)-*bracketing*

$$[x, y]_{r,s} = xy - (\omega'_{|y|}, \omega_{|x|}) yx.$$

For a Lyndon word $\ell \in \mathcal{L}$, a decomposition $\ell = \ell_1 \ell_2$ is called a *costandard factorization* if ℓ_1, ℓ_2 are nonempty, $\ell_1 \in \mathcal{L}$, and the length of ℓ_1 is the maximal possible. In this case, it is known that ℓ_2 is also a Lyndon word. Following [L, §4.1] and [CHW, §4.3], given a Lyndon word ℓ we define its bracketing $[\ell] \in \mathcal{F}$ inductively via:

- $[\ell] = \ell$ if $\ell \in I$,

- $[\ell] = [[\ell_1], [\ell_2]]_{r,s}$ if $\ell = \ell_1\ell_2$ is the costandard factorization of ℓ.

Evoking the *canonical factorization* of (5.1), we define the bracketing of any word $w \in \mathcal{W}$ via:

$$[w] = [\ell_1][\ell_2] \dots [\ell_k].$$

Finally, we also define $\Xi\colon (\mathcal{F}, \cdot) \to (\mathcal{F}, *)$ as the algebra homomorphism given by $\Xi([i_1 \dots i_d]) = i_1 * \cdots * i_d$. Then we have the following three results, whose proofs are exactly the same as those of Proposition 19, Proposition 20, and Lemma 21 in [L]:

Proposition 5.6. *For any $\ell \in \mathcal{L}$, we have $[\ell] = \ell + x$, where x is a linear combination of words $v > \ell$.*

Proposition 5.7. *The set $\big\{[w] \,\big|\, w \in \mathcal{W}\big\}$ is a basis for $\mathcal{F}$.*

Lemma 5.8. *A word $w \in \mathcal{W}$ is dominant if and only if it cannot be expressed modulo* $\ker(\Xi)$ *as a linear combination of words $v > w$.*

For any dominant word $w \in \mathcal{W}^+$, we define

$$R_w = \Xi([w]).$$

For any $x, y \in \mathcal{F}$, let us introduce the following notation:

$$x \circledast y = x *_{r,s} y - x *_{s,r} y.$$

Recall from Proposition 4.2 that $x *_{r,s} y = (\omega'_{|x|}, \omega_{|y|})^{-1} y *_{s,r} x$, so we have

$$x \circledast y = x *_{r,s} y - (\omega'_{|y|}, \omega_{|x|}) y *_{r,s} x.$$

This formula immediately implies:

Lemma 5.9. *Let $\ell \in \mathcal{L}^+$, and let $\ell = \ell_1\ell_2$ be its costandard factorization. Then*

$$R_\ell = R_{\ell_1} \circledast R_{\ell_2}.$$

For any word $w = [i_1 \dots i_d]$, set

$$\epsilon_w = i_1 * i_2 * \cdots * i_d.$$

Then completely analogously to [CHW, Proposition 4.13], we obtain:

Proposition 5.10. *For $w \in \mathcal{W}^+$, we have*

$$R_w = \epsilon_w + \sum_{v \in \mathcal{W}^+}^{v > w} \chi_{wv} \epsilon_v$$

for some $\chi_{wv} \in \mathbb{C}(r, s)$. In particular, the set $\{R_w \,|\, w \in \mathcal{W}^+\}$ is a basis for $\mathcal{U}$.

We call $\{R_w \,|\, w \in \mathcal{W}^+\}$ the *Lyndon basis* of $\mathcal{U}$. Due to Proposition 5.4, we have (cf. [L, Theorem 23]):

Proposition 5.11. *The Lyndon basis has the form*

$$\big\{R_{\ell_1} * \cdots * R_{\ell_k} \,\big|\, k \in \mathbb{Z}_{\geq 0},\ \ell_1, \dots, \ell_k \in \mathcal{L}^+,\ \ell_1 \geq \cdots \geq \ell_k\big\}.$$

Let us also recall *Leclerc's algorithm* for computing the set $\mathcal{L}^+$ of dominant Lyndon words. For each $\beta \in \Phi^+$, let

$$C(\beta) = \big\{(\beta_1, \beta_2) \in \Phi^+ \times \Phi^+ \,|\, \beta_1 + \beta_2 = \beta, \ell(\beta_1) < \ell(\beta_2)\big\},$$

where $\ell\colon \mathcal{L}^+ \to \Phi^+$ denotes the inverse of the bijection in Theorem 5.5. Then we have:

Proposition 5.12. *For any $\beta \in \Phi^+$, we have $\ell(\beta) = \max\{\ell(\beta_1)\ell(\beta_2) \,|\, (\beta_1, \beta_2) \in C(\beta)\}$.*

Proof. Because there is a unique dominant Lyndon word of weight β for each $\beta \in \Phi^+$, it will suffice to show that the set $\mathcal{L}^+$ of dominant Lyndon words coincides with the set $\mathcal{GL}$ of good Lyndon words considered in [L]; then the Proposition is just a restatement of [L, Proposition 25].

To avoid confusion with our notation, we will denote the shuffle product $*$ on $\mathcal{F}$ defined in [L] by $*_q$, and we will write $\mathcal{U}^q$ for the image of the embedding of U_q^+ into $\mathcal{F}$ constructed in [L].

Now, let $\mathcal{A}_{r,s} = \mathbb{C}[r^{\pm 1}, s^{\pm 1}]$ and let $\mathcal{A}_q = \mathbb{C}[q, q^{-1}]$. Consider the free $\mathcal{A}_{r,s}$-module $\mathcal{F}_{\mathcal{A}_{r,s}} = \bigoplus_{w \in \mathcal{W}} \mathcal{A}_{r,s} w$ and the free $\mathcal{A}_q$-module $\mathcal{F}_{\mathcal{A}_q} = \bigoplus_{w \in \mathcal{W}} \mathcal{A}_q w$. Note that $\mathcal{F}_{\mathcal{A}_{r,s}}$ and $\mathcal{F}_{\mathcal{A}_q}$ are both subrings of $\mathcal{F}$ under the products $*_{r,s}$ and $*_q$, respectively. Set $\mathcal{U}_{\mathcal{A}r,s} = \mathcal{U} \cap \mathcal{F}_{\mathcal{A}_{r,s}}$ and $\mathcal{U}^q_{\mathcal{A}_q} = \mathcal{U}^q \cap \mathcal{F}_{\mathcal{A}_q}$.

Let $\psi_0 \colon \mathcal{A}_{r,s} \to \mathcal{A}_q$ be the ring homomorphism defined by $\psi_0(r) = q$ and $\psi_0(s) = q^{-1}$. If we then define $\psi \colon \mathcal{F}_{\mathcal{A}_{r,s}} \to \mathcal{F}_{\mathcal{A}_q}$ by $\psi(\sum c_w w) = \sum \psi_0(c_w) w$, then it is immediate from the definitions of $*_{r,s}$ and $*_q$ that ψ is a ring homomorphism. Moreover, if $u = \sum_{w\in\mathcal{W}} \gamma(w) w \in \mathcal{U}_{\mathcal{A}_{r,s}}$, then we know from Proposition 4.7 that the coefficients $\gamma(w)$ satisfy the linear equations (4.8). If we apply ψ_0 to the equations in (4.8), then we precisely recover the linear equations in [L, (12)], and therefore [L, Theorem 5] implies that $\psi(u) = \sum_{w\in\mathcal{W}} \psi_0(\gamma(w)) w \in \mathcal{U}^q_{\mathcal{A}_q}$. Now, for any $\ell \in \mathcal{L}^+$, consider the homogeneous element m_ℓ of the basis $\{m_w \,|\, w \in \mathcal{W}^+\}$ constructed in Proposition 5.1. By the defining properties of $\{m_w \,|\, w \in \mathcal{W}^+\}$, the only element of $\mathcal{W}^+$ that occurs as a summand of m_ℓ is ℓ, and therefore $\ell = \max(m_\ell)$. Since ℓ also occurs with coefficient 1, we have $\max(\psi(m_\ell)) = \ell$, which means that $\ell \in \mathcal{GL}$. This proves that $\mathcal{L}^+ \subseteq \mathcal{GL}$, and since both $\mathcal{L}^+$ and $\mathcal{GL}$ are in bijection with the finite set Φ^+, the inclusion must actually be an equality. ∎

Combining Proposition 5.12 with the convexity result [L, Proposition 26], we obtain (cf. [L, Corollary 27]):

Corollary 5.13. *For $\beta \in \Phi^+$, the dominant Lyndon word $\ell(\beta)$ is the smallest dominant word of weight β.*

The following result is analogous to [CHW, Lemma 4.18] (we choose to present the proof below in order to highlight the only essential calculation):

Lemma 5.14. *Let $\ell = [i_1 \dots i_d] \in \mathcal{L}^+$. Then each word appearing in the expansion of R_ℓ starts with i_1.*

Proof. We proceed by induction on d, with $d = 1$ being trivial. Let $\ell = \ell_1\ell_2$ be the costandard factorization. Then $R_\ell = R_{\ell_1} \circledast R_{\ell_2}$, and by the induction assumption every word appearing in the expansion of R_{ℓ_1} starts with i_1. Now, by [L, Lemma 14], we may write $\ell_2 = \ell_1^k w i$, where $k \geq 0$, w is a (possibly empty) left factor of ℓ_1, and i is a letter such that $wi > \ell_1$. If $k > 0$ or w is nonempty, then ℓ_2 starts with i_1, and thus every word appearing in the expansion of R_{ℓ_2} also starts with i_1. In this case, the definition of the shuffle product implies that every word in $R_{\ell_1} \circledast R_{\ell_2}$ starts with i_1. In the remaining case $\ell_2 = i$, we have $R_{\ell_2} = i$, and for any word $\ell' = [i_1 j_2 \dots j_d]$ in the expansion of R_{ℓ_1}, we obtain:

$$\ell' \circledast i = \ell' *_{r,s} i - (\omega'_i, \omega_{|\ell'|}) i *_{r,s} \ell' = i\ell' - (\omega'_i, \omega_{|\ell'|})(\omega'_i, \omega_{|\ell'|})^{-1} i\ell' + \sum c_w w = \sum c_w w,$$

where in the last sum the words w start with i_1 and $c_w \in \mathbb{C}(r,s)$. This completes the proof. ∎

Finally, the following important lemma is completely analogous to [CHW, Lemma 4.19]:

Lemma 5.15. *For $\ell \in \mathcal{L}^+$, we have* $\max(R_\ell) = \ell$.

Combining this with Lemma 5.2, we get the following result:

Corollary 5.16. *For $w \in \mathcal{W}^+$, we have* $\max(R_w) = w$.

Proof. By Proposition 5.4, we can write $w = \ell_1 \dots \ell_k$, where $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$, and each $\ell_i \in \mathcal{L}^+$. Then by definition, we have $R_w = R_{\ell_1} * \dots * R_{\ell_k}$. We proceed by induction on k. If $k = 1$, the statement reduces to Lemma 5.15. Now suppose $k > 1$, and the result holds for smaller k. Let d be the largest integer such that $\ell_1 = \ell_2 = \dots = \ell_d$. By induction, $\max(R_{\ell_1} * \dots * R_{\ell_{d-1}}) = \ell_1 \dots \ell_{d-1} = \ell_1^{d-1}$, and we know from Lemma 5.15 that $\max(R_{\ell_d}) = \ell_d = \ell_1$. Thus, if u is any word in $R_{\ell_1^{d-1}}$ and v is any word in R_{ℓ_1}, then every word appearing in $u * v$ is less than or equal to the corresponding shuffle in $\ell_1^{d-1} * \ell_1$. Hence, $\max(R_{\ell_1} * \dots * R_{\ell_d}) = \max(\ell_1^{d-1} * \ell_1) = \ell_1^d$, by Lemma 5.2.

If $d = k$, then we are done. If $d < k$, then by the induction assumption, we have $\max(R_{\ell_{d+1}} * \dots * R_{\ell_k}) = \ell_{d+1} \dots \ell_k$, and by the choice of d, $\ell_1 > \ell_{d+1}$. We shall now argue that $\ell_1 > \ell_{d+1} \dots \ell_k$. If this inequality holds, then by using Lemma 5.2 and arguments similar to those above, we will be able to conclude that $\max(R_{\ell_1} * \dots * R_{\ell_k}) = \max(\ell_1^d * (\ell_{d+1} \dots \ell_k)) = \ell_1 \dots \ell_k = w$, which will complete the proof.

Suppose instead that we have $\ell_1 \leq \ell_{d+1} \dots \ell_k$. Then we can write $\ell_1 = \ell_{d+1} \dots \ell_t \ell'_{t+1} v$, where $d \leq t < k$ (if $t = d$, then $\ell_1 = \ell'_{d+1} v$), ℓ'_{t+1} is a proper (and possibly empty) left factor of ℓ_{t+1}, and v is a (possibly empty) word such that $\ell'_{t+1} v \leq \ell_{t+1}$. But then since ℓ_1 is Lyndon and $\ell'_{t+1} v$ is a right factor of ℓ_1, we have $\ell_1 \leq \ell'_{t+1} v \leq \ell_{t+1}$, which is a contradiction. Therefore we must have $\ell_1 > \ell_{d+1} \dots \ell_k$. ∎

5.3. **Orthogonal PBW Bases.**

The following result is an analogue of [CHW, Lemma 5.6] (and the proof is very similar):

Lemma 5.17. *For any $\ell \in \mathcal{L}^+$, we have*

$$\Delta(R_\ell) = \sum_{\ell_1,\ell_2 \in \mathcal{W}^+} \vartheta^{\ell}_{\ell_1,\ell_2} R_{\ell_2} \otimes R_{\ell_1},$$

where $\vartheta^{\ell}_{\ell_1,\ell_2} = 0$ unless $|\ell_1| + |\ell_2| = |\ell|$, $\ell_1 \leq \ell$, and $\ell \leq \ell_2$ whenever $\ell_2 \neq \emptyset$.

