STANDARD LYNDON LOOP WORDS: WEIGHTED ORDERS

SEVERYN KHOMYCH, NAZAR KORNIICHUK, KOSTIANTYN MOLOKANOV,
AND ALEXANDER TSYMBALIUK

Abstract. We generalize the study of standard Lyndon loop words from [16] to a more general class of orders on the underlying alphabet, as suggested in [16, Remark 3.15]. The main new ingredient is the exponent-tightness of these words, which also allows to generalize the construction of PBW bases of the untwisted quantum loop algebra $U_q(L\mathfrak{g})$ via the combinatorics of loop words.

1. Introduction

1.1. Summary.

An interesting basis of the free Lie algebra generated by a finite family $\{e_i\}_{i\in I}$ was constructed in the 1950s using the combinatorial notion of *Lyndon* words. A few decades later, this was generalized to any finitely generated Lie algebra $\mathfrak{a}$ in [11]. Explicitly, if $\mathfrak{a}$ is generated by $\{e_i\}_{i\in I}$, then any order on the finite alphabet I gives rise to the combinatorial basis e_ℓ as ℓ ranges through all *standard Lyndon* words.

The key application of [11] was to simple finite-dimensional $\mathfrak{g}$, or more precisely, to its maximal nilpotent subalgebra $\mathfrak{n}^+$. According to the root space decomposition:

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathbb{Q}\cdot e_\alpha\,, \qquad \Delta^+ = \Big\{\text{positive roots}\Big\}, \tag{1.1}$$

with elements e_α called *root vectors*. By the PBW theorem, we thus have

$$U(\mathfrak{n}^+) \ = \bigoplus_{\gamma_1\geq\dots\geq\gamma_k\in\Delta^+}^{k\in\mathbb{N}} \mathbb{Q}\cdot e_{\gamma_1}\dots e_{\gamma_k} \tag{1.2}$$

for any total order on Δ^+. Furthermore, a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^- \tag{1.3}$$

induces the corresponding triangular decomposition of the universal enveloping:

$$U(\mathfrak{g}) = U(\mathfrak{n}^+)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}^-)\,. \tag{1.4}$$

Moreover, the root vectors satisfy (R^* shall denote nonzero elements of a ring R)

$$[e_\alpha,e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha \in \mathbb{Q}^*\cdot e_{\alpha+\beta} \tag{1.5}$$

whenever $\alpha,\beta\in\Delta^+$ satisfy $\alpha+\beta\in\Delta^+$. Thus, formula (1.5) provides an algorithm for constructing all the root vectors (1.1) inductively starting from $e_i = e_{\alpha_i}$, where $\{\alpha_i\}_{i\in I}\subset\Delta^+$ are the simple roots of $\mathfrak{g}$. Therefore, all the root vectors $\{e_\alpha\}_{\alpha\in\Delta^+}$, and hence the PBW basis (1.2), can be read off from the combinatorics of Δ^+.

The above discussion can be naturally adapted to the quantizations. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group of $\mathfrak{g}$, a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$. For one thing, it admits a triangular decomposition similar to (1.4):

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-)\,. \tag{1.6}$$

Here, $U_q(\mathfrak{n}^+)$ is the *positive* subalgebra of $U_q(\mathfrak{g})$, explicitly generated by $\{\widetilde{e}_i\}_{i\in I}$ subject to q-Serre relations. There exists a PBW basis analogous to (1.2):

$$U_q(\mathfrak{n}^+) \;=\; \bigoplus_{\gamma_1\geq\dots\geq\gamma_k\in\Delta^+}^{k\in\mathbb{N}} \mathbb{Q}(q)\cdot\widetilde{e}_{\gamma_1}\dots\widetilde{e}_{\gamma_k}\,.$$

The q-deformed root vectors $\widetilde{e}_\alpha\in U_q(\mathfrak{n}^+)$ are defined via Lusztig's braid group action, which requires one to choose a reduced decomposition of the longest element in the Weyl group of $\mathfrak{g}$. It is well-known ([18]) that this choice precisely ensures that the order $\geq$ on Δ^+ is *convex*, in the sense of Definition 2.17. Moreover, as follows from the Levendorsky-Soibelman property [13], the q-deformed root vectors satisfy the following q-analogue of the relation (1.5):

$$[\widetilde{e}_\alpha,\widetilde{e}_\beta]_q = \widetilde{e}_\alpha\widetilde{e}_\beta - q^{(\alpha,\beta)}\widetilde{e}_\beta\widetilde{e}_\alpha \in \mathbb{Q}(q)^*\cdot\widetilde{e}_{\alpha+\beta} \tag{1.7}$$

whenever $\alpha,\beta,\alpha+\beta\in\Delta^+$ satisfy $\alpha<\alpha+\beta<\beta$ as well as the *minimality* property

$$\nexists\ \alpha',\beta'\in\Delta^+ \quad \text{s.t.} \quad \alpha<\alpha'<\beta'<\beta \quad \text{and} \quad \alpha+\beta=\alpha'+\beta'\,, \tag{1.8}$$

and $(\cdot,\cdot)$ denotes the scalar product corresponding to the root system of type $\mathfrak{g}$. Thus, similarly to the Lie algebra case, we conclude that the q-deformed root vectors can be defined (up to scalar multiples) as iterated q-commutators of $\widetilde{e}_i=\widetilde{e}_{\alpha_i}$ $(i\in I)$, using the combinatorics of Δ^+ and the chosen convex order on it.

Following [7, 21, 24], let us recall that $U_q(\mathfrak{n}^+)$ can be also defined as a subalgebra of the q-shuffle algebra:

$$U_q(\mathfrak{n}^+) \overset{\Phi}{\hookrightarrow} \mathcal{F} \;=\; \bigoplus_{i_1,\dots,i_k\in I}^{k\in\mathbb{N}} \mathbb{Q}(q)\cdot[i_1\dots i_k]\,,$$

where $\mathcal{F}$ has a basis I^*, consisting of finite length words in I, and is endowed with the *quantum shuffle* product. As mentioned above, there is a natural bijection

$$\ell\colon\Delta^+ \xrightarrow{\sim} \Big\{\text{standard Lyndon words}\Big\}, \tag{1.9}$$

established in [11]. This induces the lexicographical order on Δ^+ via

$$\alpha<\beta \quad\Longleftrightarrow\quad \ell(\alpha)<\ell(\beta) \text{ lexicographically}\,.$$

As shown in [12, 22] this total order is convex, and hence can be applied to obtain quantum root vectors $\widetilde{e}_\alpha\in U_q(\mathfrak{n}^+)$ for any positive root α, as in (1.7). Moreover, [12] shows that the quantum root vector $\widetilde{e}_\alpha$ is uniquely characterized (up to a scalar multiple) by the property that $\Phi(e_\alpha)$ is an element of $\mathrm{Im}\,\Phi$ whose leading order term $[i_1\dots i_k]$ (in the lexicographic order) is precisely $\ell(\alpha)$.

It is natural to ask if the above results can be generalized from simple $\mathfrak{g}$ to affine Lie algebras $\widehat{\mathfrak{g}}$. The main complication arises from the fact that not all root subspaces of $\widehat{\mathfrak{g}}$ are one-dimensional. In [1], an analogue of (1.9) was established and all standard Lyndon words were explicitly computed for $\widehat{\mathfrak{g}}$ with $\mathfrak{g}$ of A-type. On the other hand, considering a different (new Drinfeld) "polarization" of quantum loop algebras

$$U_q(L\mathfrak{g}) = U_q(L\mathfrak{n}^+)\otimes U_q(L\mathfrak{h})\otimes U_q(L\mathfrak{n}^-)\,,$$

the above complication disappears as $U_q(L\mathfrak{n}^+)$ is a q-deformation of the universal enveloping algebra of $\mathfrak{n}^+[t,t^{-1}]$ all of which root subspaces are one-dimensional. In particular, many of the above results were adapted to the loop setup in [16].

In this note, we are interested in the generalization of all combinatorial aspects of [16][1], excluding all shuffle algebra considerations, to the so-called "weighted" version. To this end, we order the infinite alphabet $\mathcal{I} = \{i^{(d)} \,|\, i \in I, d \in \mathbb{Z}\}$ via

$$i^{(d)} < j^{(e)} \qquad \Longleftrightarrow \qquad d/c_i > e/c_j \quad \text{or} \quad d/c_i = e/c_j \text{ and } i < j\,, \tag{1.10}$$

for any fixed collection of "weights" $\{c_i\}_{i\in I} \in \mathbb{Z}^I_{>0}$ (the case $c_i = 1\ \forall\, i$ recovers the setup of [16]). This induces the lexicographic order on the loop words $[i_1^{(d_1)} \dots i_k^{(d_k)}]$ with respect to which we may define the notion of *standard Lyndon loop words* by analogy with [11], which though requires some preliminary work similar to [16]. Then, there exists a one-to-one correspondence:

$$\ell\colon \Delta^+ \times \mathbb{Z} \xrightarrow{\sim} \Big\{\text{standard Lyndon loop words}\Big\}.$$

The lexicographic order on the right-hand side induces a convex order on the left-hand side, with respect to which one can define elements

$$e_{\ell(\alpha,d)} \in U_q(L\mathfrak{n}^+) \tag{1.11}$$

for all $(\alpha, d) \in \Delta^+ \times \mathbb{Z}$. We have the following analogue of the PBW theorem:

$$U_q(L\mathfrak{n}^+) \;=\; \bigoplus_{\ell_1 \geq \dots \geq \ell_k \text{ standard Lyndon loop words}}^{k\in\mathbb{N}} \mathbb{Q}(q)\cdot e_{\ell_1}\dots e_{\ell_k}\,. \tag{1.12}$$

There are also analogues of the constructions above with $+ \leftrightarrow -$ and $e \leftrightarrow f$.

By analogy with the results of [12, 22], the total order on $\Delta^+ \times \mathbb{Z}$ given by

$$(\alpha, d) < (\beta, e) \quad \Longleftrightarrow \quad \ell(\alpha, -d) < \ell(\beta, -e) \quad \text{lexicographically} \tag{1.13}$$

is convex, cf. Proposition 3.18. As such, this order comes from a certain reduced word in the affine Weyl group associated to $\mathfrak{g}$ (= the Coxeter group associated to $\widehat{\mathfrak{g}}$), in accordance with Theorem 4.7. Therefore, the root vectors (1.11) exactly match (up to constants) the classical construction of [2, 4, 15], once we pass it through the "affine to loop" isomorphism of Theorem 5.14.

1.2. **Outline.**

The structure of the present paper is the following:

- In Section 2, we recall the notion of (standard) Lyndon words, their basic properties, and the application to simple Lie algebras through the bijection (1.9).
- In Section 3, we study the loop Lie algebras $L\mathfrak{g}$ and generalize the results of the previous Section to the loop setup with the order given by (1.10). The key new ingredient, in comparison to [16], is played by Theorem 3.6 and Proposition 3.8.
- In Section 4, we show that the order (1.13) on $\Delta^+ \times \mathbb{Z}$ corresponds to a certain reduced decomposition in the extended affine Weyl group of $\mathfrak{g}$. We further refine this result in Propositions 4.9–4.10.
- In Section 5, we construct PBW-type bases (1.12) of the quantum loop algebra $U_q(L\mathfrak{g})$ by adapting the arguments of [16] with the help of Proposition 4.10.
- In Section 6, we adapt most of our results to more general orders (6.1) on $\mathcal{I}$.

[1]We shall be using the results of [16, Section 5] that are omitted in its journal version [17].

- In Appendix A, we provide a link to the C++ code and explain how it inductively computes standard Lyndon loop words in all types, and present some examples.

1.3. **Acknowledgement.**
This note is written as a part of the project under the Yulia's Dream initiative, a subdivision of the MIT PRIMES program aimed at Ukrainian high-school students. We are very grateful to the organizers Pavel Etingof, Slava Gerovich, and Dmytro Matvieievskyi for giving us the opportunity to participate in the Yulia's Dream program. S.K., N.K., K.M. are thankful to their parents for their help and support.

A.T. is deeply indebted to Andrei Neguţ for sharing his invaluable insights, teaching the Lyndon word's theory, and stimulating discussions through the entire project. A.T. is grateful to INdAM-GNSAGA and FAR UNIMORE project CUP-E93C2300204000 for the support and wonderful working conditions during his visit to Italy in June 2024, where the final version of this note was prepared.

The work of A.T. was partially supported by an NSF Grant DMS-2302661.

2. Combinatorial approach to Lie algebras

In this Section, we recall the results of [11] and [12] that provide a combinatorial construction of an important basis of finitely generated Lie algebras, with the main application to the maximal nilpotent subalgebra of a simple Lie algebra.

2.1. **Lyndon words.**
Let I be a finite ordered alphabet, and let I^* be the set of all finite length words in the alphabet I. For $u = [i_1 \dots i_k] \in I^*$, we define its *length* by $|u| = k$. We introduce the *lexicographical order* on I^* in a standard way:

$$[i_1 \dots i_k] < [j_1 \dots j_l] \quad \text{if} \quad \begin{cases} i_1 = j_1, \dots, i_a = j_a, i_{a+1} < j_{a+1} \text{ for some } a \geq 0 \\ \text{ or} \\ i_1 = j_1, \dots, i_k = j_k \text{ and } k < l \end{cases} .$$

Definition 2.2. *A word $\ell = [i_1 \dots i_k]$ is called* <u>Lyndon</u> *if it is smaller than all of its cyclic permutations:*

$$[i_1 \dots i_{a-1} i_a \dots i_k] < [i_a \dots i_k i_1 \dots i_{a-1}] \qquad \forall\, a \in \{2, \dots, k\} .$$

For a word $w = [i_1 \dots i_k] \in I^*$, the subwords

$$w_{a|} = [i_1 \dots i_a] \qquad \text{and} \qquad w_{|a} = [i_{k-a+1} \dots i_k]$$

with $0 \leq a \leq k$ will be called a *prefix* and a *suffix* of w, respectively. We call such a prefix or a suffix *proper* if $0 < a < k$. It is straightforward to show that Definition 2.2 is equivalent to the following one:

Definition 2.3. *A word w is* <u>Lyndon</u> *if it is smaller than all of its proper suffixes:*

$$w < w_{|a} \qquad \forall\; 0 < a < |w| .$$

The following simple result is well-known:

Lemma 2.4. *If $\ell_1 < \ell_2$ are Lyndon, then $\ell_1 \ell_2$ is also Lyndon, and so $\ell_1 \ell_2 < \ell_2 \ell_1$.*

We recall the following two basic facts from the theory of Lyndon words:

Proposition 2.5. *([14, Proposition 5.1.3]) Any Lyndon word ℓ has a factorization*

$$\ell = \ell_1 \ell_2 \tag{2.1}$$

defined by the property that ℓ_2 is the longest proper suffix of ℓ which is also a Lyndon word. Under these circumstances, ℓ_1 is also a Lyndon word.

The factorization (2.1) is called a *costandard factorization* of a Lyndon word.

Proposition 2.6. *([14, Proposition 5.1.5]) Any word w has a unique factorization*

$$w = \ell_1 \dots \ell_k\,, \tag{2.2}$$

where $\ell_1 \geq \dots \geq \ell_k$ are all Lyndon words.

The factorization (2.2) is called a *canonical factorization* of a word.

2.7. **Standard Lyndon words.**

Let $\mathfrak{a}$ be a Lie algebra generated by a finite set $\{e_i\}_{i\in I}$ labelled by the alphabet I.

Definition 2.8. *The standard bracketing of a Lyndon word ℓ is given inductively by:*

- *$e_{[i]} = e_i \in \mathfrak{a}$ for $i \in I$,*
- *$e_\ell = [e_{\ell_1}, e_{\ell_2}] \in \mathfrak{a}$, where $\ell = \ell_1\ell_2$ is the costandard factorization (2.1).*

The major importance of this definition is due to the following result of Lyndon:

Theorem 2.9. *([14, Theorem 5.3.1]) If $\mathfrak{a}$ is a free Lie algebra in the generators $\{e_i\}_{i\in I}$, then the set $\big\{e_\ell \,|\, \ell\text{−Lyndon word}\big\}$ provides a basis of $\mathfrak{a}$.*

It is natural to ask if Theorem 2.9 admits a generalization to Lie algebras $\mathfrak{a}$ generated by $\{e_i\}_{i\in I}$ but with some defining relations. The answer was provided a few decades later in [11]. To state the result, define ${}_we, e_w \in U(\mathfrak{a})$ for any $w \in I^*$:

- For a word $w = [i_1 \dots i_k] \in I^*$, we set

$$ {}_we = e_{i_1} \dots e_{i_k} \in U(\mathfrak{a})\,. \tag{2.3}$$

- For a word $w \in I^*$ with a canonical factorization $w = \ell_1 \dots \ell_k$ of (2.2), we set

$$ e_w = e_{\ell_1} \dots e_{\ell_k} \in U(\mathfrak{a})\,. \tag{2.4}$$

It is well-known that the elements (2.3) and (2.4) are connected by the following triangularity property:

$$ e_w = \sum_{v \geq w} c^v_w \cdot {}_ve \qquad \text{with} \quad c^v_w \in \mathbb{Z} \quad \text{and} \quad c^w_w = 1\,. \tag{2.5}$$

The following definition is due to [11]:

Definition 2.10. *(a) A word w is called <u>standard</u> if ${}_we$ cannot be expressed as a linear combination of ${}_ve$ for various $v > w$.*

(b) A Lyndon word ℓ is called <u>standard Lyndon</u> if e_ℓ cannot be expressed as a linear combination of e_m for various Lyndon words $m > \ell$.

The following result is nontrivial and justifies the above terminology:

Proposition 2.11. *([11]) A Lyndon word is standard iff it is standard Lyndon.*

The major importance of this definition is due to the following result:

Theorem 2.12. *([11]) For any Lie algebra $\mathfrak{a}$ generated by a finite collection $\{e_i\}_{i\in I}$, the set $\big\{e_\ell \,|\, \ell\text{−standard Lyndon word}\big\}$ provides a basis of $\mathfrak{a}$.*

We also have the following simple properties of standard words:

Proposition 2.13. *([11]) (a) Any subword of a standard word is standard.*

(b) A word w is standard iff it can be written (uniquely) as $w = \ell_1 \dots \ell_k$, where $\ell_1 \geq \dots \geq \ell_k$ are standard Lyndon words.

