Ch 2.1: Linear Equations; Integrating Factors

- A linear first order ODE has the general form $$\frac{dy}{dt} = f(t, y)$$ where f is linear in y.

Examples include equations with constant coefficients, such as those in Chapter 1,

$$y' = -ay + b$$

or equations with variable coefficients: $$\frac{dy}{dt} + p(t)y = g(t)$$

(Question) How do we solve the ODE?

(Ex) (1) $$\frac{dy}{dt} + \frac{2}{t}y = 1$$

(2) $$\frac{dy}{dt} + 2y = 10$$

(3) $$\frac{dy}{dt} + \frac{2t}{t^2+1}y = 5t^2$$

(4) $$\frac{dy}{dt} + 2ty = e^{-t^2}$$

Constant Coefficient Case

- For a first order linear equation (: separable equation) with constant coefficients,

$$\frac{dy}{dt} = -ay + b,$$

recall that we can use methods of calculus to solve:

$$\frac{dy/dt}{y - b/a} = -a$$

$$\int \frac{dy}{y - b/a} = -\int a\,dt$$

$$\ln\left|y - b/a\right| = -at + C$$

$$y = b/a + ke^{at},\quad k = \pm e^{C}$$

(Question) How do we solve ODE with variable coefficients?

Can we transform a **linear equation** into an equation like an integrable equation?

(EX)

(1) $\frac{dy}{dt} + 2ty = e^{-t^2}$ (2) $\frac{dy}{dt} + \frac{2t}{t^2+1}y = 5t^2$ (3) $t\frac{dy}{dt} + 3y = 7 \quad (t > 0)$

Linear Differential Equation

(Ex) Find the general solution of the equation:

$$y' + 2t\, y = e^{-t^2}$$

(1) idea

(2) integrating factor

Method of Integrating Factors for General First Order Linear Equation

- Next, we consider the general first order linear equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

- Multiplying both sides by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

- Next, we want $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$, from which it will follow that

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$$

Integrating Factor for General First Order Linear Equation

- Thus we want to choose $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$.

- Assuming $\mu(t) > 0$, it follows that

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \quad \Rightarrow \quad \ln \mu(t) = \int p(t)dt + k$$

- Choosing $k = 0$, we then have $\mu(t) = e^{\int p(t)dt}$,

and note $\mu(t) > 0$ as desired.

Solution for
General First Order Linear Equation

- Thus we have the following:

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

- Then

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

(Example 3) Solve the IVP

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

Example 3: General Solution (1 of 2)

- To solve the initial value problem $$ty' + 2y = 4t^2, \quad y(1) = 2$$

  first put into standard form: $$y' + \frac{2}{t}y = 4t, \quad \text{for } t \neq 0$$

- Then $$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln\left(t^2\right)} = t^2$$

  and hence

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int t^2(4t)dt + C}{t^2} = \frac{1}{t^2}\left[\int 4t^3 dt + C\right] = t^2 + \frac{C}{t^2}$$

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

Example 3: Particular Solution (2 of 2)

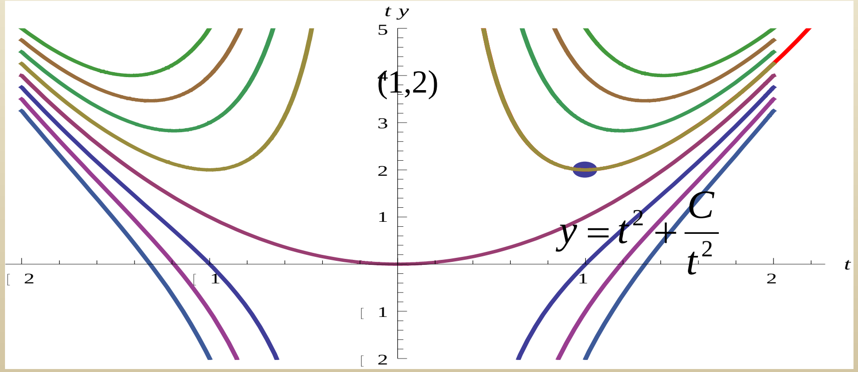
- Using the initial condition $y(1) = 2$ and general solution

it follows that $\quad y = t^2 + \dfrac{C}{t^2}, \quad y(1) = 1 + C = 2 \;\Rightarrow\; C = 1$