Proof. The condition $|\ell_1| + |\ell_2| = |\ell|$ holds because Δ preserves the grading. To prove that $\ell_1 \leq \ell$, note that Lemma 5.15 implies that we can write

$$R_\ell = \sum_{w \leq \ell} \phi_{\ell w} w$$

for some $\phi_{\ell w} \in \mathbb{C}(r,s)$. Then

$$\Delta(R_\ell) = \sum_{w_1,w_2,\, w_1w_2 = w \leq \ell} \phi_{\ell w} w_2 \otimes w_1,$$

and since $w_1 \leq w \leq \ell$, Lemma 5.15 implies that $\vartheta^{\ell}_{\ell_1,\ell_2} = 0$ unless $\ell_1 \leq \ell$. To prove that $\ell \leq \ell_2$ whenever $\ell_2 \neq \emptyset$, we proceed by induction on the length of ℓ. If ℓ is a letter, the claim is obvious. For the induction step, let $\ell = wv$ be the costandard factorization of ℓ, so that $R_\ell = R_w \circledast R_v = R_w *_{r,s} R_v - R_w *_{s,r} R_v$. Since $\{R_w \,|\, w \in \mathcal{W}^+\}$ is a basis for $\mathcal{U}$, we can write $\Delta(R_w *_{r,s} R_v) = \sum_{\mathsf{h},\mathsf{k} \in \mathcal{W}^+} z_{\mathsf{h},\mathsf{k}} R_{\mathsf{h}} \otimes R_{\mathsf{k}}$ for some $z_{\mathsf{h},\mathsf{k}} \in \mathbb{C}(r,s)$. Moreover, interchanging r and s in Proposition 4.10 shows that we have $\Delta(x *_{s,r} y) = \Delta(x) *_{s,r} \Delta(y)$, where

$$(w \otimes x) *_{s,r} (y \otimes z) = (\omega'_{|y|}, \omega_{|x|})(w *_{s,r} y) \otimes (x *_{s,r} z).$$

Thus, $\Delta(R_w *_{s,r} R_v) = \sum_{\mathsf{h},\mathsf{k} \in \mathcal{W}^+} \bar{z}_{\mathsf{h},\mathsf{k}} R_{\mathsf{h}} \otimes R_{\mathsf{k}}$, since $\bar{r} = s$ and $\bar{s} = r$. This implies that

$$\Delta(R_\ell) = \Delta(R_w *_{r,s} R_v - R_w *_{s,r} R_v) = \sum_{\mathsf{h},\mathsf{k} \in \mathcal{W}^+} (z_{\mathsf{h},\mathsf{k}} - \bar{z}_{\mathsf{h},\mathsf{k}}) R_{\mathsf{h}} \otimes R_{\mathsf{k}}.$$

On the other hand, we can also compute $\Delta(R_\ell)$ using the formula $R_\ell = R_w *_{r,s} R_v - (\omega'_{|v|}, \omega_{|w|}) R_v *_{r,s} R_w$. By the induction hypothesis, we have

$$\Delta(R_w *_{r,s} R_v) = \sum \vartheta^{w}_{w_1,w_2} \vartheta^{v}_{v_1,v_2} (R_{w_2} \otimes R_{w_1}) *_{r,s} (R_{v_2} \otimes R_{v_1}),$$

where the sum is over all $w_1, w_2, v_1, v_2 \in \mathcal{W}^+$ satisfying $w_1 \leq w \leq w_2$ unless $w_2 = \emptyset$ and $v_1 \leq v \leq v_2$ unless $v_2 = \emptyset$. Now, by Proposition 5.10, the transition matrix from the basis $\{R_w \,|\, w \in \mathcal{W}^+\}$ to the basis $\{\epsilon_w \,|\, w \in \mathcal{W}^+\}$ is triangular, and therefore we can rewrite the expression above as

$$\Delta(R_w *_{r,s} R_v) = \sum_{\substack{\mathsf{h} \geq w_2v_2 \\ \mathsf{k} \geq w_1v_1}} \Theta_{\mathsf{h},\mathsf{k}} R_{\mathsf{h}} \otimes R_{\mathsf{k}}$$

for some $\Theta_{\mathsf{h},\mathsf{k}} \in \mathbb{C}(r,s)$. Similar arguments show that

$$\Delta(R_v *_{r,s} R_w) = \sum_{\substack{\mathsf{h} \geq v_2w_2 \\ \mathsf{k} \geq v_1w_1}} \Theta'_{\mathsf{h},\mathsf{k}} R_{\mathsf{h}} \otimes R_{\mathsf{k}},$$

where $\Theta'_{\mathsf{h},\mathsf{k}} \in \mathbb{C}(r,s)$.

Since $z_{\mathsf{h},\mathsf{k}} = \Theta_{\mathsf{h},\mathsf{k}}$ by definition, it follows that $\Theta_{\mathsf{h},\mathsf{k}} = 0$ if and only if $\Theta'_{\mathsf{h},\mathsf{k}} = 0$. Indeed, if $\Theta_{\mathsf{h},\mathsf{k}} = 0$, then $z_{\mathsf{h},\mathsf{k}} = \overline{z}_{\mathsf{h},\mathsf{k}} = 0$, so the coefficient of $R_{\mathsf{h}} \otimes R_{\mathsf{k}}$ in the expansion of $\Delta(R_\ell)$ is zero. This implies that $0 = \Theta_{\mathsf{h},\mathsf{k}} - (\omega'_{|v|}, \omega_{|w|})\Theta'_{\mathsf{h},\mathsf{k}} = -(\omega'_{|v|}, \omega_{|w|})\Theta'_{\mathsf{h},\mathsf{k}}$, which forces $\Theta'_{\mathsf{h},\mathsf{k}} = 0$. Conversely, if $\Theta'_{\mathsf{h},\mathsf{k}} = 0$, then we have $\Theta_{\mathsf{h},\mathsf{k}} = z_{\mathsf{h},\mathsf{k}} = z_{\mathsf{h},\mathsf{k}} - \overline{z}_{\mathsf{h},\mathsf{k}}$, and this implies that $\overline{z}_{\mathsf{h},\mathsf{k}} = 0$, so we also have $\Theta_{\mathsf{h},\mathsf{k}} = z_{\mathsf{h},\mathsf{k}} = 0$.

Now, suppose that $z_{\mathsf{h},\mathsf{k}} - \overline{z}_{\mathsf{h},\mathsf{k}} \neq 0$, so there are dominant words w_2, v_2, such that $\mathsf{h} \geq w_2v_2$. If $w_2 \neq \emptyset$, then unless $v_2 = \emptyset$, we get $\mathsf{h} \geq w_2v_2 \geq wv \geq \ell$, as desired. If $v_2 = \emptyset$ but $w_2 > w$, then since w_2 has length at most that of w, we still have $\mathsf{h} \geq w_2 > wv = \ell$. Now suppose that $v_2 = \emptyset$ and $w_2 = w$. Then the term $R_{\mathsf{h}} \otimes R_{\mathsf{k}}$ must have come from $(R_w \otimes 1) *_{r,s} (1 \otimes R_v)$, so that $\mathsf{h} = w$ and $\mathsf{k} = v$. But since v is a right factor of the Lyndon word ℓ, we have $\ell < v$, which contradicts the fact that $\vartheta^{\ell}_{\ell_1,\ell_2} = 0$ unless $\ell_1 \leq \ell$. If $w_2 = \emptyset$, then in the case that $v_2 \neq \emptyset$, we have $\mathsf{h} \geq v_2 \geq v > \ell$ as v is a right factor of ℓ, while in the case that $v_2 = \emptyset$, we have $\mathsf{h} = \emptyset$. This completes the proof of the lemma. ∎

We shall also need the analogue of the above lemma for $\bar{R}_l$, with $^-$ introduced in Proposition 4.8:

Lemma 5.18. *For any $\ell \in \mathcal{L}^+$, we have*

$$\Delta(\bar{R}_\ell) = \sum_{\ell_1,\ell_2 \in \mathcal{W}^+} \hat{\vartheta}^{\ell}_{\ell_1,\ell_2} \bar{R}_{\ell_1} \otimes \bar{R}_{\ell_2},$$

where $\hat{\vartheta}^{\ell}_{\ell_1,\ell_2} = 0$ unless $|\ell_1| + |\ell_2| = |\ell|$, $\ell_1 \leq \ell$, and $\ell \leq \ell_2$ whenever $\ell_2 \neq \emptyset$.

Proof. This follows from Lemma 5.17 once we show that if $\Delta(w) = \sum_i u_i \otimes v_i$, then $\Delta(\bar{w}) = \sum_i \varrho_i \bar{v}_i \otimes \bar{u}_i$ for some $\varrho_i \in \mathbb{C}(r,s)$. To this end, if $w = [i_1 \dots i_d]$, then

$$\begin{aligned}\Delta(\overline{w}) &= \left(\prod_{1 \leq l < m \leq d} (\omega'_{i_m}, \omega_{i_l})^{-1} \right) \sum_{0 \leq k \leq d} [i_k \dots i_1] \otimes [i_d \dots i_{k+1}] \\ &= \sum_{0 \leq k \leq d} \left(\prod_{1 \leq l \leq k < m \leq d} (\omega'_{i_m}, \omega_{i_l})^{-1} \right) \overline{[i_1 \dots i_k]} \otimes \overline{[i_{k+1} \dots i_d]},\end{aligned}$$

as desired. ∎

For any $w \in \mathcal{W}^+$, consider its canonical factorization into dominant Lyndon words (Proposition 5.4)

$$w = w_1 w_2 \dots w_d, \quad w_1 \geq w_2 \geq \dots \geq w_d, \quad w_1, w_2, \dots, w_d \in \mathcal{L}^+. \tag{5.2}$$

Then, we define

$$\begin{aligned}\widetilde{R}_w &= R_{w_d} * R_{w_{d-1}} * \dots * R_{w_1}, \\ \bar{R}_w &= \bar{R}_{w_1} * \bar{R}_{w_2} * \dots * \bar{R}_{w_d}.\end{aligned} \tag{5.3}$$

The following is the key result of this Section (see (2.1) for the notation $r_{|\ell|}, s_{|\ell|}$):

Theorem 5.19. *Let $\ell, w \in \mathcal{W}^+$. Then $(\widetilde{R}_\ell, \bar{R}_w) = 0$ unless $\ell = w$. Moreover, if $\ell = \ell_1^{m_1} \ell_2^{m_2} \dots \ell_h^{m_h}$ with $\ell_1 > \ell_2 > \dots > \ell_h$ is the canonical factorization of ℓ into dominant Lyndon words, then we have*

$$(\widetilde{R}_\ell, \bar{R}_\ell) = \prod_{i=1}^{h} \left([m_i]_{r_{|\ell_i|}, s_{|\ell_i|}}! \, r_{|\ell_i|}^{-\frac{1}{2} m_i (m_i - 1)} (R_{\ell_i}, \bar{R}_{\ell_i})^{m_i} \right). \tag{5.4}$$

The proof of this theorem is similar to that of [CHW, Theorem 5.7]. However, we prefer to present full details because our PBW bases ($\{\widetilde{R}_\ell\}_{\ell \in \mathcal{W}^+}$ and $\{\bar{R}_\ell\}_{\ell \in \mathcal{W}^+}$) are different from those used in [CHW] (see a single basis $\{\mathsf{E}_\mathsf{i}\}_{\mathsf{i} \in \mathsf{W}^+}$ of *loc.cit.*).

Proof. We may assume that $|\ell| = |w|$, because otherwise the claim follows from the basic properties of $(\cdot,\cdot)$. We then proceed by induction on the length of ℓ (which is equal to the length of w since $|\ell| = |w|$). The case when ℓ and w have length 1 is trivial. Suppose first that $\ell \in \mathcal{L}^+$, and $w \in \mathcal{W}^+$ with $|w| = |\ell|$ but $w \neq \ell$. If $w = w_1 \dots w_d$ is the canonical factorization of (5.2), then $d > 1$, for otherwise w would be a dominant Lyndon word of the same degree as the dominant Lyndon word ℓ. Using Lemmas 3.14 and 5.17, we get:

$$\begin{aligned}(\widetilde{R}_\ell, \bar{R}_w) &= (R_\ell, \bar{R}_{w_1} * \dots * \bar{R}_{w_d}) = (\Delta(R_\ell), \bar{R}_{w_d} \otimes (\bar{R}_{w_1} * \dots * \bar{R}_{w_{d-1}})) \\ &= \sum_{\ell_1, \ell_2 \in \mathcal{W}^+} \vartheta^{\ell}_{\ell_1,\ell_2} (R_{\ell_2}, \bar{R}_{w_d})(R_{\ell_1}, \bar{R}_{w_1} * \dots * \bar{R}_{w_{d-1}}).\end{aligned}$$

Since $d > 1$, we must have $|w_d| \neq |\ell|$, so the only nonzero terms in the sum above satisfy $\ell_1 < \ell < \ell_2$, $|w_d| = |\ell_2|$, and $|\ell_1| = |w_1 \dots w_{d-1}|$. As the length of w_d is smaller than the length of w, we may apply the induction assumption to conclude that all nonzero terms in the above sum satisfy $\ell_2 = w_d$ and $\ell_1 = w_1 \dots w_{d-1}$. Then

$$w_1 \leq w_1 \dots w_{d-1} = \ell_1 < \ell_2 = w_d \leq w_1,$$

which never holds. This finally implies that $(\widetilde{R}_\ell, \bar{R}_w) = 0$ for $\ell \in \mathcal{L}^+$ and $w \neq \ell$.

Now suppose that $w \in \mathcal{L}^+$, and $\ell \in \mathcal{W}^+$ satisfies $|\ell| = |w|$ but $\ell \neq w$. Let $\ell = \ell_1 \dots \ell_d$ be the canonical factorization of ℓ, where $\ell_1 \geq \dots \geq \ell_d$ are dominant Lyndon words, cf. (5.2). Using Lemmas 3.14 and 5.17, we likewise obtain:

$$(\widetilde{R}_\ell, \bar{R}_w) = (R_{\ell_d} * \dots * R_{\ell_1}, \bar{R}_w) = \sum_{w_1, w_2 \in \mathcal{W}^+} \hat{\vartheta}^{w}_{w_1,w_2} (R_{\ell_{d-1}} * \dots * R_{\ell_1}, \bar{R}_{w_1})(R_{\ell_d}, \bar{R}_{w_2}).$$

Repeating the above argument, we conclude that the only nonzero terms in the sum above satisfy

$$\ell_1 \leq \ell_1 \dots \ell_{d-1} = w_1 < w_2 = \ell_d \leq \ell_1,$$

which is never possible. This again implies that $(\widetilde{R}_\ell, \bar{R}_w) = 0$ in the present setup.

Now, for arbitrary $\ell, w \in \mathcal{W}^+_\nu$, we proceed by induction on the height of ν. The base case $\nu \in \Pi$ is trivial, so suppose that $\nu \notin \Pi$, and for any $\mu \in Q^+$ with $\mathrm{ht}(\mu) < \mathrm{ht}(\nu)$ and any $u, v \in \mathcal{W}^+_\mu$, we have $(\widetilde{R}_u, \bar{R}_v) = 0$ unless $u = v$. Let $\ell = \ell_1\ell_2 \dots \ell_d$ and $w = w_1w_2 \dots w_e$ be the canonical factorizations of ℓ and w into products of non-increasing dominant Lyndon words. We may assume that $d, e > 1$, for otherwise $\ell \in \mathcal{L}^+$ or $w \in \mathcal{L}^+$, which are the two cases already treated above.

We shall first consider the case when $\ell_1 \leq w_1$. Then, we have:

$$\begin{aligned}(\widetilde{R}_\ell, \bar{R}_w) &= (R_{\ell_d} * \dots * R_{\ell_1}, \bar{R}_{w_1} * \dots * \bar{R}_{w_e}) = (\Delta(R_{\ell_d}) * \dots * \Delta(R_{\ell_1}), (\bar{R}_{w_2} * \dots * \bar{R}_{w_e}) \otimes \bar{R}_{w_1}) \\ &= \sum \vartheta_{\ell_{1,1},\ell_{1,2},\dots,\ell_{d,1},\ell_{d,2}} (R_{\ell_{d,2}} * \dots * R_{\ell_{1,2}}, \bar{R}_{w_2} * \dots * \bar{R}_{w_e})(R_{\ell_{d,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_1}),\end{aligned} \tag{5.5}$$

where every term in the sum above satisfies the following three properties (for all $1 \leq t \leq d$):

$$\ell_{t,1}, \ell_{t,2} \in \mathcal{W}^+, \qquad \ell_{t,1} \leq \ell_t, \qquad \ell_t \leq \ell_{t,2} \quad \text{unless} \quad \ell_{t,2} = \emptyset,$$

and $\vartheta_{\ell_{1,1},\dots,\ell_{d,2}} \in \mathbb{C}(r, s)$. For a particular term in (5.5), we shall first show that $(R_{\ell_{d,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_1}) = 0$ unless there is a unique k such that $\ell_{k,1} = w_1$ and $\ell_{t,1} = \emptyset$ for all $t \neq k$, cf. [CHW, Claim(**)].