Thus, combining the classical Poincaré–Birkhoff–Witt theorem for $U(\mathfrak{a})$ with Theorem 2.12, Proposition 2.13, and the triangularity property (2.5), we obtain the following PBW-type theorem:

$$U(\mathfrak{a}) = \bigoplus^{k\in\mathbb{N}}_{\ell_1\geq\dots\geq\ell_k \text{ standard Lyndon words}} \mathbb{Q}\cdot e_{\ell_1}\dots e_{\ell_k} = \bigoplus_{w\text{–standard words}} \mathbb{Q}\cdot e_w = \bigoplus_{w\text{–standard words}} \mathbb{Q}\cdot {}_we\,. \tag{2.6}$$

2.14. **Application to simple Lie algebras.**

Let $\mathfrak{g}$ be a simple Lie algebra with the root system $\Delta = \Delta^+ \sqcup \Delta^-$. Let $\{\alpha_i\}_{i\in I} \subset \Delta^+$ be the simple roots, and $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the root lattice. We endow Q with the symmetric pairing $(\cdot,\cdot)\colon Q\otimes Q \to \mathbb{Z}$ so that the Cartan matrix $(a_{ij})_{i,j\in I}$ and the symmetrized Cartan matrix $(d_{ij})_{i,j\in I}$ of $\mathfrak{g}$ are given by

$$a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \qquad \text{and} \qquad d_{ij} = (\alpha_i,\alpha_j)\,.$$

Explicitly, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i\in I}$ subject to the following defining relations:

$$\underbrace{[e_i,[e_i,\cdots,[e_i,e_j]\cdots]]}_{1-a_{ij} \text{ Lie brackets}} = 0 \qquad \text{if } i\neq j\,, \tag{2.7}$$

$$[h_i,e_j] = d_{ij}e_j\,, \qquad [h_i,h_j] = 0\,, \tag{2.8}$$

as well as the opposite relations with e's replaced by f's, and finally the relation:

$$[e_i,f_j] = \delta_{ij}h_i\,. \tag{2.9}$$

We will consider the triangular decomposition (1.3), where $\mathfrak{n}^+$, $\mathfrak{h}$, $\mathfrak{n}^-$ are the Lie subalgebras of $\mathfrak{g}$ generated by the e_i, h_i, f_i, respectively. We write $Q^+ \subset Q$ for the monoid generated by $\{\alpha_i\}_{i\in I}$. The Lie algebra $\mathfrak{g}$ is naturally Q-graded via

$$\deg e_i = \alpha_i\,, \qquad \deg h_i = 0\,, \qquad \deg f_i = -\alpha_i\,.$$

The Lie algebra $\mathfrak{g}$ admits the standard *root space decomposition*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \tag{2.10}$$

with $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha\in\Delta$. We pick *root vectors* $e_\alpha \in \mathfrak{g}_\alpha$ so that $\mathfrak{g}_\alpha = \mathbb{Q}\cdot e_\alpha$. Thus, the Lie subalgebra $\mathfrak{n}^+$ decomposes into $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ and is Q^+-graded. Explicitly, $\mathfrak{n}^+$ is generated by $\{e_i\}_{i\in I}$ subject to the classical *Serre* relations (2.7).

Fix any order on the set I. According to Theorem 2.12, $\mathfrak{n}^+$ has a basis consisting of the e_ℓ's, as ℓ ranges over all standard Lyndon words. Evoking the above Q^+-grading of the Lie algebra $\mathfrak{n}^+$, it is natural to define the grading of words via

$$\deg\,[i_1\dots i_k] = \alpha_{i_1} + \dots + \alpha_{i_k} \in Q^+\,. \tag{2.11}$$

Due to the decomposition (2.10) and the fact that the root vectors $\{e_\alpha\}_{\alpha\in\Delta^+} \subset \mathfrak{n}^+$ all live in distinct degrees $\alpha \in Q^+$, we conclude that there exists a bijection (1.9):

$$\ell \colon \Delta^+ \xrightarrow{\sim} \Big\{\text{standard Lyndon words}\Big\}$$

such that $\deg \ell(\alpha) = \alpha$ for all $\alpha \in \Delta^+$, which we call the *Lalonde-Ram's bijection.*

2.15. **Results of Leclerc.**

The Lalonde-Ram's bijection (1.9) was described explicitly in [12]. To state the result, we recall that for a root $\alpha = \sum_{i\in I} k_i\alpha_i \in \Delta^+$, its *height* is $\mathrm{ht}(\alpha) = \sum_i k_i$.

Proposition 2.16. *([12, Proposition 25]) The bijection ℓ is inductively given by:*

- *for simple roots, we have $\ell(\alpha_i) = [i]$,*
- *for other positive roots, we have the following Leclerc's algorithm:*

$$\ell(\alpha) = \max\Big\{\ell(\gamma_1)\ell(\gamma_2)\,\Big|\, \alpha = \gamma_1+\gamma_2\,,\ \gamma_1,\gamma_2 \in \Delta^+\,,\ \ell(\gamma_1) < \ell(\gamma_2)\Big\}\,. \tag{2.12}$$

The formula (2.12) recovers $\ell(\alpha)$ once we know $\ell(\gamma)$ for all $\{\gamma \in \Delta^+ \,|\, \mathrm{ht}(\gamma) < \mathrm{ht}(\alpha)\}$.

We shall also need one more important property of ℓ. To the end, let us recall:

Definition 2.17. *A total order on the set of positive roots Δ^+ is convex if:*

$$\alpha < \alpha + \beta < \beta \tag{2.13}$$

for all $\alpha < \beta \in \Delta^+$ such that $\alpha+\beta$ is also a root.

Remark 2.18. It is well-known ([18]) that convex orders on Δ^+ are in bijection with the reduced decompositions of the longest element $w_0 \in W$ in the Weyl group of $\mathfrak{g}$.

The following result is [12, Proposition 26], where it was attributed to the preprint of Rosso [22] (a detailed proof can be found in [16, Proposition 2.34]):

Proposition 2.19. *Consider the order on Δ^+ induced from the lexicographical order on standard Lyndon words:*

$$\alpha < \beta \quad \Longleftrightarrow \quad \ell(\alpha) < \ell(\beta) \ \text{ lexicographically}\,. \tag{2.14}$$

This order is convex.

3. Loop standard Lyndon words

We will now extend the description above to the Lie algebra of loops into $\mathfrak{g}$:

$$L\mathfrak{g} = \mathfrak{g}[t,t^{-1}] = \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{Q}[t,t^{-1}]$$

with the Lie bracket given simply by

$$[x\otimes t^m, y\otimes t^n] = [x,y]\otimes t^{m+n} \quad \text{for any} \quad x,y \in \mathfrak{g}\,,\ m,n\in\mathbb{Z}\,.$$

The triangular decomposition (1.3) extends to a similar decomposition at the loop level $L\mathfrak{g} = L\mathfrak{n}^+ \oplus L\mathfrak{h} \oplus L\mathfrak{n}^-$, and our goal is to describe $L\mathfrak{n}^+$ along the lines of the previous Section. To this end, we think of $L\mathfrak{n}^+$ as being generated by $e_i^{(d)} = e_i \otimes t^d$ for all $i \in I, d\in\mathbb{Z}$. Associate to $e_i^{(d)}$ the letter $i^{(d)}$, and call d the *exponent* of $i^{(d)}$.

We thus obtain the infinite alphabet $\mathcal{I} = \{i^{(d)} \,|\, i \in I, d \in \mathbb{Z}\}$ and any word in these letters will be called a *loop word*:

$$\Big[i_1^{(d_1)} \dots i_k^{(d_k)}\Big]\,. \tag{3.1}$$

We shall now introduce a family of total orders on $\mathcal{I}$, which will thus induce lexicographical orderings on loop words (3.1). To this end, we fix a total order on I and choose a tuple of positive integers $\{c_i\}_{i\in I} \in \mathbb{Z}^I_{>0}$ (we call c_i the *weight* of i). Following [16, Remark 3.15], we shall compare the loop letters of $\mathcal{I}$ via (1.10):

$$i^{(d)} < j^{(e)} \qquad \Longleftrightarrow \qquad \frac{d}{c_i} > \frac{e}{c_j} \quad \text{or} \quad \frac{d}{c_i} = \frac{e}{c_j} \text{ and } i < j\,.$$

Due to its importance, we shall call the ratio d/c_i the *relative exponent* of $i^{(d)} \in \mathcal{I}$. We also define the *weighted height* of roots via:

$$f(\alpha) = \sum_{i\in I} k_i \cdot c_i \quad \text{for any} \quad \alpha = \sum_{i\in I} k_i \alpha_i \in \Delta^+\,. \tag{3.2}$$

All the results of Subsection 2.1 continue to hold in the present setup, so we have a notion of Lyndon loop words. Since $L\mathfrak{n}^+$ is $Q^+ \times \mathbb{Z}$-graded via $\deg e_i^{(d)} = (\alpha_i, d)$, it makes sense to extend this grading to loop words via

$$\deg\left[i_1^{(d_1)} \ldots i_k^{(d_k)}\right] = (\alpha_{i_1} + \cdots + \alpha_{i_k}, d_1 + \cdots + d_k)\,.$$

The obvious generalization of (1.1) is:

$$L\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \bigoplus_{d\in\mathbb{Z}} \mathbb{Q}\cdot e_\alpha^{(d)}$$

with $e_\alpha^{(d)} = e_\alpha \otimes t^d$ for all $\alpha \in \Delta^+, d \in \mathbb{Z}$. We note that $L\mathfrak{n}^+$ still has one-dimensional $Q^+ \times \mathbb{Z}$-graded pieces, which is essential for the treatment of [11] to carry through.

On the other hand, the definition of standard (Lyndon) loop words in the present setup is a non-trivial task since the alphabet $\mathcal{I}$ is infinite. Motivated by the treatment of [16] in the case when all $c_i = 1$, we shall likewise consider a filtration by finitely generated Lie algebras $L^{(s)}\mathfrak{n}^+$ of (3.4), corresponding to the finite alphabets

$$\mathcal{I}^{(s)} = \left\{ i^{(d)} \,\middle|\, i \in I, -s\cdot c_i \le d \le s \cdot c_i \right\} \qquad \forall\, s \in \mathbb{Z}_{\ge 0}\,. \tag{3.3}$$

We will establish some basic properties of the corresponding standard Lyndon loop words for $L^{(s)}\mathfrak{n}^+$ which ultimately imply that the notion of a "standard Lyndon loop word" does not actually depend on the particular $L^{(s)}\mathfrak{n}^+$ with respect to which it is defined. We shall thus obtain the loop analogue (3.13) of the bijection (1.9).

3.1. **Filtration and basic properties.**

We now wish to extend Definition 2.10 in order to obtain a notion of standard (Lyndon) loop words, but here we must be careful as the alphabet $\mathcal{I}$ is infinite. In particular, the key assumption "for any word v, there are only finitely many words u of the same length and $> v$ in the lexicographical order" of [11, §2] clearly fails. To deal with this issue, we consider the increasing filtration:

$$L\mathfrak{n}^+ = \bigcup_{s=0}^{\infty} L^{(s)}\mathfrak{n}^+$$

defined with respect to the finite-dimensional Lie subalgebras (see notation (3.2)):

$$L\mathfrak{n}^+ \supset L^{(s)}\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \bigoplus_{d=-s\cdot f(\alpha)}^{s\cdot f(\alpha)} \mathbb{Q}\cdot e_\alpha^{(d)} \qquad \forall\, s \in \mathbb{Z}_{\ge 0}\,. \tag{3.4}$$

As a Lie algebra, $L^{(s)}\mathfrak{n}^+$ is generated by $\{e_i^{(d)} \mid i \in I, |d| \leq s \cdot c_i\}$. We may thus apply Definition 2.10 to yield a notion of standard (Lyndon) loop words with respect to the finite-dimensional Lie algebras $L^{(s)}\mathfrak{n}^+$, with the words made up only of $i^{(d)} \in \mathcal{I}^{(s)}$.

The following result is proved completely analogously to [16, Proposition 2.23] (which in turn is an adaptation of the analogous results from [12], cf. (2.12)):

Proposition 3.2. *There exists a bijection:*

$$\ell \colon \Big\{ (\alpha, d) \in \Delta^+ \times \mathbb{Z} \,\Big|\, |d| \leq s \cdot f(\alpha) \Big\} \xrightarrow{\sim} \Big\{ \text{standard Lyndon loop words for } L^{(s)}\mathfrak{n}^+ \Big\}, \tag{3.5}$$

determined by $\ell(\alpha_i, d) = \big[i^{(d)}\big]$ *and the following (generalized)* <u>*Leclerc's algorithm:*</u>

$$\ell(\alpha, d) \;=\; \max_{\substack{(\gamma_1,d_1)+(\gamma_2,d_2)=(\alpha,d) \\ \gamma_k \in \Delta^+,\ |d_k| \leq s \cdot f(\gamma_k) \\ \ell(\gamma_1,d_1)<\ell(\gamma_2,d_2)}} \Big\{ \text{concatenation } \ell(\gamma_1, d_1)\ell(\gamma_2, d_2) \Big\}. \tag{3.6}$$

Since standard Lyndon loop words give rise to bases of the finite-dimensional Lie algebras $L^{(s)}\mathfrak{n}^+$, then the analogue of property (2.6) gives us:

$$U(L^{(s)}\mathfrak{n}^+) \;=\; \bigoplus^{k \in \mathbb{N}}_{\substack{\ell_1 \geq \cdots \geq \ell_k \text{ standard Lyndon loop words} \\ \text{with all relative exponents in } [-s,s]}} \mathbb{Q} \cdot e_{\ell_1} \dots e_{\ell_k} \;=\; \bigoplus_{\substack{w\text{–standard loop words with} \\ \text{all relative exponents in } [-s,s]}} \mathbb{Q} \cdot e_w \;=\; \bigoplus_{\substack{w\text{–standard loop words with} \\ \text{all relative exponents in } [-s,s]}} \mathbb{Q} \cdot {}_w e \,. \tag{3.7}$$

We shall next establish some properties of the bijection (3.5). We start with the following *monotonicity* property:

Proposition 3.3. *Fix $s \in \mathbb{Z}_{>0}$. Then for any positive root $\alpha \in \Delta^+$ and any integer $d \in [-s \cdot f(\alpha) + 1, s \cdot f(\alpha)]$, the bijection (3.5) satisfies the following inequality:*

$$\ell(\alpha, d) < \ell(\alpha, d-1)\,. \tag{3.8}$$

Proof. The proof if completely analogous to that of [16, Proposition 2.25]. □

3.4. **Exponent tightness.**

While many properties of the bijection (3.5) can be established very similarly to the special case (when $c_i = 1$ for all i) of [16], the naive generalization of [16, Proposition 2.26] shall not suffice. We discuss the key upgrades in this Subsection.

We start with the following definition:

Definition 3.5. *A loop word $w = \Big[i_1^{(d_1)} \dots i_n^{(d_n)}\Big]$ is called* <u>*exponent-tight*</u> *if*

$$i_k^{(d_k)} \geq i_r^{(d_r+1)} \qquad \text{for all} \quad 1 \leq k, r \leq n\,. \tag{3.9}$$

When w is a Lyndon loop word, it clearly suffices to verify (3.9) only for $k = 1$. The following is the main result of this Subsection:

Theorem 3.6. *For any root $\alpha \in \Delta^+$ and any integer $d \in \{-s \cdot f(\alpha), \dots, s \cdot f(\alpha)\}$, the standard Lyndon loop word $\ell(\alpha, d)$ is exponent-tight.*

The proof of this result relies on Lemma 3.7 and Proposition 3.8 proved below. In what follows, we write $i^{(d)} \in w$ to denote that w contains the letter $i^{(d)} \in \mathcal{I}$. If a loop word w has a $Q \times \mathbb{Z}$-degree $\deg w = (\alpha, d)$, then we will use the notation

$$\text{hdeg } w = \alpha \qquad \text{and} \qquad \text{vdeg } w = d\,, \tag{3.10}$$

and call these two notions the *horizontal* and the *vertical* degree, respectively.

Lemma 3.7. *Any two exponent-tight loop words v and w of the same $Q \times \mathbb{Z}$-degree contain the same multisets of letters.*

Proof. First, let us show that if $i^{(k)} \in w$ then also $i^{(k)} \in v$. Assuming the contradiction, we must have $i^{(k')} \in v$ for some $k' \neq k$, as hdeg $v =$ hdeg w. Without loss of generality, we may assume that $k' \geq k+1$, so that $i^{(k')} \leq i^{(k+1)}$. As vdeg $v =$ vdeg w, there are two letters $j^{(t)} \in w$ and $j^{(t')} \in v$, such that $t' \leq t-1$, so that $j^{(t)} \leq j^{(t'+1)}$. Since both words v and w are exponent-tight, we also have

$$i^{(k+1)} \leq j^{(t)} \qquad \text{and} \qquad j^{(t'+1)} \leq i^{(k')}\,.$$

Combining the above inequalities, we obtain:

$$j^{(t)} \leq j^{(t'+1)} \leq i^{(k')} \leq i^{(k+1)} \leq j^{(t)}\,,$$

so that $j^{(t)} = j^{(t'+1)} = i^{(k')} = i^{(k+1)}$. Hence $i^{(k)} = j^{(t')} \in v$, a contradiction.

