Fourfold Symmetric Solutions to the Ginzburg Landau Equation for d-wave Superconductors

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Abstract

We find and investigate the structure of solutions to the Ginzburg Landau equation for a high temperature superconductor with tetragonal symmetry. This is done near an isolated, rotationally symmetric d-wave vortex state with its core at the origin defined on all of $\mathbb{R}^2$. We prove that the solution's s-wave component nucleates near the vortex core for temperatures just below the d-wave critical temperature. We further show that this causes the radial symmetry to break and that the solution develops a fourfold symmetry with respect to a rotation by an angle of $\frac{\pi}{2}$.

*Research supported by NSF grants DMS-0456286 and DMS-0604839.

1 Introduction

In this paper we analyze solutions to the Ginzburg-Landau (G–L) equation for a high temperature (high-T_c) superconductor near an isolated vortex state. We are interested in superconductivity for layered materials with small inter-layer coupling. As such we work with the two-dimensional problem of a single superconducting layer ($\mathbb{R}^2$). Conventional low temperature superconductivity can be described with a simpler G–L model given in terms of a complex valued order parameter ψ and a real, two-dimensional vector field A called the magnetic potential. The energy density in this case is given as

$$e_1 = |\Pi\psi|^2 + \alpha|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + |curl\ A|^2$$

where $\Pi = \nabla - iA$ denotes the covariant derivative, $curl\ A$ is the magnetic induction, $\alpha = \kappa^2(T - T_c)$ such that κ is the G–L parameter, T is the (constant) material temperature, and T_c is the critical transition temperature. Superconducting equilibria nucleate for $T < T_c$. Motivated by this the energy is normalized by applying the transformation $\tilde{\mathbf{x}} = \sqrt{(T_c - T)}\ \mathbf{x}, \tilde{\psi} = \frac{1}{\sqrt{(T_c - T)}}\psi$, and $\tilde{A} = \frac{1}{\sqrt{(T_c - T)}}A$. Suppressing the tilde, the normalized G–L energy is defined as

$$\mathcal{E}_1(\psi, A) = \int \varepsilon_1 = \int_{\mathbb{R}^2} \left(|\Pi\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |curl\ A|^2 \right). \tag{1.1}$$

For a vector valued $B = (B_1, B_2)$, $curl\ B = \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}$ and for a scalar valued ξ, $curl\ \xi = (\partial_2\xi, -\partial_1\xi)$. The Euler Lagrange equation of the functional $\mathcal{E}_1$ is

$$\mathbf{F}_0(\psi, A) \equiv \begin{pmatrix} -\Pi^2\psi - \kappa^2(1 - |\psi|^2)\psi \\ curlcurlA + \frac{1}{2}i(\psi^*\Pi\psi - \psi\Pi^*\psi^*) \end{pmatrix} = 0. \tag{1.2}$$

For $\kappa > 0$ the equation (1.2) has a well known vortex solution

$$\psi(x) = d_0 = f_1(r)e^{i\theta}, \quad A(x) = A_0 = \frac{a_1(r)}{r}\hat{x}^\perp$$

where $\hat{x}^\perp = \frac{1}{r}(-x_2, x_1)^t$ such that the energy $\mathcal{E}_1(d_0, A_0)$ is finite. The functions f_1, a_1 are smooth and have following properties (see [3], [8], [14]).

$$\begin{aligned} &0 < f_1 < 1, 0 < a_1 < 1 \,\text{on}\, (0, \infty), \text{ and } f_1', a_1' > 0, \\ &f_1 \sim cr, a_1 \sim dr^2 \,\text{as}\, r \to 0, \text{ and} \\ &1 - f_1, 1 - a_1 \to 0 \text{ at exponential rates as } r \to \infty. \end{aligned}$$

Moreover if $\kappa \geq \sqrt{2}$ then f_1 and a_1 are uniquely determined (see [2]). Vortices in solutions to (1.2) are zeroes of the superconducting density $|\psi|^2$, about which supercurrent circulate. They are a stable feature in type II superconductors (i.e. for $\kappa > \frac{1}{\sqrt{2}}$), and the vortex solution is an accurate local description of a degree one vortex in a general stable solution for κ sufficiently large (see [15]). The model for low temperature superconductivity described above is isotropic and a signature of this is that the level curves of $|\psi|^2$ for the vortex solution are radial.

In contrast to this, most high-T_c superconductors have anisotropic mean field features. Here we investigate solutions to a model for certain high-T_c materials having tetragonal crystal symmetry using a G–L theory with two order parameters ψ_s and ψ_d, each with a critical temperature T_c^s and T_c^d (see [10]). The corresponding energy density takes the form

$$\begin{aligned} e_2 &= |\Pi\psi_d|^2 + K\,|\Pi\psi_s|^2 \\ &+ \mu\left(\Pi_x\psi_d\Pi_x^*\psi_s^* - \Pi_y\psi_d\Pi_y^*\psi_s^* + \Pi_x^*\psi_d^*\Pi_x\psi_s - \Pi_y^*\psi_d^*\Pi_y\psi_s\right) \\ &+ \alpha_s|\psi_s|^2 + \alpha_d|\psi_d|^2 + \frac{\gamma}{2}|\psi_s|^4 + \frac{\kappa^2}{2}|\psi_d|^4 + |curl\ A|^2. \end{aligned}$$