To this end, let k be the maximal integer such that $\ell_{k,1} \neq \emptyset$. Then

$$\begin{aligned}(R_{\ell_{d,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_1}) &= (R_{\ell_{k,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_1}) \\ &= \sum_{w_{1,1}, w_{1,2} \in \mathcal{W}^+} \hat{\vartheta}^{w_1}_{w_{1,1},w_{1,2}} (R_{\ell_{k-1,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_{1,1}})(R_{\ell_{k,1}}, \bar{R}_{w_{1,2}}).\end{aligned}$$

Suppose that a particular term

$$\hat{\vartheta}^{w_1}_{w_{1,1},w_{1,2}} (R_{\ell_{k-1,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_{1,1}})(R_{\ell_{k,1}}, \bar{R}_{w_{1,2}})$$

is nonzero. Then $|\ell_{k,1}| \leq |\ell_k| < |\ell|$ and $|w_{1,2}| \leq |w_1| < |w|$, so by induction on the weight, we have $(R_{\ell_{k,1}}, \bar{R}_{w_{1,2}}) = 0$ unless $\ell_{k,1} = w_{1,2}$. In this case, $w_{1,2} \neq \emptyset$, so that $w_{1,2} \geq w_1$. Thus, we obtain:

$$w_{1,2} = \ell_{k,1} \leq \ell_k \leq \ell_1 \leq w_1 \leq w_{1,2},$$

which shows that $\ell_{k,1} = w_{1,2} = w_1$, and so $w_{1,1} = \emptyset$. This also implies that $\ell_{t,1} = \emptyset$ for all $1 \leq t < k$.

Thus, if $(R_{\ell_{d,1}} * \dots * R_{\ell_{1,1}}, \bar{R}_{w_1}) \neq 0$, then there is a unique k such that $\ell_{k,1} = w_1$ and $\ell_{t,1} = \emptyset$ for $t \neq k$, and since $w_1 = \ell_{k,1} \leq \ell_k \leq \ell_1 \leq w_1$, we also have $\ell_{k,1} = \ell_k = \ell_1 = w_1$. This means that for the corresponding words $\ell_{t,2}$, we have $\ell_{t,2} = \ell_t$ if $t \neq k$, and $\ell_{k,2} = \emptyset$.

Now, let m_1 be the largest integer such that $\ell_1 = \ell_2 = \dots = \ell_{m_1}$. Then combining what we proved above and using the obvious equalities $\vartheta^\ell_{\ell,\emptyset} = \vartheta^\ell_{\emptyset,\ell} = 1$, (5.5) reduces to

$$\begin{aligned}(\widetilde{R}_\ell, \bar{R}_w) &= (R_{\ell_d} * \dots * R_{\ell_1}, \bar{R}_{w_1} * \dots * \bar{R}_{w_e}) \\ &= \sum_{k=1}^{m_1} \Big((R_{\ell_d} \otimes 1) * \dots * (R_{\ell_{k+1}} \otimes 1) * (1 \otimes R_{\ell_1}) * (R^{*(k-1)}_{\ell_1} \otimes 1), (\bar{R}_{w_2} * \dots * \bar{R}_{w_e}) \otimes \bar{R}_{\ell_1} \Big) \\ &= \left(\sum_{k=1}^{m_1} (\omega'_{|\ell_1|}, \omega^{k-1}_{|\ell_1|})^{-1} \right) \left(R_{\ell_d} * \dots * R_{\ell_2}, \bar{R}_{w_2} * \dots * \bar{R}_{w_e} \right) (R_{\ell_1}, \bar{R}_{\ell_1}),\end{aligned}$$

where we used (4.12) in the last equality. So by induction, $(\widetilde{R}_\ell, \bar{R}_w) = 0$ unless $\ell = w$, and if $\ell = w$, then using $(\omega'_{|\ell_1|}, \omega_{|\ell_1|})^{-1} = r^{-1}_{|\ell_1|} s_{|\ell_1|}$ (cf. (2.1)) and the induction assumption, we obtain

$$\begin{aligned}&(\widetilde{R}_\ell, \bar{R}_\ell) \\ &= \frac{[m_1]_{r_{|\ell_1|}, s_{|\ell_1|}}}{r^{m_1-1}_{|\ell_1|}} \prod_{i=2}^{h} \left([m_i]_{r_{|\ell_i|}, s_{|\ell_i|}}! \, r_{|\ell_i|}^{-\frac{1}{2}m_i(m_i-1)} (R_{\ell_i}, \bar{R}_{\ell_i})^{m_i} \right) [m_1 - 1]_{r_{|\ell_i|}, s_{|\ell_i|}}! \, r_{|\ell_i|}^{-\frac{1}{2}(m_1-1)(m_1-2)} (R_{\ell_1}, \bar{R}_{\ell_1})^{m_1} \\ &= \prod_{i=1}^{h} \left([m_i]_{r_{|\ell_i|}, s_{|\ell_i|}}! \, r_{|\ell_i|}^{-\frac{1}{2}m_i(m_i-1)} (R_{\ell_i}, \bar{R}_{\ell_i})^{m_i} \right),\end{aligned}$$

where $\ell = \ell_1^{m_1}\ell_2^{m_2}\dots\ell_h^{m_h}$ is the canonical factorization of ℓ into dominant Lyndon words with $\ell_1 > \dots > \ell_h$.

We now consider the case when $\ell_1 > w_1$. Then

$$\begin{aligned}(\widetilde{R}_\ell, \bar{R}_w) &= (R_{\ell_d} * \dots * R_{\ell_1}, \bar{R}_{w_1} * \dots * \bar{R}_{w_e}) = (R_{\ell_1} \otimes (R_{\ell_d} * \dots * R_{\ell_2}), \Delta(\bar{R}_{w_1}) * \dots * \Delta(\bar{R}_{w_e})) \\ &= \sum \hat{\vartheta}_{w_{1,1},w_{1,2},\dots,w_{e,1},w_{e,2}} (R_{\ell_1}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{e,1}})(R_{\ell_d} * \dots * R_{\ell_2}, \bar{R}_{w_{1,2}} * \dots * \bar{R}_{w_{e,2}}),\end{aligned}$$

and as in (5.5) every term in the sum above satisfies (for all $1 \le t \le e$):

$$w_{t,1}, w_{t,2} \in \mathcal{W}^+, \qquad w_{t,1} \le w_t, \qquad w_t \le w_{t,2} \quad \text{unless} \quad w_{t,2} = \emptyset.$$

Suppose that $(R_{\ell_1}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{e,1}}) \neq 0$ for some term in the above sum. Since $\ell \neq \emptyset$, $\ell_1 \neq \emptyset$, we have $w_{t,1} \neq \emptyset$ for at least one integer t; let k be the maximal such integer. Then, we have:

$$\begin{aligned}(R_{\ell_1}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{e,1}}) &= (R_{\ell_1}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{k,1}}) \\ &= \sum_{\ell_{1,1},\ell_{1,2} \in \mathcal{W}^+} \vartheta^{\ell_1}_{\ell_{1,1},\ell_{1,2}} (R_{\ell_{1,2}}, \bar{R}_{w_{k,1}})(R_{\ell_{1,1}}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{k-1,1}}).\end{aligned}$$

Suppose that a particular term

$$\vartheta^{\ell_1}_{\ell_{1,1},\ell_{1,2}} (R_{\ell_{1,2}}, \bar{R}_{w_{k,1}})(R_{\ell_{1,1}}, \bar{R}_{w_{1,1}} * \dots * \bar{R}_{w_{k-1,1}})$$

is nonzero. We have $|w_{k,1}| \le |w_k| < |w|$ and $|\ell_{1,2}| \le |\ell_1| < |\ell|$, so by induction on the weight, it follows that $\ell_{1,2} = w_{k,1}$ for this term. In particular, this shows that $\ell_{1,2} \neq 0$, so that $\ell_{1,2} \ge \ell_1$. But then we obtain

$$w_1 < \ell_1 \le \ell_{1,2} = w_{k,1} \le w_k \le w_1,$$

a contradiction. Hence, $(\widetilde{R}_\ell, \bar{R}_w) = 0$ when $\ell_1 > w_1$, which exhausts all cases and proves the theorem. ■

In the next Section, we shall explicitly evaluate the constants $\{(R_\ell, \bar{R}_\ell)\}_{\ell \in \mathcal{L}^+}$ for a specific ordering on I. Combining this with the theorem above, we shall derive our main Theorems 7.1 and 7.2 in the last Section.

6. Root Vectors and their Pairings

In this Section, we explicitly compute R_ℓ and the pairing $(R_\ell, \bar{R}_\ell)$ for each dominant Lyndon word $\ell \in \mathcal{L}^+$, with the ordering $1 < \dots < n$ on $I = \{1, \dots, n\}$. Similarly to [CHW, §6], we treat classical types case-by-case. We shall use the notation for positive roots that was introduced in (2.16)–(2.19), and we also follow (2.15).

6.1. Type A_n.

For the order $1 < 2 < \dots < n$, the dominant Lyndon words in type A_n are given by (cf. [L, §8.1]):

Lemma 6.1. *The set of dominant Lyndon words is*

$$\mathcal{L}^+ = \big\{[i \dots j] \,\big|\, 1 \le i \le j \le n\big\}.$$

We shall now explicitly evaluate the corresponding elements R_ℓ:

Proposition 6.2. *For $\ell = [i \dots j]$ with $1 \le i \le j \le n$, we have*

$$R_\ell = (r - s)^{j-i}[i \dots j].$$

Proof. We proceed by induction on $j - i$, with the case $j - i = 0$ being obvious. For $j - i > 0$, the costandard factorization $\ell = \ell_1\ell_2$ is explicitly given by $\ell_1 = [i \dots (j-1)]$ and $\ell_2 = [j]$. Thus, by the induction assumption, we have:

$$\begin{aligned}R_\ell &= R_{\ell_1} \circledast R_{\ell_2} = (r - s)^{j-1-i}[i \dots j-1] \circledast [j] \\ &= (r - s)^{j-1-i}\left([i \dots j-1] * [j] - (\omega'_j, \omega_{\gamma_{i,j-1}})[j] * [i \dots j-1]\right) \\ &= (r - s)^{j-1-i}\Big((\omega'_{\gamma_{i,j-1}}, \omega_j)^{-1} - (\omega'_j, \omega_{\gamma_{i,j-1}})\Big)[i \dots j] + (r - s)^{j-1-i}([i \dots (j-2)] \circledast [j])[j-1],\end{aligned}$$

which yields the result because $(\omega'_{\gamma_{i,j-1}}, \omega_j)^{-1} = r$, $(\omega'_j, \omega_{\gamma_{i,j-1}}) = s$, and $[i \dots (j-2)] \circledast [j] = 0$. ■

Finally, let us derive the formula for the pairing of the above elements:

Corollary 6.3. *For $\ell = [i \dots j]$ with $1 \le i \le j \le n$, we have*

$$(R_\ell, \bar{R}_\ell) = (r - s)^{j-i}.$$

Proof. We proceed by induction on $j-i$, the case $j=i$ being clear. Evoking the costandard factorization of ℓ (with $\ell_1=[i\dots(j-1)]$, $\ell_2=[j]$) and Lemma 3.14, we obtain:

$$\begin{aligned}(R_\ell,\bar{R}_\ell)&=(r-s)^{j-i}([i\dots j],\bar{R}_{\ell_1}*[j]-r[j]*\bar{R}_{\ell_1})=(r-s)^{j-i}(\Delta([i\dots j]),[j]\otimes\bar{R}_{\ell_1}-r\bar{R}_{\ell_1}\otimes[j])\\&=(r-s)^{j-i}([j]\otimes[i\dots(j-1)],[j]\otimes\bar{R}_{\ell_1})=(r-s)(R_{\ell_1},\bar{R}_{\ell_1})=(r-s)^{j-i},\end{aligned}$$

where the last equality follows from $R_{\ell_1}=(r-s)^{j-i-1}[i\dots(j-1)]$ and the induction hypothesis. ∎

6.2. **Type B_n.**

For the order $1<2<\dots<n$, the dominant Lyndon words in type B_n are given by (cf. [CHW, §6.2]):

Lemma 6.4. *The set of dominant Lyndon words is*

$$\mathcal{L}^+=\big\{[i\dots j]\,\big|\,1\le i\le j\le n\big\}\cup\big\{[i\dots n\,n\dots j]\,\big|\,1\le i<j\le n\big\}.$$

We shall now explicitly evaluate the corresponding elements R_ℓ:

Proposition 6.5. *(1) For $\ell=[i\dots j]$ with $1\le i\le j\le n$, we have*

$$R_\ell=(r^2-s^2)^{j-i}[i\dots j].$$

(2) For $\ell=[i\dots n\,n\dots j]$ with $1\le i<j\le n$, we have

$$R_\ell=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1}[i\dots n\,n\dots j].$$

Proof. Because $(\omega'_j,\omega_{\gamma_{i,j-1}})=s^2$ and $(\omega'_{\gamma_{i,j-1}},\omega_j)^{-1}=r^2$ for all $1\le i\le j\le n$, the proof of part (1) is exactly the same as the proof of Proposition 6.2. For (2), we proceed by induction on $n-j$. If $n-j=0$, then $\ell=[i\dots n\,n]$, and its costandard factorization is $\ell=\ell_1\ell_2$ where $\ell_1=[i\dots n]$ and $\ell_2=[n]$. Thus:

$$\begin{aligned}R_\ell&=R_{\ell_1}\circledast R_{\ell_2}=(r^2-s^2)^{n-i}\left([i\dots n]*[n]-(\omega'_n,\omega_{\gamma_{in}})[n]*[i\dots n]\right)\\&=(r^2-s^2)^{n-i}\left((\omega'_{\gamma_{in}},\omega_n)^{-1}[i\dots n\,n]+([i\dots(n-1)]*[n])[n]\right)\\&\quad-(r^2-s^2)^{n-i}\left((\omega'_n,\omega_{\gamma_{in}})[i\dots n\,n]+(\omega'_n,\omega_{\gamma_{in}})(\omega'_n,\omega_n)^{-1}([n]*[i\dots(n-1)])[n]\right)\\&=(r^2-s^2)^{n-i}\left(([i\dots(n-1)]*[n])[n]-s^2([n]*[i\dots(n-1)])[n]\right)\\&=(r^2-s^2)R_{\ell_1}[n]=(r^2-s^2)^{n+1-i}[i\dots n\,n],\end{aligned}$$

where we use the equalities $(\omega'_{\gamma_{in}},\omega_n)^{-1}=(\omega'_n,\omega_{\gamma_{in}})=rs$ and $(\omega'_n,\omega_n)^{-1}=r^{-1}s$ in the fourth line.