Thus any letter of w is contained in v and vice-versa. It remains to show that multiplicities of all letters in w and v are the same. Since hdeg $v =$ hdeg w, the sum of all multiplicities of $i^{(\bullet)} \in w$ is the same as that of $i^{(\bullet)} \in v$ for any $i \in I$. Thus, the claim is obvious if both w and v contain $i^{(k)}$ and no other $i^{(k')}$ for $k' \neq k$. Assume now that w (and hence also v) contains $i^{(k)}, i^{(k')}$ for $k' > k$. Then $k' = k+1$, due to $i^{(k')} \geq i^{(k+1)}$. In this case, we may not have $j^{(t)}, j^{(t+1)} \in w$ for any $j \neq i$ and $t \in \mathbb{Z}$. Otherwise we would have $i^{(k+1)} \geq j^{(t+1)} \geq i^{(k+1)}$, due to exponent-tightness, and so $j^{(t)} = i^{(k)}$, a contradiction with $j \neq i$. Thus, for any $j \neq i$, there is only one value of exponent such that $j^{(\bullet)}$ is contained in w (and hence in v). As $\deg v = \deg w$, we thus also conclude that multiplicities of $i^{(k)}, i^{(k+1)}$ in w and v are the same. □

Proposition 3.8. *Let $v = [i_1^{(k_1)} \dots i_m^{(k_m)}]$ and $w = [j_1^{(t_1)} \dots j_m^{(t_m)}]$ be two exponent-tight loop words such that hdeg $w =$ hdeg v and vdeg $w =$ vdeg $v+1$. Then:*

(a) The first letter $j_1^{(t_1)}$ of the loop word w equals $\max_{1 \leq a \leq m}\{i_a^{(d_a+1)}\}$.

(b) The multisets of the other letters coincide: $\{i_a^{(k_a)}\}_{a=1}^m - \{j_1^{(t_1-1)}\} = \{j_a^{(t_a)}\}_{a=2}^m$.

Proof. Let $i_r^{(d_r+1)} = \max_{1 \leq a \leq m}\{i_a^{(d_a+1)}\}$. Since v is exponent-tight, so is any loop word u formed by the letters $\{i_a^{(d_a)}\}_{a \neq r} \cup \{i_r^{(d_r+1)}\}$ (a loop word is exponent-tight iff any loop word formed by the same multiset of letters is exponent-tight). But then w and u must have the same multisets of letters, according to Lemma 3.7. Since the loop word u satisfies both properties (a) and (b), so does w. □

Remark 3.9. Following the setup of Proposition 3.8, one may vice-versa express the multiset of letters of v through the one for w: $\{i_a^{(k_a)}\}_{a=1}^m = \{j_1^{(t_1-1)}\} \cup \{j_a^{(t_a)}\}_{a=2}^m$.

Now we are ready to present the proof of Theorem 3.6.

Proof of Theorem 3.6. The proof proceeds by induction on the height $n = \mathrm{ht}(\alpha)$.

The base case of the induction is $n = 2$. Let $\ell(\alpha, d) = [i_1^{(k_1)} i_2^{(k_2)}]$, where $i_1^{(k_1)} < i_2^{(k_2)}$ and $i_1 \neq i_2$. We claim that $i_1^{(k_1-1)} > i_2^{(k_2+1)}$, as otherwise we would get $\ell(\alpha, d) = [i_1^{(k_1)} i_2^{(k_2)}] < [i_1^{(k_1-1)} i_2^{(k_2+1)}]$, a contradiction with Leclerc's algorithm (3.6). But then, invoking (3.6), we obtain $\ell(\alpha, d) \geq i_2^{(k_2+1)} i_1^{(k_1-1)}$. This implies the desired inequality $i_1^{(k_1)} \geq i_2^{(k_2+1)}$, establishing the base of the induction.

Let us now prove the step of the induction, assuming the assertion holds for all roots of height $< n$. If not, then for some root $\alpha \in \Delta^+$ of height n and some $d \in \mathbb{Z}$, we have $\ell(\alpha, d) = [i_1^{(d_1)} \dots i_n^{(d_n)}]$ with $i_r^{(d_r+1)} > i_1^{(d_1)}$ for some $1 < r \leq n$. Let us consider the costandard factorization of $\ell(\alpha, d)$:

$$\ell(\alpha, d) = \ell(\gamma_1, k_1)\ell(\gamma_2, k_2)\,,$$

where $\alpha = \gamma_1 + \gamma_2$, $d = k_1 + k_2$, $\ell(\gamma_1, k_1) < \ell(\gamma_2, k_2)$, and roots γ_1, γ_2 have height $< n$. By the induction hypothesis, $i_r^{(d_r)} \notin \ell(\gamma_1, k_1)$, so that $i_r^{(d_r)} \in \ell(\gamma_2, k_2)$. Arguing as above, we claim that $\ell(\gamma_1, k_1 - 1) > \ell(\gamma_2, k_2 + 1)$, as otherwise according to (3.8) we would get $\ell(\alpha, d) = \ell(\gamma_1, k_1)\ell(\gamma_2, k_2) < \ell(\gamma_1, k_1 - 1)\ell(\gamma_2, k_2 + 1)$, a contradiction with (3.6). The inequality $\ell(\gamma_1, k_1 - 1) > \ell(\gamma_2, k_2 + 1)$ implies

$$\ell(\alpha, d) \geq \ell(\gamma_2, k_2 + 1)\ell(\gamma_1, k_1 - 1)\,, \tag{3.11}$$

due to (3.6). Since $\mathrm{ht}(\gamma_2) < n$, both words $\ell(\gamma_2, k_2)$ and $\ell(\gamma_2, k_2 + 1)$ are exponent-tight by the induction hypothesis. Therefore, the first letter of $\ell(\gamma_2, k_2 + 1)$ is $i_t^{(d_t+1)} = \max_{\mathrm{ht}(\gamma_1)<a\leq n}\{i_a^{(d_a+1)}\}$, due to Proposition 3.8. Note that $i_t^{(d_t+1)} \leq i_1^{(d_1)}$, according to (3.11). Therefore, we get $i_r^{(d_r+1)} \leq i_t^{(d_t+1)} \leq i_1^{(d_1)}$, a contradiction. □

Remark 3.10. Let us emphasize that applying directly the argument from the proof of [16, Proposition 2.26], one rather gets a weaker statement:

$$\ell(\alpha, d) = \left[i_1^{(k_1)} \dots i_n^{(k_n)}\right] \quad \text{with} \quad \left\lfloor \frac{d}{f(\alpha)} \right\rfloor \leq \frac{k_i}{c_i} \leq \left\lceil \frac{d}{f(\alpha)} \right\rceil \quad \forall\, 1 \leq i \leq n \tag{3.12}$$

with $f(\alpha)$ defined in (3.2). In particular, if $c_i = N > 1$ for all $i \in I$ (thus the order on $\mathcal{I}$ is the same as for $c_i = 1$ and so $\ell(\alpha, d)$ are the same as in [16]), then (3.12) only implies $|k_i - k_j| \leq N$, while Theorem 3.6 implies a much finer bound $|k_i - k_j| \leq 1$.

The following is a simple corollary of Theorem 3.6:

Corollary 3.11. *(a) For $\alpha \in \Delta^+, d > 0$, the first letter of $\ell(\alpha, d)$ has exponent > 0.*

(b) For $\alpha \in \Delta^+, d \leq 0$, the first letter of $\ell(\alpha, d)$ has exponent ≤ 0.

Proof. Let $\ell(\alpha, d) = [i_1^{(k_1)} \dots i_n^{(k_n)}]$. Then $i_1^{(k_1)} \leq i_r^{(k_r)}$ and so $\frac{k_1}{c_1} \geq \frac{k_r}{c_r}$ for any r. Thus if $k_1 \leq 0$, then $k_r \leq 0$ for any r, and so $d = \sum_{r=1}^n k_r \leq 0$, implying part (a).

To prove (b), we note that $i_1^{(k_1)} \geq i_r^{(k_r+1)}$ for any r by Theorem 3.6, hence $\frac{k_1}{c_1} \leq \frac{k_r+1}{c_r}$. If $k_1 > 0$, then $k_r \geq 0$ for all r, and so $d = \sum_{r=1}^n k_r > 0$, a contradiction. □

3.12. **Stabilization.**

As an important consequence of Theorem 3.6, we obtain:

Proposition 3.13. *Any loop word w with relative exponents in $[-s, s]$ is standard (Lyndon) with respect to $L^{(s)}\mathfrak{n}^+$ iff it is standard (Lyndon) with respect to $L^{(s+1)}\mathfrak{n}^+$.*

Proof. While the proof of [16, Proposition 2.28] can be directly generalized with the help of Theorem 3.6, let us present a shorter argument. Consider loop words

$$\ell = \ell(\alpha, d) \text{ of (3.5) with respect to } L^{(s)}\mathfrak{n}^+ ,$$
$$\ell' = \ell(\alpha, d) \text{ of (3.5) with respect to } L^{(s+1)}\mathfrak{n}^+ .$$

Combining (3.12) with Theorem 3.6 and Proposition 3.8, we see that both words ℓ and ℓ' contain the same multisets of letters (all thus being elements of $\mathcal{I}^{(s)}$). Additionally, their standard bracketings e_ℓ, $e_{\ell'}$ are both nonzero multiples of $e_\alpha^{(d)}$. By the very definition of standard Lyndon loop words, this implies that $\ell = \ell'$. □

The above result implies that the notion of a "standard Lyndon loop word" does not depend on the particular $L^{(s)}\mathfrak{n}^+$ with respect to which it is defined. We conclude that there exists a bijection:

$$\ell \colon \Delta^+ \times \mathbb{Z} \xrightarrow{\sim} \Big\{\text{standard Lyndon loop words}\Big\} \tag{3.13}$$

satisfying property (3.6) with $s = \infty$ as well as Theorem 3.6 and Proposition 3.8.

3.14. **Periodicity.**

While ℓ of (3.13) is a bijection between infinite sets, it is actually determined by the values of ℓ only on a finite "block" of $\Delta^+ \times \mathbb{Z}$:

$$L = \Big\{(\alpha, d) \,\Big|\, \alpha \in \Delta^+, 0 \leq d < f(\alpha)\Big\}, \tag{3.14}$$

cf. notation (3.2). More precisely, we have the following *periodicity* property:

Proposition 3.15. *For any $(\alpha, d) \in \Delta^+ \times \mathbb{Z}$, the standard Lyndon loop word $\ell(\alpha, d + f(\alpha))$ is obtained from the standard Lyndon loop word $\ell(\alpha, d)$ by increasing all exponents of its letters $i^{(\bullet)}$ by c_i (that is, increasing all relative exponents by 1).*

Proof. Let Υ denote the aforementioned bijective map on the set of loop words:

$$\Upsilon \colon \left[i_1^{(d_1)} \dots i_k^{(d_k)}\right] \mapsto \left[i_1^{(d_1 + c_{i_1})} \dots i_k^{(d_k + c_{i_k})}\right] . \tag{3.15}$$

Note that $u < v$ iff $\Upsilon(u) < \Upsilon(v)$ in accordance with (1.10). Thus, (3.15) preserves the property of a loop word being Lyndon. Likewise, if $\ell = \ell_1\ell_2$ is the costandard factorization of ℓ, then $\Upsilon(\ell) = \Upsilon(\ell_1)\Upsilon(\ell_2)$ is the costandard factorization of $\Upsilon(\ell)$. This also implies that $e_{\Upsilon(\ell)} = \widetilde{\Upsilon}(e_\ell)$, where $\widetilde{\Upsilon}$ is the Lie algebra isomorphism:

$$\widetilde{\Upsilon} \colon L\mathfrak{n}^+ \xrightarrow{\sim} L\mathfrak{n}^+ \qquad \text{given by} \qquad e_\alpha^{(d)} \mapsto e_\alpha^{(d + f(\alpha))} .$$

Hence, (3.15) also preserves the property of a Lyndon loop word being standard. □

Similarly to [16, Proposition 2.31], we also note the following simple property:

Proposition 3.16. *The restriction of (3.13) to $\Delta^+ \times \{0\}$ matches (1.9).*

Proof. This is simply the $s = 0$ case of Proposition 3.13. □

Since $U(L\mathfrak{n}^+)$ is the direct limit as $s \to \infty$ of the $U(L^{(s)}\mathfrak{n}^+)$, then (3.7) implies:

$$U(L\mathfrak{n}^+) \;=\; \bigoplus_{\ell_1 \geq \dots \geq \ell_k \text{ standard Lyndon loop words}}^{k \in \mathbb{N}} \mathbb{Q} \cdot e_{\ell_1} \dots e_{\ell_k} \;=\; \bigoplus_{w\text{–standard loop words}} \mathbb{Q} \cdot e_w \;=\; \bigoplus_{w\text{–standard loop words}} \mathbb{Q} \cdot {}_w e \,. \tag{3.16}$$

3.17. **Convexity and minimality.**
We conclude this Section with a few fundamental properties of the total order on $\Delta^+ \times \mathbb{Z}$ induced by transporting the lexicographic order on loop words via the bijection (3.13). A straightforward generalization of [16, Proposition 2.34] establishes that this order is *convex*, a notion that is a direct generalization of Definition 2.17:

Proposition 3.18. *For all $(\alpha, d), (\beta, e), (\alpha+\beta, d+e) \in \Delta^+ \times \mathbb{Z}$, we have:*

$$\ell(\alpha, d) < \ell(\alpha+\beta, d+e) < \ell(\beta, e) \tag{3.17}$$

if $\ell(\alpha, d) < \ell(\beta, e)$.

This result admits the following natural generalization:

Corollary 3.19. *Consider any $k, k' \geq 1$ and any*

$$(\gamma_1, d_1), \dots, (\gamma_k, d_k), (\gamma'_1, d'_1), \dots, (\gamma'_{k'}, d'_{k'}) \in \Delta^+ \times \mathbb{Z}$$

such that $(\gamma_1, d_1) + \dots + (\gamma_k, d_k) = (\gamma'_1, d'_1) + \dots + (\gamma'_{k'}, d'_{k'})$. Then we have:

$$\min \Big\{ \ell(\gamma_1, d_1), \dots, \ell(\gamma_k, d_k) \Big\} \leq \max \Big\{ \ell(\gamma'_1, d'_1), \dots, \ell(\gamma'_{k'}, d'_{k'}) \Big\}.$$

Proof. The proof is completely analogous to that of [16, Corollary 2.37]. □

An important consequence of this Corollary is the following result, which will play a crucial role in our proof of Theorem 5.8 below:

Proposition 3.20. *If $\ell_1 < \ell_2$ are standard Lyndon loop words such that $\ell_1 \ell_2$ is also a standard Lyndon loop word, then we cannot have:*

$$\ell_1 < \ell'_1 < \ell'_2 < \ell_2$$

for standard Lyndon loop words ℓ'_1, ℓ'_2 such that $\deg \ell_1 + \deg \ell_2 = \deg \ell'_1 + \deg \ell'_2$.

Proof. The proof is completely analogous to that of [16, Proposition 2.38]. □

4. Lyndon words and Weyl groups

In this Section, we show that the lexicographic order (1.13) on $\Delta^+ \times \mathbb{Z}$ induced by (3.13) is related to the construction of [19, 20] applied to a reduced decomposition of a translation element in the extended affine Weyl group encoding the weights c_i.

4.1. **Affine Lie algebras.**
In this Section, we recall the next simplest class of Kac-Moody Lie algebras after the simple ones, the affine Lie algebras. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $\{\alpha_i\}_{i \in I}$ be the simple roots, and $\theta \in \Delta^+$ be the highest root. The *labels* of the Dynkin diagram of $\mathfrak{g}$ are the positive integers $\{\theta_i\}_{i \in I}$ such that

$$\theta = \sum_{i \in I} \theta_i \alpha_i \,. \tag{4.1}$$

We define $\widehat{I} = I \sqcup \{0\}$. Consider the affine root lattice $\widehat{Q}$ with the generators $\{\alpha_i\}_{i \in \widehat{I}}$ which admits a natural identification

$$\widehat{Q} \xrightarrow{\sim} Q \times \mathbb{Z} \qquad \text{with} \qquad \alpha_i \mapsto (\alpha_i, 0) \ \ \forall\, i \in I, \qquad \alpha_0 \mapsto (-\theta, 1)\,. \tag{4.2}$$

We endow $\widehat{Q}$ with the symmetric pairing defined by:

$$\big((\alpha, n), (\beta, m)\big) = (\alpha, \beta) \qquad \forall\ \alpha, \beta \in Q\,,\ n, m \in \mathbb{Z}.$$

As opposed from the non-degenerate pairing on $\mathfrak{g}$ itself, the pairing on affine type root systems has a one-dimensional kernel, which is spanned by the minimal *imaginary root* $\delta = \alpha_0 + \theta = (0, 1) \in Q \times \mathbb{Z}$. This implies the fact that:

$$(\alpha_0 + \theta, -) = 0 \quad \Longleftrightarrow \quad d_{0j} + \sum_{i \in I} \theta_i d_{ij} = 0 \qquad \forall\, j \in I\,, \tag{4.3}$$

where $\{d_{ij}\}_{i,j \in \widehat{I}}$ is the symmetrized affine Cartan matrix. Let $(a_{ij})_{i,j \in \widehat{I}}$ be the affine Cartan matrix, giving rise to the *affine Lie algebra* $\widehat{\mathfrak{g}}$ generated by $\{e_i, f_i, h_i\}_{i \in \widehat{I}}$ with the defining relations (2.7)–(2.9). We note that (4.3) implies that

$$c = h_0 + \sum_{i \in I} \theta_i h_i \qquad \text{is a central element of } \ \widehat{\mathfrak{g}}\,. \tag{4.4}$$

The associated affine root system $\widehat{\Delta} = \widehat{\Delta}^+ \sqcup \widehat{\Delta}^-$ has the following description:

$$\widehat{\Delta}^+ = \big\{\Delta^+ \times \mathbb{Z}_{\geq 0}\big\} \sqcup \big\{0 \times \mathbb{Z}_{>0}\big\} \sqcup \big\{\Delta^- \times \mathbb{Z}_{>0}\big\}, \tag{4.5}$$

$$\widehat{\Delta}^- = \big\{\Delta^- \times \mathbb{Z}_{\leq 0}\big\} \sqcup \big\{0 \times \mathbb{Z}_{<0}\big\} \sqcup \big\{\Delta^+ \times \mathbb{Z}_{<0}\big\}. \tag{4.6}$$

With this notation, we have the following root space decomposition, cf. (2.10):

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \widehat{\Delta}} \widehat{\mathfrak{g}}_\alpha \quad \text{where} \quad \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}} - \text{Cartan subalgebra}\,. \tag{4.7}$$

The rich theory of affine Lie algebras is mainly based on the following key result:

Claim 4.2. There exists a Lie algebra isomorphism:

$$\widehat{\mathfrak{g}} \xrightarrow{\ \sim\ } L\mathfrak{g} \tag{4.8}$$

determined on the generators by the following formulas:

$$\begin{aligned} &e_i \mapsto e_i \otimes t^0 && f_i \mapsto f_i \otimes t^0 && h_i \mapsto h_i \otimes t^0 \qquad \forall\, i \in I\,, \\ &e_0 \mapsto f_\theta \otimes t^1 && f_0 \mapsto e_\theta \otimes t^{-1} && h_0 \mapsto -[e_\theta, f_\theta] \otimes t^0\,, \end{aligned}$$

where e_θ and f_θ are root vectors of degrees θ and $-\theta$, respectively.