- The graphs below show solution curves for the differential equation, including a particular solution whose graph contains the initial point (1,2). Notice that when C=0, we get the parabolic solution (shown) and that solution separates the solutions into those that are asymptotic to the positive versus negative y-axis.

$$y = t^2 + \frac{1}{t^2}$$

$$y = t^2$$



Example 4

- Solve the initial value problem:

$$2y' + ty = 2, \qquad y(0) = 1$$

Example 4: A Solution in Integral Form (1 of 2)

- To solve the initial value problem $$2y' + ty = 2, \quad y(0) = 1,$$

  first put into standard form: $$y' + \frac{t}{2}y = 1$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{2}dt} = e^{\frac{t^2}{4}}$$

and hence

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds + C\right) = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds\right) + Ce^{-t^2/4}$$

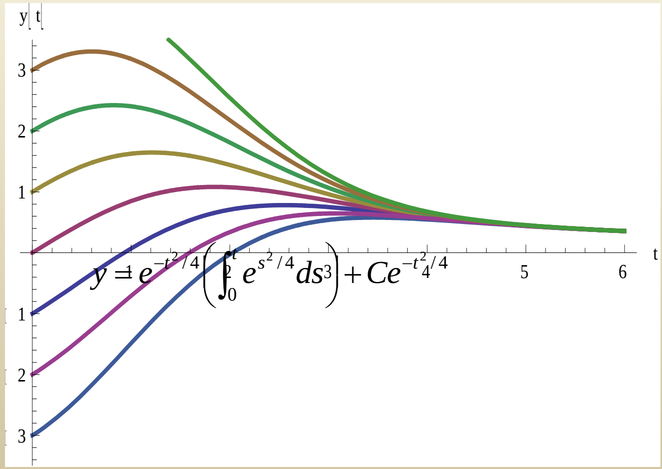
$$2y' + ty = 2, \quad y(0) = 1,$$

Example 4: A Solution in Integral Form (2 of 2)

- Notice that this solution must be left in the form of an integral, since there is no closed form for the integral.

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}\,ds\right) + Ce^{-t^2/4}$$

- Using software such as *Mathematica* or Matlab, we can approximate the solution for the given initial conditions as well as for other initial conditions.
- Several solution curves are shown.



Example 5

- Solve the ODE $$\frac{dy}{dt} - 2y = 4 - t$$

Example 5: General Solution (1 of 2)

- We can solve the following equation $\frac{dy}{dt} - 2y = 4 - t$

  using the formula derived on the previous slide:

$$y = e^{-at}\int e^{at}g(t)dt + Ce^{-at} = e^{2t}\int e^{-2t}(4-t)dt + Ce^{2t}$$

- Integrating by parts,

$$\begin{aligned}\int e^{-2t}(4-t)dt &= \int 4e^{-2t}dt - \int te^{-2t}dt \\ &= -2e^{t/5} - \left[-\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}dt\right] \\ &= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}\end{aligned}$$

- Thus

$$y = e^{2t}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right) + Ce^{2t} = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$

$$\frac{dy}{dt} - 2y = 4 - t$$

Example 5: Graphs of Solutions (2 of 2)

- The graph shows the direction field along with several integral curves. If we set $C = 0$, the exponential term drops out and you should notice how the solution in that case, through the point (0, -7/4), separates the solutions into those that grow exponentially in the positive direction from those that grow exponentially in the negative direction.

$$\frac{dy}{dt} - 2y = 4 - t$$

$$y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$