Now suppose that $n-j>0$ and the result holds for all larger j. Then, the costandard factorization of $\ell=[i\dots n\,n\dots j]$ is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots n\,n\dots(j+1)]$ and $\ell_2=[j]$. Thus, by the induction assumption:

$$\begin{aligned}R_\ell&=R_{\ell_1}\circledast R_{\ell_2}\\&=(rs)^{2(j+1-n)}(r^2-s^2)^{2n-i-j}([i\dots n\,n\dots(j+1)]*[j]-(\omega'_j,\omega_{\beta_{i,j+1}})[j]*[i\dots n\,n\dots(j+1)])\\&=(rs)^{2(j+1-n)}(r^2-s^2)^{2n-i-j}\Big((\omega'_{\beta_{i,j+1}},\omega_j)^{-1}[i\dots n\,n\dots j]+([i\dots n\,n\dots(j+2)]*[j])[j+1]\Big)\\&\quad-(rs)^{2(j+1-n)}(r^2-s^2)^{2n-i-j}\left((\omega'_j,\omega_{\beta_{i,j+1}})[i\dots n\,n\dots j]-(\omega'_j,\omega_{\beta_{i,j+2}})([j]*[i\dots n\,n\dots(j+2)])[j+1]\right)\\&=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1}[i\dots n\,n\dots j]\\&\quad+(rs)^{2(j+1-n)}(r^2-s^2)^{2n-i-j}\left([i\dots n\,n\dots(j+2)]*[j]-[j]*[i\dots n\,n\dots(j+2)]\right)[j+1],\end{aligned}$$

where we used the equalities $(\omega'_{\beta_{i,j+1}},\omega_j)^{-1}=s^{-2}$, $(\omega'_j,\omega_{\beta_{i,j+1}})=r^{-2}$, and $(\omega'_j,\omega_{\beta_{i,j+2}})=1$ in the last line. Thus, it suffices to prove that

$$[i\dots n\,n\dots(j+2)]*[j]-[j]*[i\dots n\,n\dots(j+2)]=0$$

(for $j=n-1$, the above equality should be rather interpreted as $[i\dots n]*[n-1]-[n-1]*[i\dots n]=0$). Since $(\omega'_{\beta_{ik}},\omega_j)^{-1}=1$, $(\omega'_{\gamma_{ik}},\omega_j)^{-1}=1$, and $(\omega'_j,\omega_k)^{-1}=1$ for $j+2\le k\le n$, we have

$$\begin{aligned}&[i\dots n\,n\dots(j+2)]*[j]-[j]*[i\dots n\,n\dots(j+2)]=\\&\qquad\qquad([i\dots(j+1)]*[j]-[j]*[i\dots(j+1)])\,[(j+2)\dots n\,n\dots(j+2)]\end{aligned}$$

and thus it remains to verify $[i\dots(j+1)]*[j]-[j]*[i\dots(j+1)]=0$ for $i<j\le n-1$. To this end, we have:

$$\begin{aligned}
&[i\dots(j+1)]*[j]-[j]*[i\dots(j+1)]\\
&=(\omega'_{\gamma_{i,j+1}},\omega_j)^{-1}[i\dots(j+1)j]-[i\dots(j+1)j]+([i\dots j]*[j])[j+1]-(\omega'_j,\omega_{j+1})^{-1}([j]*[i\dots j])[j+1]\\
&=([i\dots j]*[j]-r^2[j]*[i\dots j])[j+1]\\
&=((\omega'_{\gamma_{ij}},\omega_j)^{-1}[i\dots j\,j]+([i\dots(j-1)]*[j])[j]-r^2[i\dots j\,j]-s^2([j]*[i\dots(j-1)])[j])[j+1]\\
&=(s^2-r^2)[i\dots j\,j(j+1)]+\frac{1}{(r^2-s^2)^{j-1-i}}R_{[i\dots j]}[j(j+1)]\\
&=(s^2-r^2)[i\dots j\,j(j+1)]+(r^2-s^2)[i\dots j\,j(j+1)]=0.
\end{aligned}$$

This completes the proof of (2). ■

Finally, let us derive the formula for the pairing of the above elements:

Corollary 6.6. *(1) For $\ell=[i\dots j]$ with $1\le i\le j\le n$, we have*

$$(R_\ell,\bar R_\ell)=(r^2-s^2)^{j-i}.$$

(2) For $\ell=[i\dots n\,n\dots j]$ with $1\le i<j\le n$, we have

$$(R_\ell,\bar R_\ell)=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1}.$$

Proof. The proof of (1) is the same as the proof of Corollary 6.3. For (2), we proceed by induction on $n-j$. If $n-j=0$, then $\ell=[i\dots n\,n]$ and its costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots n]$ and $\ell_2=[n]$. Therefore, we get:

$$\begin{aligned}
(R_\ell,\bar R_\ell)&=(r^2-s^2)^{n+1-i}([i\dots n\,n],\bar R_{\ell_1}*[n]-rs[n]*\bar R_{\ell_1})\\
&=(r^2-s^2)^{n+1-i}(\Delta([i\dots n\,n]),[n]\otimes\bar R_{\ell_1}-rs\bar R_{\ell_1}\otimes[n])\\
&=(r^2-s^2)^{n+1-i}([n]\otimes[i\dots n],[n]\otimes\bar R_{\ell_1})\\
&=(r^2-s^2)([n],[n])(R_{\ell_1},\bar R_{\ell_1})=(r^2-s^2)^{n+1-i},
\end{aligned}$$

where the last equality is a consequence of part (1).

If $n-j>0$, then the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots n\,n\dots(j+1)]$, $\ell_2=[j]$. Thus:

$$\begin{aligned}
(R_\ell,\bar R_\ell)&=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1}([i\dots n\,n\dots j],\bar R_{\ell_1}*[j]-s^{-2}[j]*\bar R_{\ell_1})\\
&=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1}([j]\otimes[i\dots n\,n\dots(j+1)],[j]\otimes\bar R_{\ell_1})\\
&=(rs)^{-2}(r^2-s^2)([j],[j])(R_{\ell_1},\bar R_{\ell_1})\\
&=(rs)^{2(j-n)}(r^2-s^2)^{2n-i-j+1},
\end{aligned}$$

where the last equality follows from the induction hypothesis. ■

6.3. **Type C_n.**

For the order $1<2<\dots<n$, the dominant Lyndon words in type C_n are given by (cf. [CHW, §6.3]):

Lemma 6.7. *The set of dominant Lyndon words is*

$$\mathcal{L}^+=\{[i\dots j]\,|\,1\le i\le j\le n\}\cup\{[i\dots n\dots j]\,|\,1\le i<j<n\}\cup\{[i\dots(n-1)\,i\dots n]\,|\,1\le i<n\}.$$

We shall now explicitly evaluate the corresponding elements R_ℓ:

Proposition 6.8. *(1) For $\ell=[i\dots j]$ with $1\le i\le j<n$ and $\ell=[n]$, we have*

$$R_\ell=(r-s)^{j-i}[i\dots j].$$

(2) For $\ell=[i\dots n]$ with $1\le i<n$, we have

$$R_\ell=(r-s)^{n-1-i}(r^2-s^2)[i\dots n].$$

(3) For $\ell=[i\dots n\dots j]$ with $1\le i<j<n$, we have

$$R_\ell=(rs)^{j-n}(r-s)^{2n-i-j-1}(r^2-s^2)[i\dots n\dots j].$$

(4) For $\ell=[i\dots(n-1)\,i\dots n]$ with $1\le i<n$, we have

$$R_\ell=(r-s)^{2n-2i-1}(r^2-s^2)r\,([i\dots(n-1)]*[i\dots(n-1)])[n].$$

Proof. The proofs of parts (1) and (2) are similar to the proof of Proposition 6.2, while the proof of part (3) is similar to the proof of part (2) of Proposition 6.5. For part (4), we first note that $[i\dots(n-1)]\in\mathcal{U}$ by part (1), and therefore $[i\dots(n-1)]*[i\dots(n-1)]\in\mathcal{U}$ as well. Moreover, it follows from Proposition 4.7 that also $f=([i\dots(n-1)]*[i\dots(n-1)])[n]\in\mathcal{U}$. Now, by Lemma 5.2, we have $\max(f)=\ell$. Then, since Lemmas 5.2 and 5.15 together imply that $\max(R_w)=w$ for all $w\in\mathcal{W}^+$, we may write

$$f=\sum_{w\leq\ell,\ w\in\mathcal{W}^+}\vartheta_w R_w$$

for some $\vartheta_w\in\mathbb{C}(r,s)$ with $\vartheta_\ell\neq 0$. But ℓ is the smallest dominant word of its degree, so we must have

$$R_\ell=\vartheta_\ell^{-1}f=\vartheta_\ell^{-1}([i\dots(n-1)]*[i\dots(n-1)])[n].$$

Using this, we can now compute R_ℓ. Since the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots(n-1)]$ and $\ell_2=[i\dots n]$, we have:

$$\begin{aligned}R_\ell=R_{\ell_1}\circledast R_{\ell_2}&=(r-s)^{2n-2i-2}(r^2-s^2)\left([i\dots(n-1)]*[i\dots n]-rs[i\dots n]*[i\dots(n-1)]\right)\\&=(r-s)^{2n-2i-2}(r^2-s^2)\left(([i\dots(n-2)]*[i\dots n])[n-1]-r([i\dots n]*[i\dots(n-2)])[n-1]\right)\\&\quad+(r-s)^{2n-2i-2}(r^2-s^2)(r^2-rs)([i\dots(n-1)]*[i\dots(n-1)])[n].\end{aligned}$$

Since we know that R_ℓ is a multiple of $([i\dots(n-1)]*[i\dots(n-1)])[n]$, it follows that

$$([i\dots(n-2)]*[i\dots n])[n-1]-r([i\dots n]*[i\dots(n-2)])[n-1]=0,$$

and therefore

$$R_\ell=(r-s)^{2n-2i-1}(r^2-s^2)r\,([i\dots(n-1)]*[i\dots(n-1)])[n],$$

which completes the proof. ■

Finally, let us derive the formula for the pairing of the above elements:

Corollary 6.9. *(1) For $\ell=[i\dots j]$ with $1\leq i\leq j<n$ and $\ell=[n]$, we have*

$$(R_\ell,\bar{R}_\ell)=(r-s)^{j-i}.$$

(2) For $\ell=[i\dots n]$, we have

$$(R_\ell,\bar{R}_\ell)=(r-s)^{n-1-i}(r^2-s^2).$$

(3) For $\ell=[i\dots n\dots j]$ with $1\leq i<j<n$, we have

$$(R_\ell,\bar{R}_\ell)=(r-s)^{2n-i-j-1}(r^2-s^2)(rs)^{j-n}.$$

(4) For $\ell=[i\dots(n-1)\,i\dots n]$ with $1\leq i<n$, we have

$$(R_\ell,\bar{R}_\ell)=(r-s)^{2n-2i-1}(r^2-s^2)(r+s).$$

Proof. The proofs of parts (1)–(3) are similar to the arguments given in the preceding two Subsections, so we shall only present the details for part (4). First, we note that (cf. (4.13))

$$\begin{aligned}&\Delta(([i\dots(n-1)]*[i\dots(n-1)])[n])\\&\qquad=(\Delta([i\dots(n-1)])*\Delta([i\dots(n-1)]))\cdot([n]\otimes 1)+1\otimes([i\dots(n-1)]*[i\dots(n-1)])[n].\end{aligned}$$

Then, since the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots(n-1)]$ and $\ell_2=[i\dots n]$, we have

$$\begin{aligned}(R_\ell,\bar{R}_\ell)&=(r-s)^{2n-2i-1}(r^2-s^2)r\left(\Delta(([i\dots(n-1)]*[i\dots(n-1)])[n]),\bar{R}_{\ell_2}\otimes\bar{R}_{\ell_1}-rs\bar{R}_{\ell_1}\otimes\bar{R}_{\ell_2}\right)\\&=(r-s)^{2n-2i-1}(r^2-s^2)r\left((\Delta([i\dots(n-1)])*\Delta([i\dots(n-1)]))\cdot([n]\otimes 1),\bar{R}_{\ell_2}\otimes\bar{R}_{\ell_1}\right)\\&=(r-s)^{2n-2i-1}(r^2-s^2)r\left((1+r^{-1}s)([i\dots n]\otimes[i\dots(n-1)]),\bar{R}_{\ell_2}\otimes\bar{R}_{\ell_1}\right)\\&=(r-s)^{2n-2i-1}(r^2-s^2)(r+s)\cdot\frac{1}{(r-s)^{2n-2i-2}(r^2-s^2)}(R_{\ell_2},\bar{R}_{\ell_2})(R_{\ell_1},\bar{R}_{\ell_1})\\&=(r-s)^{2n-2i-1}(r^2-s^2)(r+s),\end{aligned}$$

where the last equality follows from parts (1)–(2). ■

6.4. **Type D_n.**

For the order $1 < 2 < \cdots < n$, the dominant Lyndon words in type D_n are given by (cf. [CHW, §6.4]):

Lemma 6.10. *The set of dominant Lyndon words is*

$$\mathcal{L}^+ = \big\{[i\dots j]\,\big|\, 1\le i\le j<n\big\}\cup\big\{[i\dots(n-2)\,n]\,\big|\,1\le i\le n-2\big\}\cup \big\{[i\dots(n-2)\,n\,(n-1)\dots j]\,\big|\,1\le i<j\le n-1\big\}.$$

We shall now explicitly evaluate the corresponding elements R_ℓ:

Proposition 6.11. *(1) For $\ell=[i\dots j]$ with $1\le i\le j<n$, we have*

$$R_\ell=(r-s)^{j-i}[i\dots j].$$

(2) For $\ell=[i\dots(n-2)\,n]$ with $1\le i\le n-2$, we have

$$R_\ell=(r-s)^{n-1-i}[i\dots(n-2)\,n].$$

(3) For $\ell=[i\dots(n-2)\,n\,(n-1)\dots j]$ with $1\le i<j\le n-1$, we have

$$R_\ell=(rs)^{j+1-n}(r-s)^{2n-i-j-1}\Big([i\dots(n-1)\,n\,(n-2)\dots j]+(rs)^{-1}[i\dots(n-2)\,n\,(n-1)\dots j]\Big).$$

Proof. The computations for parts (1)–(2) are completely analogous to those in the proof of Proposition 6.2, so we shall only provide the details for part (3). We proceed by induction on $n-j$. If $n-j=1$, then the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots(n-2)\,n]$ and $\ell_2=[n-1]$. Therefore, we obtain:

$$\begin{aligned}
R_\ell &= (r-s)^{n-1-i}\left([i\dots(n-2)\,n]*[n-1]-(\omega'_{n-1},\omega_{\beta_{in}})[n-1]*[i\dots(n-2)\,n]\right)\\
&= (r-s)^{n-1-i}\left((\omega'_{\beta_{in}},\omega_{n-1})^{-1}-(\omega'_{n-1},\omega_{\beta_{in}})\right)[i\dots(n-2)\,n\,(n-1)]\\
&\quad+(r-s)^{n-1-i}\left(([i\dots(n-2)]*[n-1])[n]-r^{-1}(\omega'_{n-1},\omega_n)^{-1}([n-1]*[i\dots(n-2)])[n]\right)\\
&= (r-s)^{n-1-i}(s^{-1}-r^{-1})[i\dots(n-2)\,n\,(n-1)]\\
&\quad+(r-s)^{n-1-i}\left([i\dots(n-2)]*[n-1]-s[n-1]*[i\dots(n-2)]\right)[n]\\
&= (r-s)^{n-i}(rs)^{-1}[i\dots(n-2)\,n\,(n-1)]+(r-s)R_{[i\dots(n-1)]}[n]\\
&= (r-s)^{n-i}\big([i\dots(n-1)\,n]+(rs)^{-1}[i\dots(n-2)\,n\,(n-1)]\big).
\end{aligned}$$