4.3. **Affine Weyl groups.**

We have already mentioned in Remark 2.18 that convex orders of Δ^+ are in 1-to-1 correspondence with reduced decompositions of the longest element of the finite Weyl group W associated to $\mathfrak{g}$. To define the latter, consider the *coroot lattice*:

$$Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \cdot \alpha_i^\vee$$

where for any $\alpha \in \Delta^+$ the corresponding *coroot* $\alpha^\vee$ is defined via $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. The finite Weyl group W, i.e. the abstract Coxeter group associated to the Cartan matrix $(a_{ij})_{i,j \in I}$, acts faithfully on the coroot lattice $Q^\vee$ and the root lattice Q:

$$W \curvearrowright Q^\vee \qquad \text{and} \qquad W \curvearrowright Q \tag{4.9}$$

via the following assignments ($\forall\, i \in I$, $\mu \in Q^\vee$, $\lambda \in Q$):

$$s_i(\mu) = \mu - (\alpha_i, \mu)\alpha_i^\vee \qquad \text{and} \qquad s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i\,. \tag{4.10}$$

In the present setup, we need the *affine Weyl group*, defined as the semidirect product $\widehat{W} = W \ltimes Q^\vee$ with respect to the action (4.9). It is well-known that $\widehat{W}$ is also the Coxeter group associated to the Cartan matrix $(a_{ij})_{i,j\in\widehat{I}}$. In other words, the affine Weyl group is generated by the symbols $\{s_i\}_{i\in\widehat{I}}$ defined by:

$$s_0 = (s_\theta, -\theta^\vee) \qquad \text{and} \qquad s_i = (s_i, 0) \quad \forall\, i \in I\,.$$

The affine analogue of the action $W \curvearrowright Q$ from (4.9) is

$$\widehat{W} \curvearrowright \widehat{Q}\,, \tag{4.11}$$

where the generators of the affine Weyl group act by the following formulas:

$$s_i(\lambda, d) = (\lambda - (\lambda, \alpha_i^\vee)\alpha_i, d) \quad \forall\, i \in I\,, \tag{4.12}$$

$$s_0(\lambda, d) = (\lambda - (\lambda, \theta^\vee)\theta, d + (\lambda, \theta^\vee)) \tag{4.13}$$

for all $(\lambda, d) \in Q \times \mathbb{Z} \simeq \widehat{Q}$, see (4.2). An important feature of the affine Weyl group is that it contains a large commutative subalgebra $1 \ltimes Q^\vee \subset \widehat{W}$ which acts on the affine root lattice $\widehat{Q} \simeq Q \times \mathbb{Z}$ by translations:

$$\widehat{\mu}(\lambda, d) = (\lambda, d - (\lambda, \mu)) \qquad \forall\, \mu \in Q^\vee,\, \lambda \in Q,\, d \in \mathbb{Z}\,. \tag{4.14}$$

Henceforth, we write $\widehat{\mu}$ for the element $1 \ltimes \mu \in \widehat{W}$ and call it a *translation element*.

Finally, we also need to consider the *extended affine Weyl group*, defined as the semidirect product $\widehat{W}^{\text{ext}} = W \ltimes P^\vee$, where $P^\vee$ is the *coweight lattice*. Thus $P^\vee = \bigoplus_{i\in I} \mathbb{Z}\cdot\omega_i^\vee$ with the fundamental coweights $\omega_i^\vee$ dual to the simple roots:

$$(\alpha_j, \omega_i^\vee) = \delta_{ij}\,. \tag{4.15}$$

In particular, $Q^\vee$ is a finite index subgroup of $P^\vee$. It is well-known that:

$$\widehat{W}^{\text{ext}} \simeq \mathcal{T} \ltimes \widehat{W} \tag{4.16}$$

where the finite subgroup $\mathcal{T}$ of $\widehat{W}^{\text{ext}}$ is naturally identified with a subgroup of automorphisms of the Dynkin diagram of $\widehat{\mathfrak{g}}$. The semi-direct product (4.16) is such that $\tau s_i = s_{\tau(i)}\tau$ for any $\tau \in \mathcal{T}$ and $i \in \widehat{I}$. Finally, the action (4.11) extends to:

$$\widehat{W}^{\text{ext}} \curvearrowright \widehat{Q}$$

via $\tau(\alpha_i) = \alpha_{\tau(i)}$ for $\tau \in \mathcal{T}$, $i \in \widehat{I}$. We still have the following formula, akin to (4.14):

$$\widehat{\mu}(\lambda, d) = (\lambda, d - (\lambda, \mu)) \qquad \forall\, \mu \in P^\vee,\, \lambda \in Q,\, d \in \mathbb{Z}\,. \tag{4.17}$$

4.4. **Reduced decompositions.**

Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Recall that the <u>length</u> of an element $x \in \widehat{W}$, denoted by $l(x) \in \mathbb{N}$, is the smallest number $l \in \mathbb{N}$ such that we can write $x = s_{i_{1-l}} \dots s_{i_0}$ for various $i_{1-l}, \dots, i_0 \in \widehat{I}$. Every such factorization is called a *reduced decomposition* of x. Given such a reduced decomposition, the *terminal subset* of the affine root system is:

$$E_x = \left\{ s_{i_0} s_{i_{-1}} \dots s_{i_{k+1}}(\alpha_{i_k}) \,\middle|\, 0 \geq k > -l \right\} \subset \widehat{\Delta}\,. \tag{4.18}$$

It is well-known that E_x is independent of the reduced decomposition of x, and consists of the positive affine roots (all with multiplicity one) that are mapped to negative ones under the action of x:

$$E_x = \left\{ \widetilde{\lambda} \in \widehat{\Delta}^+ \,\middle|\, x(\widetilde{\lambda}) \in \widehat{\Delta}^- \right\}. \tag{4.19}$$

In particular, we get the following description of the length of x:

$$l(x) = \#\left\{ \widetilde{\lambda} \in \widehat{\Delta}^+ \,\middle|\, x(\widetilde{\lambda}) \in \widehat{\Delta}^- \right\}.$$

The aforementioned length function $l\colon \widehat{W} \to \mathbb{N}$ naturally extends to $\widehat{W}^{\mathrm{ext}}$ via

$$l(\tau w) = l(w) \quad \text{for any} \quad \tau \in \mathcal{T}\,,\ w \in \widehat{W}\,.$$

Thus, the length $l(x)$ of $x \in \widehat{W}^{\mathrm{ext}}$ is the smallest number l such that we can write:

$$x = \tau s_{i_{1-l}} \dots s_{i_0} \tag{4.20}$$

for various $i_{1-l}, \dots, i_0 \in \widehat{I}$ and (a uniquely determined) $\tau \in \mathcal{T}$. Given a reduced decomposition of $x \in \widehat{W}^{\mathrm{ext}}$ as in (4.20) with $l = l(x)$, define E_x via (4.18). We note that E_x is still described via (4.19) since τ acts by permuting negative affine roots. Therefore, E_x is independent of the reduced decomposition of x and we still have:

$$l(x) = \#\left\{\widetilde{\lambda} \in \widehat{\Delta}^+ \,\middle|\, x(\widetilde{\lambda}) \in \widehat{\Delta}^-\right\}.$$

The following result is well-known (cf. [16, Proposition 3.9]):

Proposition 4.5. *For any $\mu \in P^\vee$ such that $(\alpha_i, \mu) \in \mathbb{Z}_{>0}$ for all $i \in I$, we have*

$$E_{\widehat{\mu}} = \left\{(\alpha, d) \,\middle|\, \alpha \in \Delta^+, 0 \leq d < (\alpha, \mu)\right\}, \tag{4.21}$$

and consequently

$$l(\widehat{\mu}) = \sum_{\alpha \in \Delta^+} (\alpha, \mu)\,.$$

4.6. **Identification of two orders.**

We start by recalling the classical construction of [3]. Pick any $\mu \in P^\vee$ such that $(\alpha_i, \mu) \in \mathbb{Z}_{>0}$ for all $i \in I$. Let $l = l(\widehat{\mu})$ and consider any reduced decomposition:

$$\widehat{\mu} = \tau s_{i_{1-l}} s_{i_{2-l}} \dots s_{i_0}\,. \tag{4.22}$$

Extend $i_{1-l}, \dots, i_0$ to a τ-*quasiperiodic* bi-infinite sequence $\{i_k\}_{k \in \mathbb{Z}}$ via:

$$i_{k+l} = \tau(i_k) \qquad \forall\, k \in \mathbb{Z}\,. \tag{4.23}$$

To such a bi-infinite sequence (4.23), one assigns the following bi-infinite sequence of affine roots:

$$\beta_k = \begin{cases} s_{i_1} s_{i_2} \dots s_{i_{k-1}}(-\alpha_{i_k}) & \text{if } k > 0 \\ s_{i_0} s_{i_{-1}} \dots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0 \end{cases}. \tag{4.24}$$

According to [19, 20], the sequences:

$$\beta_1 > \beta_2 > \beta_3 > \cdots \tag{4.25}$$

$$\beta_0 < \beta_{-1} < \beta_{-2} < \cdots \tag{4.26}$$

give convex orders of the sets $\Delta^+ \times \mathbb{Z}_{<0}$ and $\Delta^+ \times \mathbb{Z}_{\geq 0}$, respectively. We note that if $\beta_k = (\alpha, d)$ and $\beta_{k+l} = (\alpha', d')$, then

$$\beta_{k+l} = \widehat{\mu}(\beta_k) \qquad \Longrightarrow \qquad \alpha = \alpha' \text{ and } d = d' + (\alpha, \mu)\,, \tag{4.27}$$

due to (4.17). This reveals a periodicity of the entire set $\Delta^+ \times \mathbb{Z}$.

Evoking the setup of Section 3, let us consider

$$\mu = \sum_{i \in I} c_i \omega_i^\vee \tag{4.28}$$

so that $f(\alpha) = (\alpha, \mu)$ for any $\alpha \in \Delta^+$, cf. (3.2) and (4.15). The following is the first main result of this Section, which naturally generalizes [16, Theorem 3.14]:

Theorem 4.7. *There exists a reduced decomposition of $\widehat{\mu} \in \widehat{W}^{\text{ext}}$ such that:*

- *the order (4.25) of the roots $\{(\alpha, d) \,|\, \alpha \in \Delta^+, d < 0\}$ matches the lexicographic order of the standard Lyndon loop words $\ell(\alpha, -d)$ via (1.13),*
- *the order (4.26) of the roots $\{(\alpha, d) \,|\, \alpha \in \Delta^+, d \geq 0\}$ matches the lexicographic order of the standard Lyndon loop words $\ell(\alpha, -d)$ via (1.13).*

Proof. Recall the finite subset $L = \{(\alpha, d) \,|\, \alpha \in \Delta^+, 0 \leq d < f(\alpha)\} \subset \widehat{\Delta}^+$ from (3.14), ordered via:

$$(\alpha, d) < (\beta, e) \quad \Longleftrightarrow \quad \ell(\alpha, -d) < \ell(\beta, -e)\,. \tag{4.29}$$

If $(\alpha, d), (\beta, e) \in L$ with $(\alpha, d) < (\beta, e)$ and $(\alpha + \beta, d + e) \in \widehat{\Delta}$, then clearly $(\alpha+\beta, d+e) \in L$, as well as $(\alpha, d) < (\alpha+\beta, d+e) < (\beta, e)$, due to Proposition 3.18.

Furthermore, we claim that if $\widetilde{\lambda}, \widetilde{\mu} \in \widehat{\Delta}^+$ with $\widetilde{\lambda} + \widetilde{\mu} \in L$, then at least one of $\widetilde{\lambda}$ or $\widetilde{\mu}$ belongs to L and is $< \widetilde{\lambda} + \widetilde{\mu}$. There are two cases to consider:

(1) If $\widetilde{\lambda} = (\alpha, d)$, $\widetilde{\mu} = (\beta, e)$ with $\alpha, \beta \in \Delta^+$ and $d, e \geq 0$, we can assume without loss of generality that $\ell(\alpha, -d) < \ell(\beta, -e)$. By Proposition 3.18, we have $\ell(\alpha, -d) < \ell(\alpha + \beta, -d - e) < \ell(\beta, -e)$. It remains to prove $d < f(\alpha)$. If not, then $e < f(\beta)$ as $d + e < f(\alpha + \beta)$. Hence, the first letter of $\ell(\alpha, -d)$ has a relative exponent ≤ -1 and the first letter of $\ell(\beta, -e)$ has a relative exponent > -1, due to Corollary 3.11 and Proposition 3.15, which contradicts $\ell(\alpha, -d) < \ell(\beta, -e)$.

(2) In the remaining case, we may assume $\widetilde{\lambda} = (\alpha + \beta, d), \widetilde{\mu} = (-\beta, e)$, so that $\alpha, \beta, \alpha + \beta \in \Delta^+$ and $d \geq 0, e > 0$. Then $d < d + e < f(\alpha) < f(\alpha + \beta)$, so that $\widetilde{\lambda} \in L$. It remains to verify $\ell(\alpha + \beta, -d) < \ell(\alpha, -d - e)$. Since $(\alpha+\beta, -d) = (\beta, e) + (\alpha, -d-e)$, it suffices to prove $\ell(\beta, e) < \ell(\alpha, -d-e)$, due to Proposition 3.18. But applying Corollary 3.11 once again, we see that the exponent of the first letter in $\ell(\beta, e)$ is > 0, while the exponent of the first letter in $\ell(\alpha, -d - e)$ is ≤ 0, hence, indeed $\ell(\beta, e) < \ell(\alpha, -d - e)$.

Invoking [18] (which also applies to finite subsets in affine root systems), we get:

(I) there is a unique element $x \in \widehat{W}$ such that $L = E_x$,

(II) the order of L arises via a unique reduced decomposition of x, where the set E_x of (4.18) is ordered via $\alpha_{i_0} < s_{i_0}(\alpha_{i_{-1}}) < \cdots < s_{i_0} s_{i_{-1}} \ldots s_{i_{2-l}}(\alpha_{i_{1-l}})$.

However, as follows from (4.21), we have

$$L = E_{\widehat{\mu}} = \Big\{\beta_0, \beta_{-1}, \ldots, \beta_{1-l}\Big\}. \tag{4.30}$$

There is a unique $\tau \in \mathcal{T}$ such that $\tau^{-1}\widehat{\mu} \in \widehat{W}$. Thus, we obtain $L = E_{\widehat{\mu}} = E_{\tau^{-1}\widehat{\mu}}$. Therefore, in view of the uniqueness statement of (I), the result of (II) implies that there exists a reduced decomposition (4.22) of $\widehat{\mu}$ such that the ordered finite sequence $\beta_0 < \beta_{-1} < \cdots < \beta_{1-l}$ exactly coincides with L ordered via (4.29).

The proof of Theorem 4.7 now follows by combining (4.27), Proposition 3.15, and Theorem 3.6, precisely as in [16]. Indeed, let us split $\Delta^+ \times \mathbb{Z}$ into the blocks:

$$L_N = \Big\{(\alpha, d) \,\Big|\, \alpha \in \Delta^+,\, N \cdot f(\alpha) \leq d < (N+1) f(\alpha)\Big\}$$

so that

$$\bigsqcup_{N \geq 0} L_N = \Delta^+ \times \mathbb{Z}_{\geq 0} = \{\beta_k\}_{k \leq 0}\,, \qquad \bigsqcup_{N < 0} L_N = \Delta^+ \times \mathbb{Z}_{<0} = \{\beta_k\}_{k > 0}\,.$$

According to (4.27) and $L_0 = L = \{\beta_0, \ldots, \beta_{1-l}\}$, we have:

$$L_N = \Big\{\beta_{-Nl}, \beta_{-Nl-1}, \ldots, \beta_{1-(N+1)l}\Big\} \qquad \forall\, N \in \mathbb{Z}\,.$$

For any $(\alpha, d) \in L_N$, the relative exponent of the first letter in $\ell(\alpha, -d)$ lies in $(-N-1; N]$, due to Corollary 3.11 and Proposition 3.15. Thus, for any $(\alpha, d) \in L_M$, $(\beta, e) \in L_N$ with $M > N$, we have $\ell(\alpha, -d) > \ell(\beta, -e)$. As for the affine roots from the same block, consider $\beta_{r-Nl}, \beta_{s-Nl} \in L_N$ with $1 - l \leq s < r \leq 0$. If $\beta_r = (\alpha, d)$ and $\beta_s = (\beta, e)$, then $\beta_{r-Nl} = (\alpha, d + N \cdot f(\alpha))$ and $\beta_{s-Nl} = (\beta, e + N \cdot f(\beta))$, due to (4.27). On the other hand, the words $\ell(\alpha, -d - N \cdot f(\alpha))$ and $\ell(\beta, -e - N \cdot f(\beta))$ are obtained from $\ell(\alpha, -d)$ and $\ell(\beta, -e)$, respectively, by decreasing each relative exponent by N, due to Proposition 3.15. Since the latter operation obviously preserves the lexicographic order, and $\ell(\alpha, -d) < \ell(\beta, -e)$ as a consequence of $r > s$, we obtain the required inequality $\ell(\alpha, -d - N \cdot f(\alpha)) < \ell(\beta, -e - N \cdot f(\beta))$. $\square$

Remark 4.8. Since $\ell(\alpha, -d) < \ell(\beta, -e)$ if $d < 0 \leq e$, a consequence of Corollary 3.11, we actually have the stronger result that the order of $\Delta^+ \times \mathbb{Z}$ given by:

$$\cdots < \beta_3 < \beta_2 < \beta_1 < \beta_0 < \beta_{-1} < \beta_{-2} < \cdots \tag{4.31}$$

matches the lexicographic order of the standard Lyndon loop words $\ell(\alpha, -d)$.