Now, if $n-j>1$, the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\dots(n-2)\,n\,\dots(j+1)]$ and $\ell_2=[j]$. Thus, by induction hypothesis, we have:

$$\begin{aligned}
R_\ell &= (rs)^{j+2-n}(r-s)^{2n-i-j-2}[i\dots(n-2)(n-1)\,n\,(n-2)\dots(j+1)]\circledast[j]\\
&\quad+(rs)^{j+1-n}(r-s)^{2n-i-j-2}[i\dots(n-2)\,n\,(n-1)(n-2)\dots(j+1)]\circledast[j].
\end{aligned}$$

Note first that

$$\begin{aligned}
&[i\dots(n-1)\,n\,(n-2)\dots(j+1)]\circledast[j]\\
&=[i\dots(n-1)\,n\,(n-2)\dots(j+1)]*[j]-r^{-1}[j]*[i\dots(n-1)\,n\,(n-2)\ \dots(j+1)]\\
&=(s^{-1}-r^{-1})[i\dots(n-1)\,n\,(n-2)\dots j]+([i\dots(n-1)\,n\,(n-2)\dots(j+2)]\circledast[j])[j+1].
\end{aligned}$$

Moreover, as $(\omega'_{\beta_{ik}},\omega_j)^{-1}=1$, $(\omega'_k,\omega_j)^{-1}=1$, and $(\omega'_j,\omega_k)^{-1}=1$ for all $k\ge j+2$, we have:

$$[i\dots(n-1)\,n\,(n-2)\dots(j+2)]\circledast[j]=([i\dots(j+1)]*[j]-[j]*[i\dots(j+1)])[(j+2)\dots(n-1)\,n\,(n-2)\dots(j+2)].$$

Finally, computations analogous to those in the proof of Proposition 6.5(2) show that

$$[i\dots(j+1)]*[j]-[j]*[i\dots(j+1)]=0\qquad\text{for all}\quad i<j\le n-2,$$

so that

$$[i\dots(n-1)\,n\,(n-2)\dots(j+1)]\circledast[j]=(s^{-1}-r^{-1})[i\dots(n-1)\,n\,(n-2)\dots j].$$

Similarly,

$$\begin{aligned}
&[i\dots(n-2)\,n\,(n-1)\dots(j+1)]\circledast[j]\\
&=(s^{-1}-r^{-1})[i\dots(n-2)\,n\,(n-1)\dots j]+([i\dots(n-2)\,n\,(n-1)\dots(j+2)]\circledast[j])[j+1],
\end{aligned}$$

and as above, we have

$$[i\ldots(n-2)\,n\,(n-1)\ldots(j+2)]\circledast[j]= \\ ([i\ldots(j+1)]*[j]-[j]*[i\ldots(j+1)])[(j+2)\ldots(n-2)\,n\,(n-1)\ldots(j+2)]=0.$$

Thus, we obtain:

$$\begin{aligned}R_\ell &= (rs)^{j+2-n}(r-s)^{2n-i-j-2}(s^{-1}-r^{-1})[i\ldots(n-2)(n-1)\,n\,(n-2)\ldots j]\\ &\quad+(rs)^{j+1-n}(r-s)^{2n-i-j-2}(s^{-1}-r^{-1})[i\ldots(n-2)\,n\,(n-1)(n-2)\ldots j]\\ &=(rs)^{j+1-n}(r-s)^{2n-i-j-1}\big([i\ldots(n-1)\,n\,(n-2)\ldots j]+(rs)^{-1}[i\ldots(n-2)\,n\,(n-1)\ldots j]\big),\end{aligned}$$

which completes the proof. ∎

Finally, let us derive the formula for the pairing of the above elements:

Corollary 6.12. *(1) For $\ell=[i\ldots j]$ with $1\le i\le j<n$, we have*

$$(R_\ell,\bar R_\ell)=(r-s)^{j-i}.$$

(2) For $\ell=[i\ldots(n-2)\,n]$ with $1\le i\le n-2$, we have

$$(R_\ell,\bar R_\ell)=(r-s)^{n-1-i}.$$

(3) For $\ell=[i\ldots(n-2)\,n\,(n-1)\ldots j]$ with $1\le i<j\le n-1$, we have

$$(R_\ell,\bar R_\ell)=(r-s)^{2n-i-j-1}(rs)^{j-n}.$$

Proof. The proofs of parts (1)–(2) are similar to the previous computations, so we shall omit the details. For part (3), we proceed by induction on $n-j$. If $n-j=1$, then the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\ldots(n-2)\,n]$ and $\ell_2=[n-1]$, so that

$$\begin{aligned}(R_\ell,\bar R_\ell)&=(\Delta(R_\ell),[n-1]\otimes\bar R_{\ell_1}-s^{-1}\bar R_{\ell_1}\otimes[n-1])\\ &=(r-s)^{n-i}(\Delta([i\ldots(n-1)\,n]+(rs)^{-1}[i\ldots(n-2)\,n\,(n-1)]),[n-1]\otimes\bar R_{\ell_1}-s^{-1}\bar R_{\ell_1}\otimes[n-1])\\ &=(r-s)^{n-i}(rs)^{-1}([n-1]\otimes[i\ldots(n-2)\,n],[n-1]\otimes\bar R_{\ell_1})\\ &=(r-s)(rs)^{-1}(R_{\ell_1},\bar R_{\ell_1})=(r-s)^{n-i}(rs)^{-1}.\end{aligned}$$

If $n-j>1$, then the costandard factorization is $\ell=\ell_1\ell_2$ with $\ell_1=[i\ldots(n-2)\,n\,\ldots(j+1)]$ and $\ell_2=[j]$, so that we have

$$\begin{aligned}(R_\ell,\bar R_\ell)&=(\Delta(R_\ell),[j]\otimes\bar R_{\ell_1}-s^{-1}\bar R_{\ell_1}\otimes[j])\\ &=(rs)^{j+1-n}(r-s)^{2n-i-j-1}(\Delta([i\ldots(n-1)\,n\,(n-2)\ldots j]),[j]\otimes\bar R_{\ell_1}-s^{-1}\bar R_{\ell_1}\otimes[j])\\ &\quad+(rs)^{j+1-n}(r-s)^{2n-i-j-1}((rs)^{-1}\Delta([i\ldots(n-2)\,n\,(n-1)\ldots j]),[j]\otimes\bar R_{\ell_1}-s^{-1}\bar R_{\ell_1}\otimes[j])\\ &=(rs)^{j+1-n}(r-s)^{2n-i-j-1}([i\ldots(n-1)\,n\,(n-2)\ldots(j+1)],\bar R_{\ell_1})\\ &\quad+(rs)^{j+1-n}(r-s)^{2n-i-j-1}((rs)^{-1}[i\ldots(n-2)\,n\,(n-1)\ldots(j+1)],\bar R_{\ell_1})\\ &=(rs)^{-1}(r-s)(R_{\ell_1},\bar R_{\ell_1})=(rs)^{j-n}(r-s)^{2n-i-j-1},\end{aligned}$$

where the last equality follows from the induction hypothesis. ∎

7. Orthogonal PBW Bases for $U_{r,s}(\mathfrak{g})$

In this Section, we transfer (using Theorem 3.17) the orthogonal bases constructed in Section 5 to $U_{r,s}(\mathfrak{g})$, which, along with the computations from Section 6, proves our main results (Theorem 7.1 and Theorem 7.2).

To state our results, we first need to introduce the corresponding notion of *quantum root vectors* of $U_{r,s}(\mathfrak{g})$. To this end, we recall the notation $\ell\colon\Phi^+\to\mathcal{L}^+$ for the inverse of the bijection from Theorem 5.5. Then, for $\gamma\in\Phi^+$, we define

$$e_\gamma=\overline{\Psi^{-1}\left(\bar R_{\ell(\gamma)}\right)}\qquad\text{and}\qquad f_\gamma=\varphi\left(\Psi^{-1}\left(R_{\ell(\gamma)}\right)\right),$$

where the $\mathbb{C}$-algebra bar-involution $^-$ and the $\mathbb{C}(r,s)$-algebra anti-automorphism φ were introduced in Proposition 2.6. Explicitly, if $\alpha,\beta\in\Phi^+$ are such that $\ell(\gamma)=\ell(\alpha)\ell(\beta)$ is the costandard factorization of $\ell(\gamma)$, then:

$$e_\gamma=e_\alpha e_\beta-(\omega'_\beta,\omega_\alpha)e_\beta e_\alpha\qquad\text{and}\qquad f_\gamma=f_\beta f_\alpha-(\omega'_\alpha,\omega_\beta)^{-1}f_\alpha f_\beta.$$

We also note that the lexicographical ordering on $\mathcal{L}^+$ induces, via the bijection ℓ, a total ordering on Φ^+:

$$\alpha < \beta \quad \Longleftrightarrow \quad \ell(\alpha) < \ell(\beta) \ \text{ lexicographically.} \tag{7.1}$$

For the order $1 < 2 < \cdots < n$, one can easily verify (using Lemmas 6.1, 6.4, 6.7, 6.10) that the corresponding orderings on Φ^+ are as follows:

- *Type A_n*
$$\alpha_1 < \alpha_1 + \alpha_2 < \cdots < \alpha_1 + \cdots + \alpha_n < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_{n-1} + \alpha_n < \alpha_n.$$
- *Type B_n*
$$\begin{aligned} \alpha_1 < \alpha_1 + \alpha_2 < \cdots < \alpha_1 + \cdots + \alpha_n \\ < \alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n < \cdots < \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n \\ < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_{n-1} + \alpha_n < \alpha_{n-1} + 2\alpha_n < \alpha_n. \end{aligned}$$
- *Type C_n*
$$\begin{aligned} \alpha_1 < \alpha_1 + \alpha_2 < \cdots < \alpha_1 + \cdots + \alpha_{n-1} < 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n < \alpha_1 + \cdots + \alpha_n \\ < \alpha_1 + \cdots + \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n < \cdots < \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n \\ < \alpha_2 < \cdots < \alpha_{n-1} < 2\alpha_{n-1} + \alpha_n < \alpha_{n-1} + \alpha_n < \alpha_n. \end{aligned}$$
- *Type D_n*
$$\begin{aligned} \alpha_1 < \alpha_1 + \alpha_2 < \cdots < \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1} < \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n < \alpha_1 + \cdots + \alpha_n \\ < \alpha_1 + \cdots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n < \cdots < \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ < \alpha_2 < \cdots < \alpha_{n-2} < \alpha_{n-2} + \alpha_{n-1} < \alpha_{n-2} + \alpha_n < \alpha_{n-2} + \alpha_{n-1} + \alpha_n < \alpha_{n-1} < \alpha_n. \end{aligned}$$

We may now state our first main result, which corresponds to parts (a) and (b) of [MT, Theorem 5.12]:

Theorem 7.1. *(1) The ordered products*

$$\left\{ \overleftarrow{\prod_{\gamma\in\Phi^+}} e_\gamma^{m_\gamma} \,\Big|\, m_\gamma \geq 0 \right\} \quad \text{and} \quad \left\{ \overleftarrow{\prod_{\gamma\in\Phi^+}} f_\gamma^{m_\gamma} \,\Big|\, m_\gamma \geq 0 \right\} \tag{7.2}$$

are bases for $U^+_{r,s}(\mathfrak{g})$ and $U^-_{r,s}(\mathfrak{g})$, respectively. Here and below, the arrow $\leftarrow$ over the product signs refers to the total order (7.1) on Φ^+, thus ordering the positive roots in decreasing order.

(2) The Hopf pairing (2.9) is orthogonal with respect to these bases. More explicitly, we have:

$$\left(\overleftarrow{\prod_{\gamma\in\Phi^+}} f_\gamma^{n_\gamma}, \overleftarrow{\prod_{\gamma\in\Phi^+}} e_\gamma^{m_\gamma} \right)_H = \prod_{\gamma\in\Phi^+} \left(\delta_{n_\gamma,m_\gamma} [m_\gamma]_{r_\gamma,s_\gamma}! s_\gamma^{-\frac{1}{2}m_\gamma(m_\gamma-1)} (f_\gamma, e_\gamma)_H^{m_\gamma} \right). \tag{7.3}$$

Proof. For part (1), we first recall from Proposition 5.11 that

$$\{R_w \,|\, w \in \mathcal{W}^+\} = \{R_{\ell_1} * \cdots * R_{\ell_k} \,|\, k \in \mathbb{Z}_{\geq 0},\, \ell_1, \ldots, \ell_k \in \mathcal{L}^+,\, \ell_1 \geq \cdots \geq \ell_k\}$$

is a basis for $\mathcal{U}$. Since $x \mapsto \bar{x}$ is a $\mathbb{C}$-algebra automorphism, we find that

$$\{\bar{R}_w \,|\, w \in \mathcal{W}^+\} = \{\bar{R}_{\ell_1} * \cdots * \bar{R}_{\ell_k} \,|\, k \in \mathbb{Z}_{\geq 0},\, \ell_1, \ldots, \ell_k \in \mathcal{L}^+,\, \ell_1 \geq \cdots \geq \ell_k\}. \tag{7.4}$$

is also a basis for $\mathcal{U}$. Then, evoking (5.3), the set

$$\{\widetilde{R}_w \,|\, w \in \mathcal{W}^+\} = \{R_{\ell_k} * \cdots * R_{\ell_1} \,|\, k \in \mathbb{Z}_{\geq 0},\, \ell_1, \ldots, \ell_k \in \mathcal{L}^+,\, \ell_1 \geq \cdots \geq \ell_k\} \tag{7.5}$$

is yet another basis for $\mathcal{U}$ because, by Theorem 5.19, it is orthogonal to $\{\bar{R}_w \,|\, w \in \mathcal{W}^+\}$ with respect to the non-degenerate pairing $(\cdot,\cdot)$ on $\mathcal{U}$.

Thus, applying Ψ^{-1} followed by the bar involution $x \mapsto \bar{x}$ on $U^+_{r,s}(\mathfrak{g})$ to (7.4), we obtain the basis

$$\{e_{|\ell_1|} \ldots e_{|\ell_k|} \,|\, k \in \mathbb{Z}_{\geq 0},\, \ell_1, \ldots, \ell_k \in \mathcal{L}^+,\, \ell_1 \geq \cdots \geq \ell_k\} = \{e_{\gamma_1} \ldots e_{\gamma_k} \,|\, k \in \mathbb{Z}_{\geq 0},\, \gamma_1, \ldots, \gamma_k \in \Phi^+,\, \gamma_1 \geq \cdots \geq \gamma_k\}$$

for $U^+_{r,s}(\mathfrak{g})$, where we use the bijection ℓ and the ordering it induces on Φ via (7.1). Similarly, as $\varphi\colon U^+_{r,s}\to U^-_{r,s}$ is an anti-isomorphism, applying $\varphi\circ(x\mapsto\bar{x})\circ\Psi^{-1}$ to (7.5) yields a basis of $U^-_{r,s}(\mathfrak{g})$:

$$\left\{f_{|\ell_1|}\dots f_{|\ell_k|}\,\middle|\,k\in\mathbb{Z}_{\geq 0},\,\ell_1,\dots,\ell_k\in\mathcal{L}^+,\,\ell_1\geq\dots\geq\ell_k\right\}=\\ \left\{f_{\gamma_1}\dots f_{\gamma_k}\,\middle|\,k\in\mathbb{Z}_{\geq 0},\,\gamma_1,\dots,\gamma_k\in\Phi^+,\,\gamma_1\geq\dots\geq\gamma_k\right\}$$

This completes the proof of part (1).

To prove part (2), we shall use Theorem 3.17. Given sequences $(n_\gamma)_{\gamma\in\Phi^+},(m_\gamma)_{\gamma\in\Phi^+}\in(\mathbb{Z}_{\geq 0})^{\Phi^+}$, consider dominant words $\ell=\overleftarrow{\prod\limits_{\gamma\in\Phi^+}}\ell(\gamma)^{n_\gamma}$ and $w=\overleftarrow{\prod\limits_{\gamma\in\Phi^+}}\ell(\gamma)^{m_\gamma}$. Furthermore, for any $\mu=\sum_{i=1}^n c_i\alpha_i\in\Phi^+$, we set

$$C_\mu=\prod_{i=1}^n\frac{1}{(s_i-r_i)^{c_i}}.$$

We may assume that $|\ell|=|w|$, because otherwise the claim is obvious. Then, since $\overline{\varphi(f_\gamma)}=\Psi^{-1}(R_{\ell(\gamma)})$ and $\bar{e}_\gamma=\Psi^{-1}(\bar{R}_{\ell(\gamma)})$, Theorem 3.17 implies that

$$\left(\overleftarrow{\prod_{\gamma\in\Phi^+}}f_\gamma^{n_\gamma},\overleftarrow{\prod_{\gamma\in\Phi^+}}e_\gamma^{m_\gamma}\right)_H=C_{|\ell|}\overline{\left(\overrightarrow{\prod_{\gamma\in\Phi^+}}\Psi^{-1}\left(R^{n_\gamma}_{\ell(\gamma)}\right),\overleftarrow{\prod_{\gamma\in\Phi^+}}\Psi^{-1}\left(\bar{R}^{m_\gamma}_{\ell(\gamma)}\right)\right)}=C_{|\ell|}\overline{\left(\widetilde{R}_\ell,\bar{R}_w\right)}. \tag{7.6}$$

The last term in (7.6) is zero unless $\ell=w$, i.e. $n_\gamma=m_\gamma$ for all $\gamma\in\Phi^+$, due to the first part of Theorem 5.19. Moreover, if $\ell=w$, then according to the second part of Theorem 5.19, we have

$$(\widetilde{R}_\ell,\bar{R}_w)=\prod_{\gamma\in\Phi^+}\left([m_\gamma]_{r_\gamma,s_\gamma}!\,r_\gamma^{-\frac{1}{2}m_\gamma(m_\gamma-1)}(R_{\ell(\gamma)},\bar{R}_{\ell(\gamma)})^{m_\gamma}\right),$$

and therefore

$$\begin{aligned}C_{|\ell|}\overline{\left(\widetilde{R}_\ell,\bar{R}_w\right)}&=\prod_{\gamma\in\Phi^+}\left([m_\gamma]_{r_\gamma,s_\gamma}!\,s_\gamma^{-\frac{1}{2}m_\gamma(m_\gamma-1)}C_\gamma^{m_\gamma}\overline{(R_{\ell(\gamma)},\bar{R}_{\ell(\gamma)})}^{m_\gamma}\right)\\&=\prod_{\gamma\in\Phi^+}\left([m_\gamma]_{r_\gamma,s_\gamma}!\,s_\gamma^{-\frac{1}{2}m_\gamma(m_\gamma-1)}(f_\gamma,e_\gamma)_H^{m_\gamma}\right),\end{aligned}$$

where the last equality follows from $(f_\gamma,e_\gamma)_H=C_\gamma\overline{(R_{\ell(\gamma)},\bar{R}_{\ell(\gamma)})}$, due to (7.6), a corollary of Theorem 3.17.

This completes the proof. ■

By above theorem, the Hopf pairing $(\cdot,\cdot)_H$ is completely determined by nonzero constants $\{(f_\gamma,e_\gamma)_H\}_{\gamma\in\Phi^+}$. Our second main result is the explicit evaluation of these constants. Below, we use the notation for the positive roots that was introduced in (2.16)–(2.19).

Theorem 7.2. *(1) In type A_n (that is, $\mathfrak{g}=\mathfrak{sl}_{n+1}$), we have*

$$(f_{\gamma_{ij}},e_{\gamma_{ij}})_H=\frac{1}{s-r}\qquad\text{for}\quad 1\leq i\leq j\leq n.$$

(2) In type B_n (that is, $\mathfrak{g}=\mathfrak{so}_{2n+1}$), we have

$$\begin{aligned}(f_{\gamma_{ij}},e_{\gamma_{ij}})_H&=\frac{1}{s^2-r^2}\qquad\text{for}\quad 1\leq i\leq j<n,\\(f_{\gamma_{in}},e_{\gamma_{in}})_H&=\frac{1}{s-r}\qquad\text{for}\quad 1\leq i\leq n,\\(f_{\beta_{ij}},e_{\beta_{ij}})_H&=\frac{[2]^2_{r,s}(rs)^{2(j-n)}}{s^2-r^2}\qquad\text{for}\quad 1\leq i<j\leq n.\end{aligned}$$

(3) In type C_n (that is, $\mathfrak{g} = \mathfrak{sp}_{2n}$), we have

$$(f_{\gamma_{ij}}, e_{\gamma_{ij}})_H = \frac{1}{s-r} \qquad \text{for} \quad 1 \le i \le j \le n \quad \text{with} \quad (i,j) \neq (n,n),$$
$$(f_{\gamma_{nn}}, e_{\gamma_{nn}})_H = \frac{1}{s^2-r^2},$$
$$(f_{\beta_{ii}}, e_{\beta_{ii}})_H = \frac{[2]_{r,s}^2}{s^2-r^2} \qquad \text{for} \quad 1 \le i < n,$$
$$(f_{\beta_{ij}}, e_{\beta_{ij}})_H = \frac{(rs)^{j-n}}{s-r} \qquad \text{for} \quad 1 \le i < j < n.$$

(4) In type D_n (that is, $\mathfrak{g} = \mathfrak{so}_{2n}$), we have

$$(f_{\gamma_{ij}}, e_{\gamma_{ij}})_H = \frac{1}{s-r} \qquad \text{for} \qquad 1 \le i \le j < n,$$
$$(f_{\beta_{ij}}, e_{\beta_{ij}})_H = \frac{(rs)^{j-n}}{s-r} \qquad \text{for} \quad 1 \le i < j \le n.$$

Proof. (1) As $\gamma_{ij} = \alpha_i + \cdots + \alpha_j$ for $1 \le i \le j \le n$, combining Corollary 6.3 and Theorem 3.17, we obtain:

$$(f_{\gamma_{ij}}, e_{\gamma_{ij}})_H = \frac{1}{(s-r)^{j-i+1}} \overline{(R_{\ell(\gamma_{ij})}, \bar{R}_{\ell(\gamma_{ij})})} = \frac{1}{(s-r)^{j-i+1}} (s-r)^{j-i} = \frac{1}{s-r}.$$

(2) The computation for the roots γ_{ij} with $1 \le i \le j < n$ is the same as the one above. For the roots $\gamma_{in} = \alpha_i + \cdots + \alpha_n$, combining Corollary 6.6(1) and Theorem 3.17, we get:

$$(f_{\gamma_{in}}, e_{\gamma_{in}})_H = \frac{1}{(s^2-r^2)^{n-i}} \cdot \frac{1}{s-r} \cdot \overline{(R_{\ell(\gamma_{in})}, \bar{R}_{\ell(\gamma_{in})})} = \frac{1}{s-r}.$$

For the roots $\beta_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$, combining Corollary 6.6(2) and Theorem 3.17, we obtain:

$$(f_{\beta_{ij}}, e_{\beta_{ij}})_H = \frac{1}{(s^2-r^2)^{2n-i-j}} \cdot \frac{1}{(s-r)^2} \cdot \overline{(R_{\ell(\beta_{ij})}, \bar{R}_{\ell(\beta_{ij})})} = \frac{s^2-r^2}{(s-r)^2} (rs)^{2(j-n)} = \frac{[2]_{r,s}^2 (rs)^{2(j-n)}}{s^2-r^2}.$$

(3) We shall only carry out the verification for the roots β_{ii}, since the other formulas are proved as the ones above. For the roots $\beta_{ii} = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$, combining Corollary 6.9(4) and Theorem 3.17, we get:

$$(f_{\beta_{ii}}, e_{\beta_{ii}})_H = \frac{1}{(s-r)^{2n-2i}} \cdot \frac{1}{s^2-r^2} \cdot \overline{(R_{\ell(\beta_{ii})}, \bar{R}_{\ell(\beta_{ii})})} = \frac{1}{s-r}(r+s) = \frac{(r+s)^2}{s^2-r^2} = \frac{[2]_{r,s}^2}{s^2-r^2}.$$

(4) The computation for the roots γ_{ij} with $1 \le i \le j < n$ is the same as the one in (1). For the roots $\beta_{in} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n$, combining Corollary 6.12(2) and Theorem 3.17 gives us

$$(f_{\beta_{in}}, e_{\beta_{in}})_H = \frac{1}{(s-r)^{n-i}} \cdot \overline{(R_{\ell(\beta_{ij})}, \bar{R}_{\ell(\beta_{ij})})} = \frac{1}{s-r}.$$

For the roots $\beta_{i,n-1} = \alpha_i + \cdots + \alpha_n$ with $1 \le i < n-1$, as well as the roots $\beta_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ with $1 \le i < j < n-1$, combining Corollary 6.12(3) and Theorem 3.17 yields:

$$(f_{\beta_{ij}}, e_{\beta_{ij}})_H = \frac{1}{(s-r)^{2n-i-j}} \cdot \overline{(R_{\ell(\beta_{ij})}, \bar{R}_{\ell(\beta_{ij})})} = \frac{(rs)^{j-n}}{s-r}.$$

This completes the proof of this theorem. ∎

Appendix A. General Pairing Formulas for Root Vectors

In this Appendix, we derive a formula for the pairing $(R_\ell, \bar{R}_\ell)$ for certain types of dominant Lyndon words $\ell \in \mathcal{L}^+$ that is valid for any ordering of $I = \{1, \ldots, n\}$. The types of dominant Lyndon words ℓ that we shall mainly concern ourselves with in this Appendix are those whose *first letter occurs exactly once*. We note that this is equivalent to saying that every left factor of ℓ is also Lyndon, so in particular, the costandard factorization of ℓ has the form $\ell = \ell_1 i$, where $i \in I$ is a single letter.

To prove the main result of this Appendix, we shall need a few lemmas on root systems. As in the rest of the paper, Φ is an irreducible reduced root system with an ordered set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.

First, we remind the reader of some basic facts about root systems that we shall use frequently.

Lemma A.1. *(1) For all $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, we have $(\alpha, \beta) \leq 0$.*
(2) If $\alpha, \beta \in \Phi$ and $\alpha \neq \pm\beta$, then $\alpha + \beta \in \Phi$ whenever $(\alpha, \beta) < 0$, and $\alpha - \beta \in \Phi$ whenever $(\alpha, \beta) > 0$.

Lemma A.2. *If $\gamma \in \Phi$, and for distinct $\alpha, \beta \in \Pi$, we have $\gamma + \alpha, \gamma + \beta \in \Phi$, then $\gamma + \alpha + \beta \in \Phi \cup \{0\}$. Likewise, if $\gamma - \alpha, \gamma - \beta \in \Phi$, then $\gamma - \alpha - \beta \in \Phi \cup \{0\}$.*

Proof. If $\gamma+\alpha$ is proportional to β, then we must have $\gamma+\alpha = -\beta$, so that $\gamma+\alpha+\beta = 0$. Similarly, if $\gamma+\beta$ is proportional to α, we have $\gamma+\beta+\alpha = 0$. Thus, we may assume that neither of those two cases hold. Then if $\gamma+\alpha+\beta \notin \Phi$, we must have $(\gamma+\alpha, \beta) \geq 0$ and $(\gamma+\beta, \alpha) \geq 0$. This implies that $(\gamma, \beta)+(\gamma, \alpha) \geq -2(\alpha, \beta)$. But since $\gamma + \beta - (\gamma + \alpha) = \beta - \alpha \notin \Phi$, we must have $(\gamma + \beta, \gamma + \alpha) \leq 0$. Combining this with the previous inequality yields

$$0 \geq (\gamma, \gamma) + (\gamma, \beta) + (\gamma, \alpha) + (\alpha, \beta) \geq (\gamma, \gamma) - (\alpha, \beta).$$

But $(\alpha, \beta) \leq 0$ because α and β are distinct simple roots, and $(\gamma, \gamma) > 0$. Thus, we have a contradiction.