In the next Section, we shall need a certain generalization of (4.30). To this end, for any $i \in I$ and $d \geq 0$, we define the subset $L_{<(i,d)}$ of $\Delta^+ \times \mathbb{Z}$ via

$$L_{<(i,d)} = \Big\{(\alpha, p) \,\Big|\, \alpha \in \Delta^+,\, p \in \mathbb{Z}_{\geq 0},\, \ell(\alpha, -p) < \ell(\alpha_i, -d)\Big\}. \tag{4.32}$$

We also define a collection of nonnegative integers $\{p_j\}_{j \in I}$ via:

$$p_j = \begin{cases} d & \text{if } j = i \\ \left\lceil \frac{d \cdot c_j}{c_i} \right\rceil & \text{if } \frac{d \cdot c_j}{c_i} \notin \mathbb{Z} \\ \frac{d \cdot c_j}{c_i} & \text{if } \frac{d \cdot c_j}{c_i} \in \mathbb{Z} \text{ and } j > i \\ \frac{d \cdot c_j}{c_i} + 1 & \text{if } \frac{d \cdot c_j}{c_i} \in \mathbb{Z} \text{ and } j < i \end{cases}. \tag{4.33}$$

Finally, for any positive root $\alpha = \sum_{i \in I} k_i \alpha_i \in \Delta^+$, we define $p(\alpha) \in \mathbb{N}$ via

$$p(\alpha) = \sum_{i \in I} k_i p_i\,.$$

Proposition 4.9. *For any $i \in I$ and $d \geq 0$, we have*

$$L_{<(i,d)} = \Big\{(\alpha, p) \,\Big|\, \alpha \in \Delta^+, 0 \leq p < p(\alpha)\Big\}.$$

Proof. First, let us prove that $\ell(\alpha, -p) < \ell(\alpha_i, -d) = i^{(-d)}$ implies $p < p(\alpha)$. Let $j^{(-e)}$ be the first letter of $\ell(\alpha, -p)$, so that $j^{(-e)} < i^{(-d)}$. Hence, $e/c_j \leq d/c_i$ and the inequality is strict if $j \geq i$. This is equivalent to $e < p_j$, due to the definition (4.33). Then for any letter $\imath^{(-s)} \in \ell(\alpha, -p)$, we have $\imath^{(-s+1)} \leq j^{(-e)} \leq i^{(-d)}$ with the first inequality due to Theorem 3.6. As above, this implies $s - 1 < p_\imath$, so that $s \leq p_\imath$. Summing all these inequalities, we obtain the desired inequality $p < p(\alpha)$.

Let us prove the opposite implication by contradiction: assume that $\ell(\alpha, -p) > \ell(\alpha_i, -d)$ for some $\alpha \in \Delta^+$ and $p < p(\alpha)$. Let $j^{(-e)}$ be the first letter of $\ell(\alpha, -p)$, so that $j^{(-e)} \geq i^{(-d)}$. We consider two cases:

- *Case 1*: $j^{(-e)} = i^{(-d)}$.
  As $\ell(\alpha, -p)$ is Lyndon, any letter $\imath^{(-s)} \in \ell(\alpha, -p)$ satisfies $j^{(-e)} \leq \imath^{(-s)}$. Therefore, $s/c_\imath \geq e/c_j = d/c_i$ and the inequality is strict for $\imath < i$. Thus, $s \geq p_\imath$. Summing all these inequalities, we obtain $p \geq p(\alpha)$, a contradiction.
- *Case 2*: $j^{(-e)} > i^{(-d)}$.
  As in Case 1, any letter $\imath^{(-s)} \in \ell(\alpha, -p)$ satisfies $\imath^{(-s)} \geq j^{(-e)} > i^{(-d)}$. Hence, $s/c_\imath \geq d/c_i$ and the inequality is strict for $\imath < i$. Thus, $s \geq p_\imath$. Summing all these inequalities, we again obtain $p \geq p(\alpha)$, a contradiction.

This completes our proof of $\ell(\alpha, -p) < \ell(\alpha_i, -d)$ for any $0 \leq p < p(\alpha)$. □

In view of Proposition 4.5, the above result can be recast as follows:

Proposition 4.10. *For any $i \in I$ and $d \geq 0$, we have $L_{<(i,d)} = E_{\widehat{\omega_{i,d}}}$, where*

$$\omega_{i,d} = \sum_{j \in I} p_j \omega_j^\vee \in P^\vee \tag{4.34}$$

with p_j's defined in (4.33).

5. Quantum groups and PBW bases

In this Section, we combine the results of Subsection 4.6 with the PBW-type bases [2, 3] of quantum affine algebras (in the Drinfeld-Jimbo realization) to produce a family of PBW-type combinatorial bases of quantum loop algebras (in the new Drinfeld realization), thus generalizing the construction of [12] for the finite type.

5.1. **Quantum groups.**

We shall follow the notation of Subsection 2.14, corresponding to a simple finite-dimensional $\mathfrak{g}$. Consider the q-numbers, q-factorials, and q-binomial coefficients:

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}\,, \qquad [k]!_i = [1]_i \dots [k]_i\,, \qquad \binom{n}{k}_i = \frac{[n]!_i}{[k]!_i [n-k]!_i}$$

for any $i \in I$, where $q_i = q^{\frac{d_{ii}}{2}}$.

Definition 5.2. *The Drinfeld-Jimbo quantum group of $\mathfrak{g}$, denoted by $U_q(\mathfrak{g})$, is an associative $\mathbb{Q}(q)$-algebra generated by $\{e_i, f_i, \varphi_i^{\pm 1}\}_{i \in I}$ subject to the following defining relations (for all $i, j \in I$):*

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i e_i^k e_j e_i^{1-a_{ij}-k} = 0 \qquad \text{if } i \neq j\,, \tag{5.1}$$

$$\varphi_i e_j = q^{d_{ij}} e_j \varphi_i\,, \qquad \varphi_i \varphi_j = \varphi_j \varphi_i\,, \tag{5.2}$$

as well as the opposite relations with e's replaced by f's, and finally the relation:

$$[e_i, f_j] = \delta_{ij} \cdot \frac{\varphi_i - \varphi_i^{-1}}{q_i - q_i^{-1}}\,. \tag{5.3}$$

The algebra $U_q(\mathfrak{g})$ is naturally Q-graded via

$$\deg e_i = \alpha_i\,, \qquad \deg \varphi_i = 0\,, \qquad \deg f_i = -\alpha_i\,.$$

Furthermore, it admits the triangular decomposition (1.6):

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-)\,,$$

where $U_q(\mathfrak{n}^+)$, $U_q(\mathfrak{h})$, and $U_q(\mathfrak{n}^-)$ are the subalgebras of $U_q(\mathfrak{g})$ generated by the e_i's, $\varphi_i^{\pm 1}$'s, and f_i's, respectively. In fact, the associative algebra $U_q(\mathfrak{n}^+)$ is generated by e_i's with the defining relations (5.1), cf. e.g. [9, §4.21].

If we write $\varphi_i = q_i^{h_i}$ and take the limit $q \to 1$, then $U_q(\mathfrak{g})$ degenerates to $U(\mathfrak{g})$. It is thus natural that many features of the latter also admit q-deformations. For example, let us recall the notion of standard Lyndon words from Subsections 2.1–2.7, and consider the following q-version of the construction of (2.4) and Definition 2.8.

Definition 5.3. *([12]) For any word w, define $e_w \in U_q(\mathfrak{n}^+)$ by:*

$$e_{[i]} = e_i$$

for all $i \in I$, and then recursively by:

$$e_\ell = [e_{\ell_1}, e_{\ell_2}]_q = e_{\ell_1} e_{\ell_2} - q^{(\deg \ell_1, \deg \ell_2)} e_{\ell_2} e_{\ell_1} \tag{5.4}$$

if ℓ is a Lyndon word with the costandard factorization (2.1), and:

$$e_w = e_{\ell_1} \dots e_{\ell_k} \tag{5.5}$$

if w is an arbitrary word with the canonical factorization $\ell_1 \dots \ell_k$, as in (2.2).

We also define $f_w \in U_q(\mathfrak{n}^-)$ by replacing e's by f's in the above Definition. Then we have the following natural q-deformation of the PBW theorem (2.6):

Theorem 5.4. *We have:*

$$U_q(\mathfrak{n}^+) \; = \bigoplus_{\ell_1 \geq \dots \geq \ell_k \text{ standard Lyndon words}}^{k \in \mathbb{N}} \mathbb{Q}(q) \cdot e_{\ell_1} \dots e_{\ell_k} \; = \bigoplus_{w-\text{standard words}} \mathbb{Q}(q) \cdot e_w \,. \tag{5.6}$$

The analogous result also holds with $\mathfrak{n}^+ \leftrightarrow \mathfrak{n}^-$ and $e \leftrightarrow f$.

This result is a consequence of the usual PBW theorem for $U_q(\mathfrak{n}^\pm)$, since e_ℓ's are simply renormalizations of the standard root vectors constructed in [15] using the braid group action, according to [12, Theorem 28] (cf. also [16, Section 6.5]).

5.5. **Quantum loop algebras.**

To introduce a loop version of the above algebras, consider the generating series

$$e_i(z) = \sum_{k \in \mathbb{Z}} \frac{e_{i,k}}{z^k} \,, \qquad f_i(z) = \sum_{k \in \mathbb{Z}} \frac{f_{i,k}}{z^k} \,, \qquad \varphi_i^\pm(z) = \sum_{l=0}^{\infty} \frac{\varphi_{i,l}^\pm}{z^{\pm l}}$$

as well as the formal delta function $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$. For any $i, j \in I$, we set:

$$\zeta_{ij}\left(\frac{z}{w}\right) = \frac{z - wq^{-d_{ij}}}{z - w} \,.$$

Definition 5.6. *The quantum loop group (in the new Drinfeld realization) of $\mathfrak{g}$, denoted by $U_q(L\mathfrak{g})$, is an associative $\mathbb{Q}(q)$-algebra generated by $\{e_{i,k}, f_{i,k}, \varphi_{i,l}^\pm\}_{i \in I}^{k \in \mathbb{Z}, l \in \mathbb{N}}$ subject to the following defining relations (for all $i, j \in I$):*

$$e_i(z) e_j(w) \zeta_{ji}\left(\frac{w}{z}\right) = e_j(w) e_i(z) \zeta_{ij}\left(\frac{z}{w}\right), \tag{5.7}$$

$$\sum_{\sigma \in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i \cdot \tag{5.8}$$
$$e_i(z_{\sigma(1)}) \dots e_i(z_{\sigma(k)}) e_j(w) e_i(z_{\sigma(k+1)}) \dots e_i(z_{\sigma(1-a_{ij})}) = 0 \qquad \text{if } i \neq j\,,$$

$$\varphi_i^{\pm}(z) e_j(w) \zeta_{ji}\left(\frac{w}{z}\right) = e_j(w) \varphi_i^{\pm}(z) \zeta_{ij}\left(\frac{z}{w}\right), \tag{5.9}$$

$$\varphi_i^{\pm}(z) \varphi_j^{\pm'}(w) = \varphi_j^{\pm'}(w) \varphi_i^{\pm}(z)\,, \qquad \varphi_{i,0}^{+} \varphi_{i,0}^{-} = 1\,, \tag{5.10}$$

as well as the opposite relations with e's replaced by f's, and finally the relation:

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \delta\left(\frac{z}{w}\right) \cdot \Big(\varphi_i^{+}(z) - \varphi_i^{-}(w)\Big). \tag{5.11}$$

The algebra $U_q(L\mathfrak{g})$ is naturally $Q \times \mathbb{Z}$-graded via

$$\deg e_{i,k} = (\alpha_i, k)\,, \qquad \deg \varphi_{i,l}^{\pm} = (0, \pm l)\,, \qquad \deg f_{i,k} = (-\alpha_i, k)$$

for $i \in I, k \in \mathbb{Z}, l \in \mathbb{N}$. If $x \in U_q(L\mathfrak{g})$ has a $Q \times \mathbb{Z}$-degree $\deg x = (\alpha, d)$, then we set

$$\operatorname{hdeg} x = \alpha \qquad \text{and} \qquad \operatorname{vdeg} x = d\,, \tag{5.12}$$

and call these the *horizontal* and the *vertical* degrees of x, respectively, cf. (3.10). Finally, the algebra $U_q(L\mathfrak{g})$ also admits the triangular decomposition (cf. [8, §3.3]):

$$U_q(L\mathfrak{g}) = U_q(L\mathfrak{n}^+) \otimes U_q(L\mathfrak{h}) \otimes U_q(L\mathfrak{n}^-)\,, \tag{5.13}$$

where $U_q(L\mathfrak{n}^+)$, $U_q(L\mathfrak{h})$, and $U_q(L\mathfrak{n}^-)$ are the subalgebras of $U_q(L\mathfrak{g})$ generated by the $e_{i,k}$'s, $\varphi_{i,l}^{\pm}$'s, and $f_{i,k}$'s, respectively. In fact, the associative algebra $U_q(L\mathfrak{n}^+)$ is generated by $e_{i,k}$'s with the defining relations (5.7, 5.8).

Let us now present a loop version of Definition 5.3:

Definition 5.7. *For any loop word w, define $e_w \in U_q(L\mathfrak{n}^+)$ and $f_w \in U_q(L\mathfrak{n}^-)$ by:*

$$e_{[i^{(d)}]} = e_{i,d} \qquad \text{and} \qquad f_{[i^{(d)}]} = f_{i,-d}$$

for all $i \in I$, $d \in \mathbb{Z}$, and then recursively by:

$$e_\ell = [e_{\ell_1}, e_{\ell_2}]_q = e_{\ell_1} e_{\ell_2} - q^{(\operatorname{hdeg} \ell_1, \operatorname{hdeg} \ell_2)} e_{\ell_2} e_{\ell_1}\,, \tag{5.14}$$

$$f_\ell = [f_{\ell_1}, f_{\ell_2}]_q = f_{\ell_1} f_{\ell_2} - q^{(\operatorname{hdeg} \ell_1, \operatorname{hdeg} \ell_2)} f_{\ell_2} f_{\ell_1} \tag{5.15}$$

if ℓ is a Lyndon loop word with the costandard factorization (2.1), and:

$$e_w = e_{\ell_1} \dots e_{\ell_k} \qquad \text{and} \qquad f_w = f_{\ell_1} \dots f_{\ell_k} \tag{5.16}$$

if w is an arbitrary loop word with the canonical factorization $\ell_1 \dots \ell_k$, as in (2.2).

Note that $\deg e_w = -\deg f_w = \deg w$ for all loop words w. The following is the main result of this Section, which generalizes (3.16) as well as Theorem 5.4:

Theorem 5.8. *We have:*

$$U_q(L\mathfrak{n}^+) \;=\; \bigoplus^{k \in \mathbb{N}}_{\ell_1 \geq \dots \geq \ell_k \text{ standard Lyndon loop words}} \mathbb{Q}(q) \cdot e_{\ell_1} \dots e_{\ell_k} \;=\; \bigoplus_{w-\text{standard loop words}} \mathbb{Q}(q) \cdot e_w\,.$$

The analogous result also holds with $L\mathfrak{n}^+ \leftrightarrow L\mathfrak{n}^-$ and $e \leftrightarrow f$.

The proof of this result occupies the rest of this Section. While it looks similar to the proof of [16, Theorem 4.24], we shall crucially utilize Proposition 4.10.

5.9. **Quantum affine algebras.**
Let us recall the notion of Drinfeld-Jimbo quantum affine algebras and their relation to quantum loop algebras $U_q(L\mathfrak{g})$. We use the notations of Subsection 4.1.