The second claim follows by applying the first one to $-\gamma$. ∎

Lemma A.3. *Let $\gamma \in \Phi^+$, and suppose that for some $i \neq j$, $\beta = \gamma - \alpha_i \in \Phi^+$ and $\beta' = \gamma - \alpha_j \in \Phi^+$. Suppose further that for some $m \leq \mathrm{ht}(\gamma) - 3$, there is a sequence $\alpha_{k_1}, \ldots, \alpha_{k_m} \in \Pi$ such that $\beta - \alpha_{k_1} - \ldots - \alpha_{k_t} \in \Phi^+$ and $\beta' - \alpha_{k_1} - \ldots - \alpha_{k_t} \in \Phi^+$ whenever $0 \leq t \leq m$. Then $\gamma - \alpha_{k_1} - \ldots - \alpha_{k_t} \in \Phi^+$ whenever $0 \leq t \leq m$.*

Proof. We proceed by induction on t. If $t = 0$, then the assertion is obvious. For the induction step, suppose that $\gamma_p = \gamma - \alpha_{k_1} - \ldots - \alpha_{k_p} \in \Phi^+$ for all $p < t$. Let $\beta_p = \beta - \alpha_{k_1} - \ldots - \alpha_{k_p}$ and $\beta'_p = \beta - \alpha_{k_1} - \ldots - \alpha_{k_p}$; by assumption, both of these are positive roots. Moreover, $\beta_{t-1} = \gamma_{t-1} - \alpha_i$ and $\beta'_{t-1} = \gamma_{t-1} - \alpha_j$. Thus, Lemma A.2 implies that $\gamma'_{t-1} = \gamma_{t-1} - \alpha_i - \alpha_j \in \Phi^+$ as well. Now, because $i \neq j$, we have $k_t \neq i$ or $k_t \neq j$. If $k_t \neq i$, then since $\beta'_{t-1} - \alpha_i = \gamma'_{t-1} \in \Phi^+$ and $\beta'_{t-1} - \alpha_{k_t} = \beta'_t \in \Phi^+$, we conclude from Lemma A.2 (and the assumption that $m \leq \mathrm{ht}(\gamma) - 3$ if $t = m$) that $\gamma'_t = \gamma_{t-1} - \alpha_i - \alpha_j - \alpha_{k_t} \in \Phi^+$. Likewise, if $k_t \neq j$, then since $\beta_{t-1} - \alpha_j = \gamma'_{t-1} \in \Phi^+$ and $\beta_{t-1} - \alpha_{k_t} = \beta_t \in \Phi^+$, we again conclude that $\gamma'_t = \gamma_{t-1} - \alpha_i - \alpha_j - \alpha_{k_t} \in \Phi^+$. Finally, since $\gamma'_t + \alpha_i = \beta'_t \in \Phi^+$ and $\gamma'_t + \alpha_j = \beta_t \in \Phi^+$, we have $\gamma'_t + \alpha_i + \alpha_j = \gamma_{t-1} - \alpha_{k_t} = \gamma_t \in \Phi^+$ by Lemma A.2, which completes the proof. ∎

Lemma A.4. *Let $\gamma \in \Phi^+$ be a positive root. Then there are at most three distinct simple roots $\alpha \in \Pi$ such that $\gamma - \alpha \in \Phi^+$.*

Proof. Suppose otherwise, and let $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4} \in \Pi$ be distinct simple roots such that $\gamma - \alpha_{i_k} \in \Phi^+$ for each $k = 1, 2, 3, 4$. Then by Lemma A.2, $\gamma - \alpha_{i_k} - \alpha_{i_l} \in \Phi^+$ whenever $k \neq l$. But the element $\gamma' = 2\gamma - \alpha_{i_1} - \alpha_{i_2} - \alpha_{i_3} - \alpha_{i_4}$ is nonzero, so we have

$$0 < (\gamma', \gamma') = ((\gamma - \alpha_{i_1} - \alpha_{i_2}) + (\gamma - \alpha_{i_3} - \alpha_{i_4}), (\gamma - \alpha_{i_1} - \alpha_{i_3}) + (\gamma - \alpha_{i_2} - \alpha_{i_4})).$$

Then at least one of the values

$$(\gamma - \alpha_{i_1} - \alpha_{i_2}, \gamma - \alpha_{i_1} - \alpha_{i_3}), \qquad (\gamma - \alpha_{i_1} - \alpha_{i_2}, \gamma - \alpha_{i_2} - \alpha_{i_4}),$$
$$(\gamma - \alpha_{i_3} - \alpha_{i_4}, \gamma - \alpha_{i_1} - \alpha_{i_3}), \qquad (\gamma - \alpha_{i_3} - \alpha_{i_4}, \gamma - \alpha_{i_2} - \alpha_{i_4}),$$

must be positive; we can assume without loss of generality that $(\gamma - \alpha_{i_1} - \alpha_{i_2}, \gamma - \alpha_{i_1} - \alpha_{i_3}) > 0$. But $(\gamma - \alpha_{i_1} - \alpha_{i_2}) - (\gamma - \alpha_{i_1} - \alpha_{i_3}) = \alpha_{i_3} - \alpha_{i_2} \notin \Phi$, which contradicts Lemma A.1. ∎

We are now ready to prove two technical lemmas that we shall need for the proofs of Theorems A.8 and A.9. In the proofs below, we will make frequent use of Leclerc's algorithm; the reader should refer to Proposition 5.12 for a reminder of its statement.

Lemma A.5. *Suppose that $\ell = [i_1 \ldots i_d]$ is a dominant Lyndon word such that i_1 occurs exactly once. Suppose that $j \neq i_1, i_d$ and $|\ell| - \alpha_j \in \Phi^+$. Then if k is the smallest integer such that $i_{d-k} = j$, we have $\ell(|\ell| - \alpha_j) = [i_1 \ldots i_{d-k-1} i_{d-k+1} \ldots i_d]$.*

Proof. We first consider the case that $k = 1$. Let $\ell' = \ell(|\ell| - \alpha_j)$, and suppose that the Lemma does not hold. Since $j \neq i_1$, the first letter of ℓ' occurs exactly once, and hence the costandard factorization of ℓ' is $\ell' = \ell'_1 h$ for some $h \in I$. Because every left factor of ℓ is a dominant Lyndon word, we have $\ell(|\ell| - \alpha_{i_d} - \alpha_{i_{d-1}}) = [i_1 \ldots i_{d-2}]$, and therefore by Leclerc's algorithm, $[i_1 \ldots i_{d-2} i_d] < \ell'_1 h$, which forces $[i_1 \ldots i_{d-2}] < \ell'_1$. But using Leclerc's algorithm again, we find that $\ell'_1 h i_{d-1} < [i_1 \ldots i_d]$, which combined with the previous inequality yields

$$\ell'_1 h i_{d-1} < [i_1 \ldots i_d] < \ell'_1,$$

a contradiction.

Now, we proceed by induction on the length of ℓ. If ℓ has length 3 (every word satisfying the above assumptions has length at least 3), then we have $\ell = [i_1 i_2 i_3]$ with $j = i_2$, so this case follows from the first part of the proof.

For the induction step, suppose that the length of ℓ is $d > 3$, and that the Lemma holds for all ℓ of smaller length. We can also assume that $k \geq 2$. For $1 \leq t \leq d$, let $\ell_t = [i_1 \dots i_t]$. Note that by the choice of k, applying Lemma A.2 to $|\ell_{t+1}|$ shows that $|\ell_t| - \alpha_j \in \Phi^+$ whenever $d-k < t < d$. For each such t, set $\ell'_t = \ell(|\ell_t| - \alpha_j)$. Then the word ℓ_{d-1} satisfies the assumptions of the Lemma with i_{d-1} in place of i_d, so by induction, we have $\ell'_{d-1} = [i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-1}]$.

Now, let $\ell'_d = [j_1 \dots j_{d-1}]$. Because i_1 still occurs exactly once in ℓ'_d, and it is the smallest letter of ℓ'_d, it follows that every left factor of ℓ'_d is also dominant Lyndon. Thus, if $j_{d-1} = i_d$, then we must have $[j_1 \dots j_{d-2}] = \ell'_{d-1}$, so in this case we are done. Therefore we can assume that $j_{d-1} \neq i_d$. But then we can apply the induction hypothesis to ℓ'_d with i_d in place of j and j_{d-1} in place of i_d. If e is the smallest integer such that $j_{d-e} = i_d$, then since $\ell'_{d-1} = \ell(|\ell'_d| - \alpha_{i_d})$, the induction hypothesis yields

$$[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-1}] = \ell'_{d-1} = [j_1 \dots j_{d-e-1} j_{d-e+1} \dots j_{d-1}].$$

We now have three cases to consider: $k < e$, $k > e$, and $k = e$. Before proceeding, we introduce the notation $w_t = [j_1 \dots j_t]$ for $1 \leq t \leq d-1$ (so in particular, $w_{d-1} = \ell'_d$), which we shall use in each of the three cases below.

Suppose first that $k < e$. Then $j_p = i_p$ for $1 \leq p \leq d-e-1$ and $d-k+1 \leq p \leq d-1$, and $j_{p+1} = i_p$ for $d-e \leq p \leq d-k-1$. This implies that

$$\ell'_d = w_{d-1} = [i_1 \dots i_{d-e-1} i_d i_{d-e} i_{d-e+1} \dots i_{d-k-1} i_{d-k+1} \dots i_{d-1}].$$

Now, every left factor of ℓ'_d is dominant Lyndon, so in particular, $[i_1 \dots i_{d-e-1} i_d i_{d-e}]$ is a dominant Lyndon word. On the other hand, we also know that $[i_1 \dots i_{d-e}]$ is dominant Lyndon, and therefore Leclerc's algorithm implies that $[i_1 \dots i_{d-e} i_d] < [i_1 \dots i_{d-e-1} i_d i_{d-e}]$, i.e. $i_{d-e} < i_d$. However, this implies that

$$w_{d-1} j > [i_1 \dots i_d] = \ell,$$

contradicting Leclerc's algorithm.

Next, suppose that $k > e$. Then $j_p = i_p$ for $1 \leq p \leq d-k-1$ and $d-e+1 \leq p \leq d-1$, and $j_p = i_{p+1}$ for $d-k \leq p \leq d-e-1$. Thus, it follows that in this case

$$\ell'_d = w_{d-1} = [i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d i_{d-e+1} \dots i_{d-1}].$$

Because every left factor of ℓ'_d is Lyndon, $[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d i_{d-e+1}]$ is a dominant Lyndon word. But so is $[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_{d-e+1}]$ (it is a left factor of ℓ'_{d-1}), so it follows from Leclerc's algorithm that

$$[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_{d-e+1} i_d] < [i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d i_{d-e+1}],$$

and hence $i_d > i_{d-e+1}$.

Now, note that $\alpha_{i_{d-1}}, \alpha_{i_{d-2}}, \dots, \alpha_{i_{d-e+1}}$ is a sequence of simple roots such that $|\ell'_d| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}} \in \Phi^+$ and $|\ell_{d-1}| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}} \in \Phi^+$ whenever $1 \leq t \leq e-1$. Because $d-1 > e$, we also have $e - 1 \leq \mathrm{ht}(|\ell|) - 3 = d-3$, and therefore we are in a position to apply Lemma A.3, which tells us that for $1 \leq t \leq e-1$, $|\ell| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}} \in \Phi^+$. Let $v_{d-t} = \ell(|\ell| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}})$ for $1 \leq t \leq e-1$. Now, since i_1 still occurs exactly once in each v_{d-t}, we know that right factor of its costandard factorization is a single letter. Suppose that the last letter of v_{d-e+1} is $h \neq j$. Then since the dominant Lyndon word $[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d]$ has degree $|v_{d-e+1}| - \alpha_j$, the induction hypothesis implies $h = i_d$. Thus, the last letter of v_{d-e+1} must be either i_d or j, so it follows that v_{d-e+1} is either $[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d j]$ or $[i_1 \dots i_{d-e} i_d]$. Now, note that by Leclerc's algorithm, we have $\ell'_d j < \ell$, and therefore $i_{d-k+1} < j$. This implies that $[i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d j] < [i_1 \dots i_{d-e} i_d]$, so by Leclerc's algorithm, $v_{d-e+1} = [i_1 \dots i_{d-e} i_d]$. Let us now determine v_{d-e+2}. It follows from Lemma A.4 that v_{d-e+2} must be one of the following three words: $v_{d-e+1} i_{d-e+1} = [i_1 \dots i_{d-k-1} j i_{d-k+1} \dots i_{d-e} i_d i_{d-e+1}]$, $w_{d-e+1} j = [i_1 \dots i_{d-k-1} i_{d-k+1} \dots i_{d-e} i_d i_{d-e+1} j]$, or $\ell_{d-e+1} i_d = [i_1 \dots i_{d-k-1} j i_{d-k+1} \dots i_{d-e} i_{d-e+1} i_d]$. But then the inequalities $i_{d-k+1} < j$ and $i_{d-e+1} < i_d$ imply that

$$w_{d-e+1} j < \ell_{d-e+1} i_d < v_{d-e+1} i_{d-e+1},$$

so by Leclerc's algorithm, $v_{d-e+2} = v_{d-e+1}i_{d-e+1} = [i_1 \dots i_{d-e}i_d i_{d-e+1}]$. Continuing alike, we conclude that $v_{d-t} = [i_1 \dots i_{d-e}i_d i_{d-e+1} \dots i_{d-t-1}]$ for $1 \le t \le e-2$, so in particular, $v_{d-1} = [i_1 \dots i_{d-e}i_d i_{d-e+1} \dots i_{d-2}]$. But then since $i_d > i_{d-e+1}$, we have

$$v_{d-1}i_{d-1} = [i_1 \dots i_{d-e}i_d i_{d-e+1} \dots i_{d-1}] > [i_1 \dots i_d] = \ell,$$

which violates Leclerc's algorithm.

Finally, we consider the case $k = e$. Then we have $j_p = i_p$ for all $p \neq d-e$, and therefore

$$\ell'_d = w_{d-1} = [i_1 \dots i_{d-k-1}i_d i_{d-k+1} \dots i_{d-1}].$$

Then $[i_1 \dots i_{d-k-1}i_d i_{d-k+1}]$ is Lyndon. But so is $[i_1 \dots i_{d-k-1}i_{d-k+1}]$, so as in the previous cases we conclude that $i_d > i_{d-k+1}$.

Now, as in the case that $k > e$, we can use Lemma A.3 to conclude that $|\ell| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}} \in \Phi^+$ for $1 \le t \le k-1$, and as before we set $v_{d-t} = \ell(|\ell| - \alpha_{i_{d-1}} - \dots - \alpha_{i_{d-t}})$. Again, we know that the right factor in the costandard factorization of each v_{d-t} is a single letter, and repeating the argument used in the previous case shows that the last letter of v_{d-k+1} is either i_d or j. Therefore v_{d-k+1} is equal to either $[i_1 \dots i_{d-k-1}i_d j]$ or $[i_1 \dots i_{d-k-1}j i_d]$. Since we must have $\ell'_d j < \ell$ by Leclerc's algorithm, it follows that $i_d < j$, and hence $v_{d-k+1} = [i_1 \dots i_{d-k-1}j i_d]$. Now we determine v_{d-k+2}. Using Lemma A.4, we see that there are three possibilities: $v_{d-k+1}i_{d-k+1} = [i_1 \dots i_{d-k-1}j i_d i_{d-k+1}]$, $w_{d-k+1}j = [i_1 \dots i_{d-k-1}i_d i_{d-k+1}j]$, or $\ell_{d-k+1}i_d = [i_1 \dots i_{d-k-1}j i_{d-k+1}i_d]$. Then using the inequalities $i_d < j$ and $i_d > i_{d-k+1}$, we get

$$w_{d-k+1}j < \ell_{d-k+1}i_d < v_{d-k+1}i_{d-k+1},$$

so by Leclerc's algorithm, $v_{d-k+2} = [i_1 \dots i_{d-k-1}j i_d i_{d-k+1}]$. As in the previous case, we can continue in this manner to obtain $v_{d-t} = [i_1 \dots i_{d-k-1}j i_d i_{d-k+1} \dots i_{d-t-1}]$ for $1 \le t \le k-2$. In particular, $v_{d-1} = [i_1 \dots i_{d-k-1}j i_d i_{d-k+1} \dots i_{d-2}]$. But then since $i_d > i_{d-k+1}$, we have

$$v_{d-1}i_{d-1} > [i_1 \dots i_d] = \ell,$$

which violates Leclerc's algorithm. Thus, we have a contradiction in all three cases, so the last letter of ℓ'_d must be i_d and the proof is complete. ∎

For the rest of this Appendix, we shall only need the following slightly weaker corollary to Lemma A.5:

Corollary A.6. *Suppose that $\ell = [i_1 \dots i_d]$ is a dominant Lyndon word such that i_1 occurs exactly once. Suppose that $j \neq i_1, i_d$, and $|\ell| - \alpha_j \in \Phi^+$. Let $\ell' = \ell(|\ell| - \alpha_j)$ and $\ell'' = \ell(|\ell| - \alpha_j - \alpha_{i_d})$. Then the costandard factorization of ℓ' is $\ell' = \ell'' i_d$.*

Proof. Since i_1 occurs once in ℓ', the above statement is equivalent to saying that the last letter of ℓ' is i_d. This is a direct consequence of Lemma A.5. ∎

For the remainder of this Appendix, we will occasionally need to use the notion of the support of an element of $\mathcal{F}$. Given $x \in \mathcal{F}$ and its unique expression $x = \sum_{w \in \mathcal{W}} c_w w$ in terms of the basis $\mathcal{W}$ for $\mathcal{F}$, we define the *support* of x to be the set

$$\mathrm{supp}(x) = \{w \in \mathcal{W} \,|\, c_w \neq 0\}.$$

Below, we will also make frequent use of the notation $\mathbb{C}(r,s)^* = \mathbb{C}(r,s) \setminus \{0\}$.

Lemma A.7. *Let $\ell = [i_1 \dots i_d]$ be a dominant Lyndon word such that i_1 occurs exactly once. Then for any $j \in I \setminus \{i_1\}$ such that $|\ell| - \alpha_j \in \Phi$, we have $\epsilon'_j(R_\ell) \in \mathbb{C}(r,s)^* R_{\ell'}$, where $\ell' = \ell(|\ell| - \alpha_j)$. Furthermore, if $|\ell| - \alpha_j \notin \Phi$, then $\epsilon'_j(R_\ell) = 0$.*

Proof. We proceed by induction on the length of ℓ. If $\ell = i_1$ has length 1, then for all $j \neq i_1$, $\epsilon'_j(R_\ell) = 0$. If ℓ has length 2, then $|\ell| - \alpha_j \notin \Phi$ if and only if $j \neq i_1, i_2$, in which case we clearly have $\epsilon'_j(R_\ell) = 0$. If $j = i_2$, it is easy to verify that $\epsilon'_{i_2}(R_\ell) \in \mathbb{C}(r,s)^* i_1 = \mathbb{C}(r,s)^* R_{i_1}$.

Now suppose that ℓ has length at least 3, and the Lemma holds for any dominant Lyndon word of smaller length that satisfies the assumptions. We know that the costandard factorization of ℓ is $\ell = \ell_1 i_d$, where $\ell_1 = [i_1 \dots i_{d-1}]$. Suppose first that $j \neq i_d, i_1$ and that $|\ell| - \alpha_j \in \Phi$. By Lemma A.2, we also have $|\ell_1| - \alpha_j \in \Phi$, and therefore by Corollary A.6, the costandard factorization of $\ell' = \ell(|\ell| - \alpha_j)$ is $\ell' = \ell'' i_d$, where $\ell'' = \ell(|\ell_1| - \alpha_j)$. This means that $R_{\ell'} = R_{\ell''} * i_d - (\omega'_{i_d}, \omega_{|\ell''|}) i_d * R_{\ell''}$. On the other hand, the induction hypothesis implies that $\epsilon'_j(R_{\ell_1}) = cR_{\ell''}$ for some $c \in \mathbb{C}(r,s)^*$, so we have

$$\begin{aligned}\epsilon'_j(R_\ell) &= \epsilon'_j(R_{\ell_1} * i_d - (\omega'_{i_d}, \omega_{|\ell_1|}) i_d * R_{\ell_1}) = \epsilon'_j(R_{\ell_1}) * i_d - (\omega'_{i_d}, \omega_{|\ell_1|})(\omega'_{i_d}, \omega_j)^{-1} i_d * \epsilon'_j(R_{\ell_1}) \\ &= c(R_{\ell''} * i_d - (\omega'_{i_d}, \omega_{|\ell''|}) i_d * R_{\ell''}) = cR_{\ell'}.\end{aligned}$$

Now suppose that $j = i_d$. Since $\max(R_\ell) = \ell$ by Lemma 5.15, we know that $\epsilon'_{i_d}(R_\ell) \neq 0$, and it contains the dominant Lyndon word $\ell_1 = [i_1 \dots i_{d-1}]$ in its support. Furthermore, if w is any other word in the support of $\epsilon'_{i_d}(R_\ell)$, then we know that $wi_d < \ell$, and hence we must have $w < \ell_1$. This shows that $\max(\epsilon'_{i_d}(R_\ell)) = \ell_1$. However, if $\epsilon'_{i_d}(R_\ell) \notin \mathbb{C}(r,s)^* R_{\ell_1}$, then we can write

$$\epsilon'_{i_d}(R_\ell) = \sum_{w \in \mathcal{W}^+} c_w R_w,$$

where each w in the sum above has degree $|\ell_1|$, and $c_w \neq 0$ for some $w \in \mathcal{W}^+ \setminus \{\ell_1\}$. But by Corollary 5.16, we have $\max(R_w) = w$, and therefore $\ell_1 = \max(\epsilon'_{i_d}(R_\ell)) \geq w$. This is a contradiction, because by Corollary 5.13, ℓ_1 is the smallest dominant word of its degree. Therefore we must have $\epsilon'_{i_d}(R_\ell) \in \mathbb{C}(r,s)^* R_{\ell_1}$.

Finally, suppose that $|\ell| - \alpha_j \notin \Phi$. If $\epsilon'_j(R_\ell) \neq 0$, then we can write

$$\epsilon'_j(R_\ell) = \sum_{w \in \mathcal{W}^+} c_w \widetilde{R}_w,$$

where $|w| = |\ell| - \alpha_j$ whenever $c_w \neq 0$. Then by Theorem 5.19,

$$(\epsilon'_j(R_\ell), \bar{R}_w) = c_w(\widetilde{R}_w, \bar{R}_w)$$

for all $w \in \mathcal{W}^+$. On the other hand, since $(x, y * j) = (\epsilon'_j(x), y)$ for all $x, y \in \mathcal{U}$, we have

$$(\epsilon'_j(R_\ell), \bar{R}_w) = (R_\ell, \bar{R}_w * j).$$

Now, upon transitioning to the monomial basis via Proposition 5.10, we get

$$\bar{R}_w * j = \left(\epsilon_w + \sum_{v \in \mathcal{W}^+}^{v > w} \bar{\chi}_{v,w} \epsilon_v\right) * j = \epsilon_{wj} + \sum_{v \in \mathcal{W}^+}^{v > w} \bar{\chi}_{v,w} \epsilon_{vj},$$

for some $\chi_{v,w} \in \mathbb{C}(r,s)$. Since $v > w$ clearly implies that $vj > wj$, it follows from Proposition 5.10 again that transitioning back to the Lyndon basis yields

$$\bar{R}_w * j = \sum_{u \in \mathcal{W}^+}^{u \geq wj} c_{u,wj} \bar{R}_u,$$

for some $c_{u,wj} \in \mathbb{C}(r,s)$. However, if $c_{u,wj} \neq 0$, then we must have $|u| = |\ell|$, and since ℓ is dominant Lyndon, Corollary 5.13 implies that $\ell \leq u$. Suppose that $\ell = u$ for some $u \in \mathcal{W}^+$ such that $u \geq wj$. Then $\ell \geq wj$. But we know that i_1 occurs first in ℓ, and it is also the smallest letter of both ℓ and wj, so this inequality implies that w starts with i_1. But w is not Lyndon, so we can write $w = w_1 w_2 \dots w_t$ for some $t \geq 2$, where $w_1 \geq w_2 \geq \dots \geq w_t$ and each w_k is dominant Lyndon (see (5.2)). Then each w_k must start with i_1, which contradicts the fact that i_1 occurs only once in w. Therefore we have $\ell < u$ for all u in the sum above with $c_{u,wj} \neq 0$. But this implies that

$$c_w(\widetilde{R}_w, \bar{R}_w) = (\epsilon'_j(R_\ell), \bar{R}_w) = (R_\ell, \bar{R}_w * j) = 0,$$

for all $w \in \mathcal{W}^+$, which is a contradiction. ∎

Theorem A.8. *Let ℓ be a dominant Lyndon word such that the first letter occurs exactly once. Then*

$$\text{(A.1)} \quad \operatorname{supp}(R_\ell) = \left\{ w = [j_1 \dots j_d] \,\middle|\, |w| = |\ell|, \text{ and } j_1 \leq j_k \text{ and } \alpha_{j_1} + \dots + \alpha_{j_k} \in \Phi^+ \text{ for all } 1 \leq k \leq d \right\}.$$

Proof. Denote the set on the right-hand side of (A.1) by $\mathcal{A}_\ell$. Let us first show that $\operatorname{supp}(R_\ell) \subseteq \mathcal{A}_\ell$, which we shall do by induction on the length of ℓ. If ℓ has length 1, then the claim is obvious. Now suppose that $\ell = [i_1 \dots i_d]$ has length $d > 1$, and let $\ell_1 = [i_1 \dots i_{d-1}]$, so that $R_\ell = R_{\ell_1} * i_d - (\omega'_{i_d}, \omega_{|\ell_1|}) i_d * R_{\ell_1}$. We know from Lemma 5.14 that each $w \in \operatorname{supp}(R_\ell)$ begins with i_1, and therefore $j_1 \leq j_k$ for all k if $w = [j_1 \dots j_d] \in \operatorname{supp}(R_\ell)$. Now suppose that $c_w w$ is a term in R_ℓ, where $c_w \neq 0$ and $w = [j_1 \dots j_d]$. Then $c_w[j_1 \dots j_{d-1}]$ is a nonzero term in $\epsilon'_{j_d}(R_\ell)$, so we must have $\alpha_{j_1} + \dots + \alpha_{j_{d-1}} \in \Phi^+$, because otherwise we would have a contradiction to the second part of Lemma A.7. Then by the first part of Lemma A.7, we have $\epsilon'_{j_d}(R_\ell) = cR_{\ell'}$ where $\ell' = \ell(|\ell| - \alpha_{j_d})$ and $c \in \mathbb{C}(r,s)^*$. Then $w' = [j_1 \dots j_{d-1}] \in \mathrm{supp}(R_{\ell'})$, so by induction we have $\alpha_{j_1} + \dots + \alpha_{j_k} \in \Phi^+$ whenever $1 \le k \le d-1$. This completes the proof that $\mathrm{supp}(R_\ell) \subseteq \mathcal{A}_\ell$.

To prove the other inclusion, we again proceed by induction on the length of ℓ, with the base case being obvious. Let $w = [j_1 \dots j_d] \in \mathcal{A}_\ell$. Then $|\ell| - \alpha_{j_d} \in \Phi^+$, so by Lemma A.7, $\epsilon'_{j_d}(R_\ell) = cR_{\ell'}$ for some $c \in \mathbb{C}(r,s)^*$. By induction, $\mathrm{supp}(R_{\ell'}) = \mathcal{A}_{\ell'}$, so in particular, $[j_1 \dots j_{d-1}] \in \mathrm{supp}(R_{\ell'}) = \mathrm{supp}(\epsilon'_{j_d}(R_\ell))$. Hence $[j_1 \dots j_d] \in \mathrm{supp}(R_\ell)$. ∎

For the next theorem, we define the number

$$p_{\alpha,\beta} = \max\left\{k \ge 0 \,|\, \alpha - k\beta \in \Phi\right\}$$

associated to any pair $\alpha, \beta \in \Phi^+$. Note that if Φ is simply-laced and $\alpha + \beta \in \Phi^+$, then $p_{\alpha,\beta} = 0$.

Theorem A.9. *Let ℓ be a dominant Lyndon word such that the first letter of ℓ occurs exactly once, and let $\ell = \ell_1 i$ be its costandard factorization. Then if $p_{|\ell_1|,\alpha_i} = 0$, we have*

$$(R_\ell, \bar{R}_\ell) = \Big((\omega'_{|\ell_1|}, \omega_i)^{-1} - (\omega'_i, \omega_{|\ell_1|})\Big)\, (R_{\ell_1}, \bar{R}_{\ell_1}).$$

Proof. By the definition of $\bar{R}_\ell$, we have

$$\left(R_\ell, \bar{R}_\ell\right) = \left(R_\ell, \bar{R}_{\ell_1} * i\right) - (\omega'_{|\ell_1|}, \omega_i)^{-1} \left(R_\ell, i * \bar{R}_{\ell_1}\right).$$

For the second term, note that, if we write $R_\ell = \sum c_w w$, the definition of Δ implies that

$$\Delta(R_\ell) = \sum_{\substack{w_1, w_2 \in \mathcal{W}, \\ w = w_1 w_2}} c_w w_2 \otimes w_1.$$

By Lemma 5.14, the first letter of ℓ must also be the first letter of w_1 whenever $w = w_1 w_2$ and $c_w \neq 0$. Since i cannot be equal to the first letter of ℓ, we find that

$$(R_\ell, i * \bar{R}_{\ell_1}) = (\Delta(R_\ell), \bar{R}_{\ell_1} \otimes i) = 0.$$

Thus,

$$(R_\ell, \bar{R}_\ell) = (R_\ell, \bar{R}_{\ell_1} * i) = (\epsilon'_i(R_\ell), \bar{R}_{\ell_1}). \tag{A.2}$$

Since $p_{|\ell_1|,\alpha_i} = 0$, Lemma A.7 implies that $\epsilon'_i(R_{\ell_1}) = 0$. Therefore

$$\epsilon'_i(R_\ell) = \epsilon'_i(R_{\ell_1} * i - (\omega'_i, \omega_{|\ell_1|}) i * R_{\ell_1}) = (\omega'_{|\ell_1|}, \omega_i)^{-1} R_{\ell_1} - (\omega'_i, \omega_{|\ell_1|}) R_{\ell_1}.$$

Combining this with (A.2) completes the proof. ∎

Finally, let us describe how Theorem A.9 translates to the Hopf pairing $(\cdot,\cdot)_H$ on $U_{r,s}(\mathfrak{g})$.

Corollary A.10. *Let $\gamma \in \Phi^+$ be a positive root such that the first letter of the dominant Lyndon word $\ell(\gamma)$ occurs exactly once. Let $\alpha, \beta \in \Phi^+$ be such that $\ell(\gamma) = \ell(\alpha)\ell(\beta)$ is the costandard factorization of $\ell(\gamma)$. Then if $p_{\alpha,\beta} = 0$, we have*

$$(f_\gamma, e_\gamma)_H = \left((\omega'_\beta, \omega_\alpha) - (\omega'_\alpha, \omega_\beta)^{-1}\right) (f_\alpha, e_\alpha)_H (f_\beta, e_\beta)_H.$$

Proof. Note first that the conditions on $\ell(\gamma)$ imply that $\beta = \alpha_i \in \Pi$ for some i, and therefore $(f_\beta, e_\beta)_H = \frac{1}{s_i - r_i}$. Suppose that $\alpha = \sum_{j=1}^n c_j \alpha_j$. Then, as in the proof of Theorem 7.2, combining Theorem 3.17 with Theorem A.9 yields

$$\begin{aligned}
(f_\gamma, e_\gamma)_H &= \left(\prod_{j=1}^n \frac{1}{(s_j - r_j)^{c_j}}\right) \cdot \frac{1}{s_i - r_i} \overline{(R_{\ell(\gamma)}, \bar{R}_{\ell(\gamma)})} \\
&= \left(\prod_{j=1}^n \frac{1}{(s_j - r_j)^{c_j}}\right) \cdot \frac{1}{s_i - r_i} \overline{\Big((\omega'_\alpha, \omega_\beta)^{-1} - (\omega'_\beta, \omega_\alpha)\Big)}\, \overline{(R_{\ell(\alpha)}, \bar{R}_{\ell(\alpha)})} \\
&= \left((\omega'_\beta, \omega_\alpha) - (\omega'_\alpha, \omega_\beta)^{-1}\right) (f_\alpha, e_\alpha)_H (f_\beta, e_\beta)_H,
\end{aligned}$$

as desired. ∎

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