Definition 5.10. *The Drinfeld-Jimbo quantum affine algebra of $\widehat{\mathfrak{g}}$, denoted by $U_q(\widehat{\mathfrak{g}})$, is defined exactly as $U_q(\mathfrak{g})$ in Definition 5.2, but using $\widehat{I}$ instead of I.*

Let $U_q(\widehat{\mathfrak{n}}^+), U_q(\widehat{\mathfrak{h}}), U_q(\widehat{\mathfrak{n}}^-)$ be the subalgebras generated by the e_i's, $\varphi_i^{\pm 1}$'s, f_i's, respectively (with $i \in \widehat{I}$). We have a triangular decomposition analogous to (1.6):

$$U_q(\widehat{\mathfrak{g}}) = U_q(\widehat{\mathfrak{n}}^+) \otimes U_q(\widehat{\mathfrak{h}}) \otimes U_q(\widehat{\mathfrak{n}}^-)\,. \tag{5.17}$$

The algebra $U_q(\widehat{\mathfrak{g}})$ is naturally $\widehat{Q} \simeq Q \times \mathbb{Z}$-graded via

$$\begin{aligned} &\deg e_0 = \alpha_0 = (-\theta, 1)\,, && \deg f_0 = -\alpha_0 = (\theta, -1)\,, && \deg \varphi_0 = 0 = (0,0)\,, \\ &\deg e_i = \alpha_i = (\alpha_i, 0)\,, && \deg f_i = -\alpha_i = (-\alpha_i, 0)\,, && \deg \varphi_i = 0 = (0,0) \end{aligned}$$

for $i \in I$, where θ is the highest root of Δ^+. Invoking the positive integers $\{\theta_i\}_{i\in I}$ introduced in (4.1), we note that the following element is central in $U_q(\widehat{\mathfrak{g}})$:

$$C = \varphi_0 \prod_{i\in I} \varphi_i^{\theta_i}\,. \tag{5.18}$$

Let us now recall the construction of the root vectors of $U_q(\widehat{\mathfrak{g}})$, presented in [2, 15]. Following Subsection 4.6, pick the coweight $\mu = \sum_{i\in I} c_i \omega_i^\vee \in P^\vee$ as in (4.28), and set $\widehat{\mu} = 1 \ltimes \mu \in \widehat{W}^{\mathrm{ext}}$. We consider the reduced decomposition:

$$\widehat{\mu} = \tau s_{i_{1-l}} s_{i_{2-l}} \dots s_{i_0}$$

from Theorem 4.7 with $\tau \in \mathcal{T}$. Following (4.23), let us extend $\{i_k \mid -l < k \leq 0\}$ to a τ-*quasiperiodic* bi-infinite sequence $\{i_k\}_{k\in\mathbb{Z}}$ via $i_{k+l} = \tau(i_k)$ for any $k \in \mathbb{Z}$. We construct the following set of positive affine roots:

$$\tilde{\beta}_k = \begin{cases} s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0 \\ s_{i_0} s_{i_{-1}} \dots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0 \end{cases} = \begin{cases} -\beta_k & \text{if } k > 0 \\ \beta_k & \text{if } k \leq 0 \end{cases}, \tag{5.19}$$

with β_k defined in (4.24). Following [2], we shall order those roots as follows:

$$\tilde{\beta}_0 < \tilde{\beta}_{-1} < \tilde{\beta}_{-2} < \tilde{\beta}_{-3} < \dots < \tilde{\beta}_4 < \tilde{\beta}_3 < \tilde{\beta}_2 < \tilde{\beta}_1\,. \tag{5.20}$$

Remark 5.11. Formula (5.19) provides all real positive roots of $\widehat{\Delta}^+$:

$$\widehat{\Delta}^{\mathrm{re},+} = \Big\{\Delta^+ \times \mathbb{Z}_{\geq 0}\Big\} \sqcup \Big\{\Delta^- \times \mathbb{Z}_{>0}\Big\} \subset \widehat{\Delta}^+\,. \tag{5.21}$$

Furthermore, (5.20) induces convex orders on the corresponding halves:

$$\Delta^+ \times \mathbb{Z}_{\geq 0} = \Big\{\tilde{\beta}_0 < \tilde{\beta}_{-1} < \tilde{\beta}_{-2} < \cdots\Big\}, \Delta^- \times \mathbb{Z}_{>0} = \Big\{\cdots < \tilde{\beta}_3 < \tilde{\beta}_2 < \tilde{\beta}_1\Big\}. \tag{5.22}$$

To have a complete theory, in particular for the PBW theorem of [2],one also needs to deal with the imaginary roots, but they will not feature in the present paper.

We may define the *root vectors*:

$$E_{\pm\tilde{\beta}} \in U_q(\widehat{\mathfrak{n}}^{\pm})$$

for all $\tilde{\beta} \in \widehat{\Delta}^{\mathrm{re},+}$ of (5.21) via

$$E_{\tilde{\beta}_k} = \begin{cases} T_{i_1} \dots T_{i_{k-1}}(e_{i_k}) & \text{if } k > 0 \\ T_{i_0}^{-1} \dots T_{i_{k+1}}^{-1}(e_{i_k}) & \text{if } k \leq 0 \end{cases} \tag{5.23}$$

and

$$E_{-\tilde{\beta}_k} = \begin{cases} T_{i_1} \dots T_{i_{k-1}}(f_{i_k}) & \text{if } k > 0 \\ T_{i_0}^{-1} \dots T_{i_{k+1}}^{-1}(f_{i_k}) & \text{if } k \leq 0 \end{cases} \tag{5.24}$$

where $\{T_i\}_{i \in \widehat{I}}$ determine Lusztig's affine braid group action [15] on $U_q(\widehat{\mathfrak{g}})$.

Remark 5.12. We note that $E_{-\tilde{\beta}} \in U_q(\widehat{\mathfrak{n}}^-)$ for $\tilde{\beta} \in \widehat{\Delta}^{\mathrm{re},+}$ in [2] are defined via

$$E_{-\tilde{\beta}} := \Omega(E_{\tilde{\beta}}) \,, \tag{5.25}$$

where the $\mathbb{Q}$-algebra anti-involution Ω of $U_q(\widehat{\mathfrak{g}})$ is determined by:

$$\Omega\colon\, e_i \mapsto f_i,\ f_i \mapsto e_i,\ \varphi_i^{\pm 1} \mapsto \varphi_i^{\mp 1},\ q \mapsto q^{-1} \qquad \forall\, i \in \widehat{I}\,. \tag{5.26}$$

Formulas (5.24) and (5.25) agree, as Ω commutes with the affine braid group action:

$$\Omega \circ T_i = T_i \circ \Omega \qquad \forall\, i \in \widehat{I}\,. \tag{5.27}$$

According to [16, (5.28)] (based on [2, Proposition 7]), we have

$$[E_{\pm\tilde{\beta}}, E_{\pm\tilde{\alpha}}]_q = E_{\pm\tilde{\beta}} E_{\pm\tilde{\alpha}} - q^{(\tilde{\alpha},\tilde{\beta})} E_{\pm\tilde{\alpha}} E_{\pm\tilde{\beta}} \in \mathbb{Q}(q)^* \cdot E_{\pm(\tilde{\alpha}+\tilde{\beta})} \tag{5.28}$$

for any real positive affine roots $\tilde{\alpha} < \tilde{\beta}$ which both belong to either $\Delta^+ \times \mathbb{Z}_{\geq 0}$ or $\Delta^- \times \mathbb{Z}_{>0}$ and which also have the additional property that $\tilde{\alpha} + \tilde{\beta}$ is a positive affine root whose decomposition as the sum of $\tilde{\alpha}$ and $\tilde{\beta}$ is *minimal* in the sense that:

$$\nexists\ \tilde{\alpha}', \tilde{\beta}' \in \widehat{\Delta}^{\mathrm{re},+} \quad \text{s.t.} \quad \tilde{\alpha} < \tilde{\alpha}' < \tilde{\beta}' < \tilde{\beta} \quad \text{and} \quad \tilde{\alpha} + \tilde{\beta} = \tilde{\alpha}' + \tilde{\beta}'\,. \tag{5.29}$$

Let $U_q^{\pm}(+\infty)$ and $U_q^{\pm}(-\infty)$ denote the "quarter" subalgebras of $U_q(\widehat{\mathfrak{g}})$ generated by $\{E_{\pm\tilde{\beta}_k} \,|\, k \geq 1\}$ and $\{E_{\pm\tilde{\beta}_k} \,|\, k \leq 0\}$, respectively. According to [16, (5.35, 5.36)] (based on [2]), each of them admits a pair of opposite PBW decompositions:

$$U_q^{\pm}(+\infty) = \bigoplus_{\substack{n_1, n_2, \dots \in \mathbb{N} \\ n_1 + n_2 + \dots < \infty}} \mathbb{Q}(q) \cdot E_{\pm\tilde{\beta}_1}^{n_1} E_{\pm\tilde{\beta}_2}^{n_2} \dots = \bigoplus_{\substack{n_1, n_2, \dots \in \mathbb{N} \\ n_1 + n_2 + \dots < \infty}} \mathbb{Q}(q) \cdot \dots E_{\pm\tilde{\beta}_2}^{n_2} E_{\pm\tilde{\beta}_1}^{n_1}\,, \tag{5.30}$$

$$U_q^{\pm}(-\infty) = \bigoplus_{\substack{n_0, n_{-1}, \dots \in \mathbb{N} \\ n_0 + n_{-1} + \dots < \infty}} \mathbb{Q}(q) \cdot E_{\pm\tilde{\beta}_0}^{n_0} E_{\pm\tilde{\beta}_{-1}}^{n_{-1}} \dots = \bigoplus_{\substack{n_0, n_{-1}, \dots \in \mathbb{N} \\ n_0 + n_{-1} + \dots < \infty}} \mathbb{Q}(q) \cdot \dots E_{\pm\tilde{\beta}_{-1}}^{n_{-1}} E_{\pm\tilde{\beta}_0}^{n_0}\,. \tag{5.31}$$

5.13. **Interplay of two algebras.**
The relation between $U_q(L\mathfrak{g})$ of Definition 5.6 and $U_q(\widehat{\mathfrak{g}})$ of Definition 5.10 goes back to [2, 3, 5] and plays a crucial role in the theory of quantum affine algebras. In the present setup, it amounts to the following result, cf. [16, Theorem 5.19]:

Theorem 5.14. *There exists an algebra isomorphism:*

$$U_q(L\mathfrak{g}) \xrightarrow{\sim} U_q(\widehat{\mathfrak{g}})/(C-1) \tag{5.32}$$

with C of (5.18), determined by the following assignment for all $i \in I$ and $d \in \mathbb{Z}$:

$$\begin{aligned} e_{i,d} &\mapsto \begin{cases} o(i)^d E_{(\alpha_i,d)} & \text{if } d \geq 0 \\ -o(i)^d E_{(\alpha_i,d)}\varphi_i^{-1} & \text{if } d < 0 \end{cases}, \\ f_{i,d} &\mapsto \begin{cases} -o(i)^d \varphi_i E_{(-\alpha_i,d)} & \text{if } d > 0 \\ o(i)^d E_{(-\alpha_i,d)} & \text{if } d \leq 0 \end{cases}, \end{aligned} \tag{5.33}$$

where $o\colon I \to \{\pm 1\}$ is a map satisfying $o(i)o(j) = -1$ whenever $a_{ij} < 0$.

The proof of this result is similar to that of [16, Theorem 5.19], but it does essentially utilize Proposition 4.10 as well as simplifies some arguments from [16].

Proof of Theorem 5.14. The isomorphism (5.32) was proved in [3, Theorem 4.7] with respect to the following seemingly different formula:

$$e_{i,d} \mapsto o(i)^d T^{-d}_{\widehat{\omega_i^\vee}}(e_i)\,, \qquad f_{i,d} \mapsto o(i)^d T^{d}_{\widehat{\omega_i^\vee}}(f_i) \qquad \forall\, i \in I, d \in \mathbb{Z}\,. \tag{5.34}$$

Here, the aforementioned action of the affine braid group on $U_q(\widehat{\mathfrak{g}})$ has been extended to the extended affine braid group by adding automorphisms $\{T_\tau\}_{\tau \in \mathcal{T}}$:

$$T_\tau\colon e_i \mapsto e_{\tau(i)},\ f_i \mapsto f_{\tau(i)},\ \varphi_i^{\pm 1} \mapsto \varphi_{\tau(i)}^{\pm 1} \qquad \forall\, \tau \in \mathcal{T}, i \in \widehat{I}\,,$$

which satisfy the relations $T_\tau T_i = T_{\tau(i)} T_\tau$ for any $\tau \in \mathcal{T}$ and $i \in \widehat{I}$.
Therefore, it remains for us to show that (5.33) is equivalent to (5.34) by proving:

$$T^{-d}_{\widehat{\omega_i^\vee}}(e_i) = \begin{cases} E_{(\alpha_i,d)} & \text{if } d \geq 0 \\ -E_{(\alpha_i,d)}\varphi_i^{-1} & \text{if } d < 0 \end{cases}, \tag{5.35}$$

$$T^{d}_{\widehat{\omega_i^\vee}}(f_i) = \begin{cases} -\varphi_i E_{(-\alpha_i,d)} & \text{if } d > 0 \\ E_{(-\alpha_i,d)} & \text{if } d \leq 0 \end{cases}. \tag{5.36}$$

It suffices to prove only (5.35) since (5.36) would then follow as Ω commutes with the extended affine braid group action (due to (5.27) and $\Omega \circ T_\tau = T_\tau \circ \Omega$ for $\tau \in \mathcal{T}$).
Fix $i \in I, d \geq 0$. According to (5.22), there is a unique $k \leq 0$ such that

$$(\alpha_i, d) = \tilde{\beta}_k = \beta_k = s_{i_0} s_{i_{-1}} \dots s_{i_{k+1}}(\alpha_{i_k})\,. \tag{5.37}$$

Invoking (4.34), we claim that $\widehat{\omega_{i,d}} \in \widehat{W}^{\mathrm{ext}}$ has a reduced decomposition of the form

$$\widehat{\omega_{i,d}} = \tau s_{i_{k+1}} \dots s_{i_{-1}} s_{i_0} \qquad \text{with} \quad \tau \in \mathcal{T}\,. \tag{5.38}$$

This follows from the equality of terminal sets $E_{s_{i_{k+1}} \dots s_{i_{-1}} s_{i_0}} = E_{\widehat{\omega_{i,d}}}$ (due to Proposition 4.10 and Theorem 4.7) and the fact that $E_x = E_y$ iff $x^{-1}y \in \mathcal{T}$ (already used in the proof of Theorem 4.7). Combining (5.37) and (5.38), we thus obtain

$$(\alpha_i, d) = s_{i_0}^{-1} s_{i_{-1}}^{-1} \dots s_{i_{k+1}}^{-1}(\alpha_{i_k}) = \widehat{\omega_{i,d}}^{-1} \tau(\alpha_{i_k}) = \widehat{\omega_{i,d}}^{-1}(\alpha_{\tau(i_k)})\,.$$

In view of (4.17), this implies $\tau(i_k) = i$. Hence, we get:

$$E_{\tilde{\beta}_k} = T_{i_0}^{-1} T_{i_{-1}}^{-1} \dots T_{i_{k+1}}^{-1}(e_{i_k}) = T_{\widehat{\omega_{i,d}}}^{-1} \tau(e_{i_k}) = T_{\widehat{\omega_{i,d}}}^{-1}(e_i)\,.$$

According to Proposition 4.5, we have $l(\widehat{\omega_{i,d}}) = \sum_{j\in I} p_j l(\widehat{\omega_j^\vee})$, cf. (4.33), so that

$$T_{\widehat{\omega_{i,d}}} = T_{\widehat{\omega_i^\vee}}^{p_i} \cdot \prod_{j\neq i} T_{\widehat{\omega_j^\vee}}^{p_j} = T_{\widehat{\omega_i^\vee}}^{d} \cdot \prod_{j\neq i} T_{\widehat{\omega_j^\vee}}^{p_j}\,.$$

As $T_{\widehat{\omega_j^\vee}}^{\pm 1}(e_i) = e_i$ for $j \neq i$ by [3, Corollary 3.2], we get the desired equality:

$$E_{(\alpha_i,d)} = E_{\tilde{\beta}_k} = T_{\widehat{\omega_{i,d}}}^{-1}(e_i) = T_{\widehat{\omega_i^\vee}}^{-d}(e_i)\,.$$

For $d < 0$, the proof is similar and follows the same arguments as in [16]. □

5.15. **PBW-type bases via quarter subalgebras.**

The isomorphism (5.32) does not intertwine the triangular decompositions (5.13) and (5.17). In fact, if we think of $U_q(L\mathfrak{g})$ and $U_q(\widehat{\mathfrak{g}})/(C-1)$ as one and the same algebra, then these two decompositions are "orthogonal" as explained in [16], cf. [6]. To this end, consider the following "quarter" subalgebras following [2, Lemmas 5–6]:

$$U_q^+(L\mathfrak{n}^-) := U_q(L\mathfrak{n}^-) \cap U_q(\widehat{\mathfrak{b}}^+) = \Big\{\text{subalgebra generated by } \mathsf{e}_{\tilde{\beta}_k}, k > 0\Big\}, \tag{5.39}$$

$$U_q^+(L\mathfrak{n}^+) := U_q(L\mathfrak{n}^+) \cap U_q(\widehat{\mathfrak{b}}^+) = \Big\{\text{subalgebra generated by } \mathsf{e}_{\tilde{\beta}_k}, k \leq 0\Big\}, \tag{5.40}$$

where we define $\mathsf{e}_{\tilde{\beta}_k}$ in accordance with (5.33) via:

$$\mathsf{e}_{\tilde{\beta}_k} = \begin{cases} \varphi_{-\mathrm{hdeg}\ (\tilde{\beta}_k)} E_{\tilde{\beta}_k} & \text{if } k > 0 \\ E_{\tilde{\beta}_k} & \text{if } k \leq 0 \end{cases}. \tag{5.41}$$

Henceforth, given a homogeneous element z of degree $\big(\sum_{i\in I} k_i\alpha_i, d\big) \in Q \times \mathbb{Z}$, set

$$\varphi_{\pm\mathrm{hdeg}\ (z)} := \varphi_{\pm\sum_{i\in I} k_i\alpha_i} = \prod_{i\in I} \varphi_i^{\pm k_i} \in U_q(L\mathfrak{h})\,.$$

Formulas (5.28) still hold when the $E_{\tilde{\beta}_k}$ are replaced with the $\mathsf{e}_{\tilde{\beta}_k}$, since commuting φ's simply produces powers of q. Likewise, the PBW decompositions (5.30, 5.31) imply that the subalgebras above have the following PBW bases:

$$U_q^+(L\mathfrak{n}^-) = \bigoplus_{\substack{n_1,n_2,\dots\in\mathbb{N} \\ n_1+n_2+\dots<\infty}} \mathbb{Q}(q) \cdot \dots \mathsf{e}_{\tilde{\beta}_2}^{n_2} \mathsf{e}_{\tilde{\beta}_1}^{n_1}\,, \tag{5.42}$$

$$U_q^+(L\mathfrak{n}^+) = \bigoplus_{\substack{n_0,n_{-1},\dots\in\mathbb{N} \\ n_0+n_{-1}+\dots<\infty}} \mathbb{Q}(q) \cdot \dots \mathsf{e}_{\tilde{\beta}_{-1}}^{n_{-1}} \mathsf{e}_{\tilde{\beta}_0}^{n_0}\,. \tag{5.43}$$

Likewise, we have PBW bases for analogous "quarter" subalgebras of $U_q(\widehat{\mathfrak{b}}^-)$:

$$U_q^-(L\mathfrak{n}^-) := U_q(L\mathfrak{n}^-) \cap U_q(\widehat{\mathfrak{b}}^-) = \bigoplus_{\substack{n_0,n_{-1},\dots\in\mathbb{N} \\ n_0+n_{-1}+\dots<\infty}} \mathbb{Q}(q) \cdot \mathsf{e}_{-\tilde{\beta}_0}^{n_0} \mathsf{e}_{-\tilde{\beta}_{-1}}^{n_{-1}} \dots\,, \tag{5.44}$$

$$U_q^-(L\mathfrak{n}^+) := U_q(L\mathfrak{n}^+) \cap U_q(\widehat{\mathfrak{b}}^-) = \bigoplus_{\substack{n_1,n_2,\dots\in\mathbb{N} \\ n_1+n_2+\dots<\infty}} \mathbb{Q}(q) \cdot \mathsf{e}_{-\tilde{\beta}_1}^{n_1} \mathsf{e}_{-\tilde{\beta}_2}^{n_2} \dots\,, \tag{5.45}$$

where we define:

$$\mathsf{e}_{-\tilde{\beta}_k} = \Omega(\mathsf{e}_{\tilde{\beta}_k}) = \begin{cases} E_{-\tilde{\beta}_k}\varphi_{\mathrm{hdeg}\,(\tilde{\beta}_k)} & \text{if } k>0 \\ E_{-\tilde{\beta}_k} & \text{if } k\leq 0 \end{cases}. \tag{5.46}$$

The following result allows to construct the PBW bases of $U_q(L\mathfrak{n}^{\pm})$:

Proposition 5.16. [16, Proposition 5.23] *The multiplication map induces a vector space isomorphism:*

$$U_q^+(L\mathfrak{n}^-)\otimes U_q^-(L\mathfrak{n}^-) \xrightarrow{\sim} U_q(L\mathfrak{n}^-)\,. \tag{5.47}$$

To make the presentation uniform, let us switch from $\tilde{\beta}_k$ of (5.19) to β_k of (4.24), so that $U_q^+(L\mathfrak{n}^-)$ and $U_q^-(L\mathfrak{n}^-)$ are generated by $\{\mathsf{e}_{-\beta_k}\}_{k\geq 1}$ and $\{\mathsf{e}_{-\beta_k}\}_{k\leq 0}$, respectively (note $\{-\beta_k\}_{k\in\mathbb{Z}} = \Delta^-\times\mathbb{Z}$). Combining Proposition 5.16 with the PBW decompositions (5.42, 5.44), we obtain the PBW basis for $U_q(L\mathfrak{n}^-)$, cf. [16, (5.69)]:

Proposition 5.17. *(a) The subalgebra $U_q(L\mathfrak{n}^-)$ admits the following PBW basis:*

$$U_q(L\mathfrak{n}^-) \;=\; \bigoplus_{\substack{\cdots,n_{-1},n_0,n_1,n_2,\cdots\in\mathbb{N} \\ \cdots+n_{-1}+n_0+n_1+n_2+\cdots<\infty}} \mathbb{Q}(q)\cdot \ldots \mathsf{e}^{n_2}_{-\beta_2}\mathsf{e}^{n_1}_{-\beta_1}\mathsf{e}^{n_0}_{-\beta_0}\mathsf{e}^{n_{-1}}_{-\beta_{-1}}\cdots \tag{5.48}$$

(b) For any $s<r$, the root vectors $\mathsf{e}_{-\beta_s}$ and $\mathsf{e}_{-\beta_r}$ satisfy

$$\mathsf{e}_{-\beta_s}\mathsf{e}_{-\beta_r} - q^{(\beta_s,\beta_r)}\mathsf{e}_{-\beta_r}\mathsf{e}_{-\beta_s} \;\in\; \bigoplus_{n_{r-1},\ldots,n_{s+1}\in\mathbb{N}} \mathbb{Q}(q)\cdot \mathsf{e}^{n_{r-1}}_{-\beta_{r-1}}\cdots\mathsf{e}^{n_{s+1}}_{-\beta_{s+1}} \tag{5.49}$$

where the sum is finite as it is taken over all tuples $n_{r-1},\ldots,n_{s+1}\in\mathbb{Z}_{\geq 0}$ such that:

$$n_{r-1}\beta_{r-1}+\cdots+n_{s+1}\beta_{s+1} = \beta_r+\beta_s\,.$$

A similar result also holds for $U_q(L\mathfrak{n}^+)$ with $\mathsf{e}_{-\beta_s}$ replaced by e_{β_s}.

5.18. **Proof of Theorem 5.8.**

Similarly to [16, Subsection 5.28], we shall now see that Theorem 5.8 is equivalent to the PBW decomposition (5.48). Recall the reduced decomposition of $\widehat{\mu}$ produced by Theorem 4.7, see Remark 4.8, so that the ordered set of roots

$$\cdots<\beta_2<\beta_1<\beta_0<\beta_{-1}<\cdots \tag{5.50}$$

coincides with $\Delta^+\times\mathbb{Z}$ ordered in accordance with the bijection (3.13) via:

$$\cdots<\ell(\overline{\beta}_2)<\ell(\overline{\beta}_1)<\ell(\overline{\beta}_0)<\ell(\overline{\beta}_{-1})<\cdots$$

where for any $(\alpha,d)\in\Delta^+\times\mathbb{Z}$ we set $\overline{(\alpha,d)}=(\alpha,-d)$.

Let ϖ be the anti-involution of $U_q(L\mathfrak{g})$ defined via

$$\varpi\colon e_{i,k}\mapsto f_{i,k}\,,\quad f_{i,k}\mapsto e_{i,k}\,,\quad \varphi^{\pm}_{i,l}\mapsto\varphi^{\pm}_{i,l}$$

for any $i\in I$, $k\in\mathbb{Z}$, $l\in\mathbb{N}$. Applying ϖ to (5.48), we obtain:

$$U_q(L\mathfrak{n}^+) \;=\; \bigoplus^{k\in\mathbb{N}}_{\gamma_1\geq\cdots\geq\gamma_k\in\Delta^+\times\mathbb{Z}} \mathbb{Q}(q)\cdot\varpi(\mathsf{e}_{-\gamma_1})\ldots\varpi(\mathsf{e}_{-\gamma_k}) \tag{5.51}$$

with the above order on $\Delta^+ \times \mathbb{Z}$ being (5.50). On the other hand, due to (5.49), we obtain:

$$[\varpi(\mathsf{e}_{-\overline{\beta}'}), \varpi(\mathsf{e}_{-\overline{\beta}})]_q \ \in \bigoplus_{\substack{\ell(\beta)>\ell(\gamma_1)\geq\dots\geq\ell(\gamma_k)>\ell(\beta') \\ \gamma_1+\dots+\gamma_k=\beta+\beta'}}^{k\in\mathbb{N}} \mathbb{Q}(q)\cdot\varpi(\mathsf{e}_{-\overline{\gamma}_1})\dots\varpi(\mathsf{e}_{-\overline{\gamma}_k})$$

for any $\beta, \beta' \in \Delta^+ \times \mathbb{Z}$ such that $\overline{\beta}' < \overline{\beta}$, or equivalently $\ell(\beta') < \ell(\beta)$. In particular, if $\beta + \beta' \in \Delta^+ \times \mathbb{Z}$ and β, β' are minimal in the sense:

$$\nexists\ \alpha, \alpha' \in \Delta^+ \times \mathbb{Z} \quad \text{s.t.} \quad \overline{\beta}' < \overline{\alpha}' < \overline{\alpha} < \overline{\beta} \quad \text{and} \quad \alpha + \alpha' = \beta + \beta' \tag{5.52}$$

we have

$$[\varpi(\mathsf{e}_{-\overline{\beta}'}), \varpi(\mathsf{e}_{-\overline{\beta}})]_q \in \mathbb{Q}(q)^* \cdot \varpi(\mathsf{e}_{-\overline{\beta}-\overline{\beta}'})\,. \tag{5.53}$$

We claim that Theorem 5.8 follows from (5.51). To this end, it suffices to show:

$$e_{\ell(\beta)} \in \mathbb{Q}(q)^* \cdot \varpi(\mathsf{e}_{-\overline{\beta}}) \tag{5.54}$$

for any $\beta = (\alpha, d) \in \Delta^+ \times \mathbb{Z}$. We prove (5.54) by induction on the height of $\alpha \in \Delta^+$. The base case $\alpha = \alpha_i$ (with $i \in I$) is immediate, due to (5.33, 5.41, 5.46):

$$e_{\left[i^{(d)}\right]} = e_{i,d} = \varpi(f_{i,d}) = \pm\varpi(\mathsf{e}_{(-\alpha_i,d)})\,.$$

For the induction step, consider the costandard factorization $\ell = \ell_1\ell_2$ of $\ell = \ell(\alpha, d)$. Since factors of standard loop words are standard, we have $\ell_1 = \ell(\gamma_1, d_1)$ and $\ell_2 = \ell(\gamma_2, d_2)$ for some $(\gamma_1, d_1), (\gamma_2, d_2) \in \Delta^+ \times \mathbb{Z}$ such that $\alpha = \gamma_1 + \gamma_2, d = d_1 + d_2$. By the induction hypothesis, we have $e_{\ell_k} \in \mathbb{Q}(q)^* \cdot \varpi(\mathsf{e}_{(-\gamma_k,d_k)})$ for $k \in \{1, 2\}$. However, we note that $(\gamma_1, d_1) < (\alpha, d) < (\gamma_2, d_2)$ is a minimal decomposition in the sense of (5.52), according to Proposition 3.20. Therefore, comparing (5.14) with (5.53), we obtain:

$$e_\ell = [e_{\ell_1}, e_{\ell_2}]_q \in \mathbb{Q}(q)^* \cdot \varpi([\mathsf{e}_{(-\gamma_2,d_2)}, \mathsf{e}_{(-\gamma_1,d_1)}]_q) = \mathbb{Q}(q)^* \cdot \varpi(\mathsf{e}_{(-\alpha,d)})$$

as we needed to prove. This completes our proof of Theorem 5.8.

6. Generalization to other orders

In this Section, we generalize our main results to a larger family of orders on the alphabet $\mathcal{I} = \{i^{(d)}\}_{i\in I}^{d\in\mathbb{Z}}$. Consider a collection of functions $\mathsf{f}_i : \mathbb{Z} \to \mathbb{R}$ such that

- $\mathsf{f}_i(0) = 0$
- all f_i are strictly increasing unbounded functions
- there are infinitely many N (both in $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$) such that there exist $\{N_i\}_{i\in I} \subset \mathbb{Z}^I$ satisfying $\mathsf{f}_i(N_i) = N$ for all $i \in I$.

We then define an order on $\mathcal{I}$ (hence a lexicographical order on the loop words) via:

$$i^{(d)} < j^{(e)} \qquad \Longleftrightarrow \qquad \mathsf{f}_i(d) > \mathsf{f}_j(e) \quad \text{or} \quad \mathsf{f}_i(d) = \mathsf{f}_j(e) \text{ and } i < j\,. \tag{6.1}$$

In the special case $\mathsf{f}_i(d) = \frac{d}{c_i}$ (with $c_i \in \mathbb{Z}_{>0}$) this recovers (1.10) considered above.

• First, we need to update the filtration (3.4) of the loop algebra $L\mathfrak{n}^+$. To this end, we fix an increasing sequence $\{N^{(+,s)}\}_{s\in\mathbb{N}}$ of non-negative numbers (respectively, a decreasing sequence $\{N^{(-,s)}\}_{s\in\mathbb{N}}$ of non-positive numbers) such that $N^{(\pm,0)} = 0$ and there exist $\{N_i^{(\pm,s)}\}_{i\in I} \subset (\pm\mathbb{N})^I$ satisfying $\mathsf{f}_i(N_i^{(\pm,s)}) = N^{(\pm,s)}$ for all $i \in I$. Then, we define $L^{(s)}\mathfrak{n}^+$ as the finite-dimensional Lie subalgebra of $L\mathfrak{n}^+$ generated by

$$\left\{ e_i^{(d)} \,\middle|\, i \in I,\, N_i^{(-,s)} \leq d \leq N_i^{(+,s)} \right\}.$$

We also amend our former definition of $\mathcal{I}^{(s)}$ in (3.3) by rather redefining

$$\mathcal{I}^{(s)} = \left\{ i^{(d)} \,\middle|\, i \in I,\, N_i^{(-,s)} \leq d \leq N_i^{(+,s)} \right\} \qquad \forall\, s \in \mathbb{Z}_{\geq 0}\,.$$

We may thus apply Definition 2.10 to yield a notion of standard (Lyndon) loop words with respect to $L^{(s)}\mathfrak{n}^+$, with the words made up only of $i^{(d)} \in \mathcal{I}^{(s)}$.

- For N as above, so that there exist $\{N_i\}_{i\in I}$ satisfying $N = \mathsf{f}_i(N_i)\ \forall\, i$, we define

$$f_N \colon \Delta^+ \to \mathbb{Z} \quad \text{via} \quad f_N(\alpha) = \sum_{i\in I} k_i \cdot N_i \quad \text{for any} \quad \alpha = \sum_{i\in I} k_i\alpha_i \in \Delta^+\,.$$

With this definition at hand, the subalgebra $L^{(s)}\mathfrak{n}^+$ can be explicitly written as

$$L^{(s)}\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \bigoplus_{d=f_{N^{(-,s)}}(\alpha)}^{f_{N^{(+,s)}}(\alpha)} \mathbb{Q}\cdot e_\alpha^{(d)}\,.$$

Then, the results of Proposition 3.2 and Proposition 3.3 still hold true with the only change that $-sf(\alpha) \leq d \leq sf(\alpha)$ is replaced with $f_{N^{(-,s)}}(\alpha) \leq d \leq f_{N^{(+,s)}}(\alpha)$.

- As before, we call a loop word $w = \left[i_1^{(d_1)} \ldots i_n^{(d_n)} \right]$ *exponent-tight* if (3.9) holds. Then, the results of Theorem 3.6, Lemma 3.7, Proposition 3.8, and Proposition 3.13 still hold true (the proofs are the same). Therefore, we still have a bijection (3.13)

$$\ell \colon \Delta^+ \times \mathbb{Z} \xrightarrow{\sim} \left\{ \text{standard Lyndon loop words} \right\}$$

satisfying property (3.6) with $s = \infty$ as well as Theorem 3.6 and Proposition 3.8.

- On the other hand, the periodicity of Proposition 3.15 no longer holds in this generality. Instead, we can only express $\ell(\alpha, f_N(\alpha))$ via $\ell(\alpha, 0)$:

Lemma 6.1. *If $\ell(\alpha,0) = [i_1^{(0)} \ldots i_n^{(0)}]$, then $\ell(\alpha, f_N(\alpha)) = [i_1^{(N_{i_1})} \ldots i_n^{(N_{i_n})}]$.*

Proof. First, we note that Theorem 3.6 together with Proposition 3.8 for $N > 0$ (respectively, Remark 3.9 for $N < 0$) guarantee that the multiset of letters constituting $\ell(\alpha, f_N(\alpha))$ is exactly $\{i_1^{(N_{i_1})}, \ldots, i_n^{(N_{i_n})}\}$. Indeed, assuming the contradiction for some $N > 0$ (the case $N < 0$ is treated analogously), there exists $0 \leq d < f_N(\alpha)$ such that $\ell(\alpha, d+1)$ starts with $i_k^{(N_{i_k}+1)}$ for some k. As the sum of exponents equals $d+1 \leq f_N(\alpha)$, the word $\ell(\alpha, d+1)$ and hence $\ell(\alpha, d)$ also contains a letter $i_l^{(e)}$ with $e < N_{i_l}$. This provides a contradiction with Proposition 3.8, since $i_l^{(e+1)} > i_k^{(N_{i_k}+1)}$.

On the other hand, we note that $i^{(N_i)} < j^{(N_j)}$ iff $i < j$, which guarantees that the loop word $[j_1^{(N_{j_1})} \ldots j_n^{(N_{j_n})}]$ is (standard) Lyndon iff the loop word $[j_1^{(0)} \ldots j_n^{(0)}]$ is (standard) Lyndon, cf. the proof of Proposition 3.15. This completes the proof. $\square$

We can now prove the following slight generalization of Corollary 3.11:

Lemma 6.2. *Fix N, $\{N_i\}_{i\in I}$, $\alpha \in \Delta^+$, $f_N(\alpha) \in \mathbb{Z}$ as above.*

(a) For $d > f_N(\alpha)$, the loop word $\ell(\alpha, d)$ starts with some $j^{(e)}$ such that $e > N_j$.

(b) For $d \leq f_N(\alpha)$, the loop word $\ell(\alpha, d)$ starts with some $j^{(e)}$ such that $e \leq N_j$.

Proof. Let $\ell(\alpha,d)=[i_1^{(k_1)}\dots i_n^{(k_n)}]$. Then $i_1^{(k_1)}\le i_r^{(k_r)}$ and so $\mathsf{f}_{i_1}(k_1)\ge \mathsf{f}_{i_r}(k_r)$ for any r. If $k_1\le N_{i_1}$, then we would have $\mathsf{f}_{i_r}(N_{i_r})=\mathsf{f}_{i_1}(N_{i_1})\ge \mathsf{f}_{i_1}(k_1)\ge \mathsf{f}_{i_r}(k_r)$ and so $k_r\le N_{i_r}$ for any r. That would imply $d=\sum_{r=1}^n k_r\le f_N(\alpha)$, a contradiction.

To prove (b), we note that $i_1^{(k_1)}\ge i_r^{(k_r+1)}$ for any r by Theorem 3.6. If $k_1>N_{i_1}$, then we would have $\mathsf{f}_{i_r}(N_{i_r})=\mathsf{f}_{i_1}(N_{i_1})<\mathsf{f}_{i_1}(k_1)\le \mathsf{f}_{i_r}(k_r+1)$ so that $k_r\ge N_{i_r}$ for all r. That would imply $d=\sum_{r=1}^n k_r>f_N(\alpha)$, a contradiction. □

• The major difference will take place in the generalization of Theorem 4.7 to the present setup. As the periodicity of Proposition 3.15 no longer holds, the bi-infinite sequence $\{i_k\}_{k\in\mathbb{Z}}$ of (4.23) shall rather be constructed as a limit of finite sequences. Explicitly, to define $\{i_k\}_{k\le 0}$, consider $L_s=\{(\alpha,d)\,|\,\alpha\in\Delta^+, 0\le d<f_{N^{(+,s)}}(\alpha)\}$ instead of L from (3.14). Then, $L_s=L_{\widehat{\mu}^{(+,s)}}$ with $\mu^{(+,s)}=\sum_{i\in I}N_i^{(+,s)}\omega_i^\vee$. Applying the arguments from the proof of Theorem 4.7, we get a reduced decomposition (4.22) of $\widehat{\mu}^{(+,s)}$ such that the ordered finite sequence $\beta_0<\beta_{-1}<\dots<\beta_{1-l(\widehat{\mu}^{(+,s)})}$ coincides with L_s ordered via (4.29). By uniqueness, such a sequence for L_{s+1} refines the one for L_s. Furthermore, the roots $\beta_k=s_{i_0}s_{i_{-1}}\dots s_{i_{k+1}}(\alpha_{i_k})$ for $k\le 0$ are all distinct and satisfy $\{\beta_k\}_{k\le 0}=\Delta^+\times\mathbb{Z}_{\ge 0}$. The construction of $\{i_k\}_{k>0}$ is similar.

• With the above update of Theorem 4.7, the results of Propositions 4.9, 4.10 still hold (for any fixed $i\in I, d\in\mathbb{Z}$), once $\{p_j\}_{j\in I}$ from (4.33) are rather redefined via:

$$p_j \ \text{ is the unique integer satisfying } \ j^{(-p_j)}\ge i^{(-d)}>j^{(-p_j+1)}\,. \tag{6.2}$$

• With the above update, the analogue of (5.38) and the paragraph afterwards hold, implying (5.35, 5.36). Thus, the main result of Section 5, the construction of PBW bases of $U_q(L\mathfrak{n}^+)$ from Theorem 5.8 still holds (with e_ℓ, e_w as in Definition 5.7).

Appendix A. Computer code

In this Appendix, we present some interesting examples of standard Lyndon loop words that nicely illustrate the key properties of Theorem 3.6 and Proposition 3.8. We also provide a link to our code used to evaluate standard Lyndon loop words.

A.1. Examples.

The first version of our code did not use the key results (Theorem 3.6 and Proposition 3.8), but was rather based on Remark 3.10, which is a simple generalization of [16, Proposition 2.26]. Thus, when evaluating $\ell(\alpha,d)$, the code simply goes through all the ways to split α into an ordered sum of simple roots, and distribute d between the exponents of these simple roots while satisfying (3.12). In the table below, we present examples of standard Lyndon loop words computed through this code (which also nicely illustrate the results of Theorem 3.6 and Proposition 3.8).

Type	Order	Weights	d	$\ell(\theta,d)$	$\ell(\theta,d+1)$	$\ell(\theta,d+2)$
A_4	1234	1 1 1 1	0	$[1^{(0)}2^{(0)}3^{(0)}4^{(0)}]$	$[4^{(1)}3^{(0)}2^{(0)}1^{(0)}]$	$[3^{(1)}2^{(0)}1^{(0)}4^{(1)}]$
A_5	51324	4 3 1 8 5	19	$[3^{(1)}4^{(8)}5^{(4)}2^{(3)}1^{(3)}]$	$[1^{(4)}2^{(3)}3^{(1)}4^{(8)}5^{(4)}]$	$[5^{(5)}4^{(8)}3^{(1)}2^{(3)}1^{(4)}]$
B_2	21	7 8	18	$[1^{(6)}2^{(6)}2^{(6)}]$	$[2^{(7)}1^{(6)}2^{(6)}]$	$[2^{(7)}2^{(7)}1^{(6)}]$
B_3	123	4 3 1	10	$[2^{(3)}1^{(3)}3^{(1)}3^{(1)}2^{(2)}]$	$[2^{(3)}3^{(1)}3^{(1)}2^{(3)}1^{(3)}]$	$[1^{(4)}2^{(3)}3^{(1)}3^{(1)}2^{(3)}]$
C_3	312	4 3 6	8	$[1^{(2)}2^{(1)}1^{(2)}2^{(1)}3^{(2)}]$	$[3^{(3)}2^{(1)}2^{(1)}1^{(2)}1^{(2)}]$	$[2^{(2)}1^{(2)}3^{(3)}2^{(1)}1^{(2)}]$
C_3	321	1 10 3	17	$[2^{(8)}1^{(0)}3^{(2)}2^{(7)}1^{(0)}]$	$[2^{(8)}1^{(0)}2^{(8)}1^{(0)}3^{(2)}]$	$[2^{(9)}1^{(0)}3^{(2)}2^{(8)}1^{(0)}]$
D_4	3124	4 3 7 5	8	$[3^{(3)}2^{(1)}1^{(1)}4^{(2)}2^{(1)}]$	$[1^{(2)}2^{(1)}4^{(2)}3^{(3)}2^{(1)}]$	$[3^{(4)}2^{(1)}4^{(2)}1^{(2)}2^{(1)}]$
G_2	21	2 3	11	$[2^{(3)}1^{(2)}2^{(2)}2^{(2)}1^{(2)}]$	$[2^{(3)}1^{(2)}2^{(3)}1^{(2)}2^{(2)}]$	$[2^{(3)}2^{(3)}1^{(2)}2^{(3)}1^{(2)}]$

Let us clarify the conventions in this table:

- In the column "Order", the elements $i \in I$ are listed in the increasing order,
- In the column "Weights", the weights c_i are listed with i ordered as in [23],
- In all these examples, we choose to consider only the highest root $\alpha = \theta$.

Let us also provide examples of standard Lyndon loop words for the remaining exceptional types (these were evaluated using our second code presented below):

Type	Order	Weights	d	$\ell(\theta, d)$
F_4	1234	1 2 3 2	17	$[3^{(3)}2^{(1)}2^{(1)}4^{(2)}3^{(3)}2^{(1)}1^{(0)}3^{(3)}2^{(1)}1^{(0)}4^{(2)}]$
F_4	1234	1 2 3 2	18	$[2^{(2)}1^{(0)}3^{(3)}2^{(1)}1^{(0)}4^{(2)}3^{(3)}2^{(1)}2^{(1)}3^{(3)}4^{(2)}]$
E_6	142653	1 2 1 2 2 1	9	$[5^{(2)}4^{(1)}3^{(1)}6^{(0)}2^{(1)}1^{(0)}3^{(1)}2^{(1)}4^{(1)}3^{(1)}6^{(0)}]$
E_6	142653	1 2 1 2 2 1	10	$[6^{(1)}3^{(1)}2^{(1)}1^{(0)}4^{(1)}3^{(1)}2^{(1)}5^{(2)}4^{(1)}3^{(1)}6^{(0)}]$
E_7	1234567	4 5 3 7 3 2 5	25	$[4^{(3)}5^{(1)}6^{(0)}3^{(1)}7^{(2)}4^{(2)}2^{(2)}1^{(1)}3^{(1)}2^{(2)}4^{(3)}5^{(1)}6^{(0)}3^{(1)}7^{(2)}4^{(2)}5^{(1)}]$
E_7	1234567	4 5 3 7 3 2 5	26	$[4^{(3)}5^{(1)}3^{(1)}7^{(2)}4^{(2)}2^{(2)}1^{(1)}3^{(1)}2^{(2)}4^{(3)}5^{(1)}6^{(0)}3^{(1)}7^{(2)}4^{(3)}5^{(1)}6^{(0)}]$
E_8	14572386	1 32 13 3 10 9 6 15	46	$[8^{(3)}5^{(1)}4^{(0)}6^{(1)}5^{(1)}3^{(2)}4^{(0)}7^{(1)}6^{(1)}5^{(1)}2^{(6)}1^{(0)}3^{(2)}4^{(0)}2^{(6)}3^{(2)}$ $8^{(3)}5^{(1)}4^{(0)}6^{(1)}5^{(1)}3^{(2)}4^{(0)}7^{(1)}6^{(1)}5^{(1)}8^{(2)}2^{(6)}1^{(0)}]$
E_8	14572386	1 32 13 3 10 9 6 15	47	$[8^{(3)}5^{(1)}4^{(0)}6^{(1)}5^{(1)}3^{(2)}7^{(1)}6^{(1)}2^{(6)}1^{(0)}8^{(3)}5^{(1)}4^{(0)}6^{(1)}5^{(1)}3^{(2)}$ $4^{(0)}7^{(1)}2^{(6)}3^{(2)}8^{(3)}5^{(1)}4^{(0)}6^{(1)}5^{(1)}3^{(2)}4^{(0)}2^{(6)}1^{(0)}]$

A.2. **The code.**

The second version of our code was written using Proposition 3.2 as well as Proposition 3.8 which provides an inductive way to compute exponents of $\ell(\alpha, d)$. This drastically improves the code performance, allowing us to compute words for much larger values of the degree d and the weights c_i. This code can be used at the following clickable link (the interested reader can use this code to check the results of this note as well as to compute standard Lyndon loop words):[2]

- C++ Code 2

A.3. **Divisible weights in type A.**

In this Subsection, we consider a special setup for type A_n (naturally generalizing [16, Section 7.3]): the order is $1 < 2 < \dots < n$, and the weights $c_1, c_2, \dots, c_n \in \mathbb{Z}_{>0}$ are such that c_i divides c_{i+1} for any $1 \leq i < n$. By induction on n and the periodicity of Proposition 3.15, it suffices to evaluate $\ell(\theta, d)$ for $0 < d \leq c_1 + \dots + c_n$. Let $a^{(k)}$ be the first letter of the standard Lyndon loop word $\ell(\theta, d)$. Then, we have:

$$\ell(\theta, d) = \Big[a^{(k)}(a-1)^{(k_2)} \dots 1^{(k_a)}\, (a+1)^{(k_{a+1})}\, (a+2)^{(k_{a+2})} \dots n^{(k_n)}\Big]$$
$$\text{with} \quad k_i = \left\lceil k \cdot \frac{c_{a-i+1}}{c_a} - 1 \right\rceil \quad \text{if} \quad 1 < i \leq a\,, \quad k_i = k \cdot \frac{c_i}{c_a} \quad \text{if} \quad a < i \leq n\,.$$

It thus suffices to describe the first letter $a^{(k)}$. This is uniquely determined by a sequence encoding the underlying element $a \in \{1, \dots, n\}$ as d increases from 1 up to $c_1 + c_2 + \dots + c_n$. Indeed, the exponent k of a (as well as the exponent of any other i) is then equal to the number of times this a (respectively i) appears among the first d terms of that sequence, due to Proposition 3.8. One can depict this sequence by a table placing each n in the top of a new column to the right

[2]The user should press the "Run" button and they will see the instructions and a small example afterwards. Type in the input in the console afterwards, following the instructions. Names of Lie algebra types for input are: A, B, C, D, G2, F4, E6, E7, E8. This code was written using C++23.

and then moving top-to-bottom until getting to the next n. Let us now present a general rule for the construction of this table:

1. At the first step, place n in the top-left corner;
2. At the i-th step (with $2 \leq i \leq n$), copy the current table and paste it to the right $\frac{c_{n-i+2}}{c_{n-i+1}} - 1$ times. After that, add an extra entry $n - i + 1$ at the bottom of the right-most column;
3. Copy the resulting table and paste it to the right $c_1 - 1$ times.

Let us illustrate it with some examples. For $n = 4$ and $c_1 = 1, c_2 = 2, c_3 = 6, c_4 = 12$, the sequence is 4 4 3 4 4 3 4 4 3 2 4 4 3 4 4 3 4 4 3 2 1, and so the table is:

$$
\begin{array}{cccccccccccc}
4&4&4&4&4&4&4&4&4&4&4&4\\
&3&&3&&3&&3&&3&&3\\
&&&&&2&&&&&&2\\
&&&&&&&&&&&1
\end{array}
$$

For $n = 3$ and $c_1 = 1, c_2 = 3, c_3 = 15$, the sequence is 3 3 3 3 3 2 3 3 3 3 3 2 3 3 3 3 3 2 1, which is thus encoded by the following table:

$$
\begin{array}{ccccccccccccccc}
3&3&3&3&3&3&3&3&3&3&3&3&3&3&3\\
&&&&2&&&&&2&&&&&2\\
&&&&&&&&&&&&&&1
\end{array}
$$

Likewise, for $n = 4$ and $c_1 = 1, c_2 = 3, c_3 = 9, c_4 = 27$, we get the following table:

$$
\begin{array}{ccccccccccccccccccccccccccc}
4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4\\
&&3&&&3&&&3&&&3&&&3&&&3&&&3&&&3&&&3\\
&&&&&&&&2&&&&&&&&&2&&&&&&&&&2\\
&&&&&&&&&&&&&&&&&&&&&&&&&&1
\end{array}
$$

Remark A.4. We note that similar tables can also be produced for other classical types B_n, C_n, D_n with the order $1 < 2 < \cdots < n$. By induction on n, the periodicity of Proposition 3.15, and the A-type case treated above, it suffices to evaluate $\ell(\alpha, d)$ for the roots $\alpha = m_1\alpha_1 + \cdots + m_n\alpha_n \in \Delta^+$ with $m_1, \ldots, m_n \geq 1$ and $0 < d \leq m_1c_1 + \cdots + m_nc_n$. The only difference between the corresponding tables and those for A_n-type, is that now when adding each i we shall be adding it m_i times. Explicitly, the corresponding table is constructed by the following algorithm:

1. At the first step, build a column of height m_n with all entries equal to n;
2. At the i-th step (with $2 \leq i \leq n$), copy the current table and paste it to the right $\frac{c_{n-i+2}}{c_{n-i+1}} - 1$ times. After that, add m_{n-i+1} times the number $n - i + 1$ at the bottom of the right-most column;
3. Copy the resulting table and paste it to the right $c_1 - 1$ times.

As an example, consider type C_4 with the weights $c_1 = 1, c_2 = 2, c_3 = 6, c_4 = 12$, and $\alpha = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = \theta$. Then, we get the following table:

$$
\begin{array}{cccccccccccc}
4&4&4&4&4&4&4&4&4&4&4&4\\
&3&&3&&3&&3&&3&&3\\
&3&&3&&3&&3&&3&&3\\
&&&&&2&&&&&&2\\
&&&&&2&&&&&&2\\
&&&&&&&&&&&1\\
&&&&&&&&&&&1
\end{array}
$$

The multiset of all letters appearing in $\ell(\alpha, d)$ is easily determined by this table: if $p_i = m_i d_i + r_i$ ($d_i \in \mathbb{N}, 0 \leq r_i < m_i$) denotes the number of times i appears among the first d terms of the table, then r_i exponents of i are $d_i + 1$ and the rest are d_i.

References

[1] Avdieiev Y., Tsymbaliuk A. *Affine standard Lyndon words: A-type*, arχiv:2305.16299.

[2] Beck J. *Convex bases of PBW type for quantum affine algebras*, Comm. Math. Phys. 165 (1994), no. 1, 193–199.

[3] Beck J. *Braid group action and quantum affine algebras*, Comm. Math. Phys. 165 (1994), no. 3, 555–568.

[4] Damiani I. *A basis of type Poincare-Birkhoff-Witt for the quantum algebra of $\widehat{sl}(2)$*, J. Algebra 161 (1993), no. 2, 291–310.

[5] Damiani I. *Drinfeld realization of affine quantum algebras: the relations*, Publ. Res. Inst. Math. Sci. 48 (2012), no. 3, 661–733.

[6] Enriquez B., Khoroshkin S., Pakuliak S. *Weight functions and Drinfeld currents*, Comm. Math. Phys. 276 (2007), no. 3, 691–725.

[7] Green J. *Quantum groups, Hall algebras and quantized shuffles*, Finite reductive groups (Luminy 1994), 273–290, Birkhäuser Prog. Math. 141, 1997.

[8] Hernandez D. *Representations of quantum affinizations and fusion product*, Transform. Groups 10 (2005), no. 2, 163–200.

[9] Jantzen J. *Lectures on quantum groups*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1996.

[10] Kac V. *Infinite-dimensional Lie algebras (3d edition)*, Cambridge University Press, Cambridge, 1990.

[11] Lalonde P., Ram A. *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1821–1830.

[12] Leclerc B. *Dual canonical bases, quantum shuffles and q-characters*, Math. Z. 246 (2004), no. 4, 691–732.

[13] Levendorsky S., Soibelman Y. *Some applications of the quantum Weyl groups*, J. Geom. Phys. 7 (1990), no. 2, 241–254.

[14] Lothaire M. *Combinatorics of words*, Cambridge University Press, Cambridge, 1997.

[15] Lusztig G. *Introduction to Quantum Groups*, Boston, Birkhäuser, 1993.

[16] Neguţ A., Tsymbaliuk A. *Quantum loop groups and shuffle algebras via Lyndon words*, arχiv:2102.11269v3.

[17] Neguţ A., Tsymbaliuk A. *Quantum loop groups and shuffle algebras via Lyndon words*, Adv. Math. 439 (2024), Paper No. 109482.

[18] Papi P. *A characterization of a special ordering in a root system*, Proc. Amer. Math. Soc. 120 (1994), no. 3, 661–665.

[19] Papi P. *Convex orderings in affine root systems*, J. Algebra 172 (1995), no. 3, 613–623.

[20] Papi P. *Convex orderings in affine root systems, II*, J. Algebra 186 (1996), no. 1, 72–91.

[21] Rosso M. *Quantum groups and quantum shuffles*, Invent. Math. 133 (1998), no. 2, 399–416.

[22] Rosso M. *Lyndon bases and the multiplicative formula for R-matrices*, unpublished.

[23] Samelson H. *Notes on Lie algebras*, Springer-Verlag, New York, 1990.

[24] Schauenburg P. *A characterization of the Borel-like subalgebras of quantum enveloping algebras*, Comm. Algebra 24 (1996), no. 9, 2811–2823.

S.K.: SEVERYN.KHOMYCH@GMAIL.COM

N.K.: N.KORNIYCHUK.A@GMAIL.COM

K.M.: KOSTYAMOLOKANOV@GMAIL.COM

A.T.: SASHIKTS@GMAIL.COM