Here, Π_x, Π_y are the components of the covariant gradient operator $\Pi = \nabla - iA$ along the crystallographic axes x and y. The coefficients K, κ, γ, and μ are constants independent of T such that K, κ, and γ are positive. An interpretation of e_2 is given in [7]. The superconductor is viewed at the microscopic level as a square lattice of atoms representing the crystalline structure forming the layer. Two complex valued order parameters, v and h are introduced representing the mean field distributions of superconducting electron pairs generated by the respective vertical and horizontal nearest neighbor bonds of the lattice. Thus $|v|^2$ ($|h|^2$) represents the number density and $\frac{v^*\Pi v - v\Pi^* v^*}{2i}$ ($\frac{h^*\Pi h - h\Pi^* h^*}{2i}$) represents the supercurrent density of the pairings due to these bonds. The material is defined to be in a pure *d-wave* phase if $h \equiv -v$ and it is in a pure *s-wave* phase if $h \equiv v$. Based on these definitions we set $\psi_s = \frac{(h+v)}{\sqrt{2}}$ and $\psi_d = \frac{(h-v)}{\sqrt{2}}$. The coupling constant μ is a measure of the anisotropy of the lattice's atomic composition. The signs of the quadratic coefficients determine which phase is dominant. If $\alpha_s > 0$ and $\alpha_d < 0$ then the term $\alpha_s|\psi_s|^2$ promotes $|\psi_s| \ll 1$ and the material is classified as a d-wave superconductor. If the signs are reversed it is labeled a s-wave superconductor. Most low temperature superconductors are isotropic s-wave materials characterized by $\mu = 0$, $\alpha_s < 0$, and $\alpha_d > 0$. In this case e_2 is uncoupled in ψ_s and ψ_d. We see it is energetically favorable to seek a pure s-wave state $(\psi_d, \psi_s, A) = (0, \psi_s, A)$, and e_2 reduces to a density equivalent to e_1.

Let υ be a positive constant. Here we investigate an anisotropic d-wave model where we assume $\mu \neq 0$, $T_c^s < T < T_c^d$, $\alpha_s = \upsilon(T - T_c^s) > 0$, and $\alpha_d = \kappa^2(T - T_c^d) < 0$. Normalizing e_2 as before, in this instance with respect to T_c^d, we are lead to

$$\mathcal{E}_2(\psi_d, \psi_s, A) = \int \varepsilon_2 = \tag{1.3}$$
$$\int_{\mathbb{R}^2} \Big(|\Pi\psi_d|^2 + K|\Pi\psi_s|^2 + \mu(\Pi_x\psi_d\Pi_x^*\psi_s^* - \Pi_y\psi_d\Pi_y^*\psi_s^* + c.c.)$$
$$+\beta|\psi_s|^2 + \frac{\gamma}{2}|\psi_s|^4 + \frac{\kappa^2}{2}(1-|\psi_d|^2)^2 + |curl\ A|^2 \Big)$$

where $\beta = \frac{\upsilon(T-T_c^s)}{(T_c^d-T)} > 0$. The corresponding Euler Lagrange equations are

$$-\Pi^2\psi_d - \mu(\Pi_x^2 - \Pi_y^2)\psi_s - \kappa^2\left(1-|\psi_d|^2\right)\psi_d = 0 \tag{1.4}$$
$$-K\Pi^2\psi_s - \mu(\Pi_x^2 - \Pi_y^2)\psi_d + \beta\psi_s + \gamma|\psi_s|^2\psi_s = 0 \tag{1.5}$$
$$curl curl A + \frac{1}{2}i(\psi_d^*\Pi\psi_d - \psi_d\Pi^*\psi_d^*) + \frac{K}{2}i(\psi_s^*\Pi\psi_s - \psi_s\Pi^*\psi_s^*) \tag{1.6}$$
$$+\frac{\mu}{2}\begin{pmatrix} -i\psi_s\Pi_x^*\psi_d^* + i\psi_d^*\Pi_x\psi_s + c.c. \\ i\psi_s\Pi_y^*\psi_d^* - i\psi_d^*\Pi_y\psi_s + c.c. \end{pmatrix} = 0.$$

We find and investigate the structure of solutions (ψ_d, ψ_s, A) that are near the isolated, rotationally symmetric d-wave vortex state $(d_0, 0, A_0)$ as $T \uparrow T_c^d$. Note that $\beta(T) \to \infty$, forcing $\psi_s \to 0$ as $T \uparrow T_c^d$. It is predicted that ψ_s nucleates away from 0 as T decreases from T_c^d with the nucleation concentrating near the vortex core of ψ_d. It is believed that this causes the rotational symmetry of the vortex state to break and that for $T < T_c^d$ the solution develops a fourfold invariance with respect to a rotation through angle of $\frac{\pi}{2}$. This scenario was proposed and studied in a series of papers [1],[4],[5],[6],[7],[10],[12],[17] by examining reduced models, applying numerical simulations, and using formal asympototics. The most direct evidence of symmetry breaking has come from examining the level curves to $|\psi_s|$. In [17], based on numerical simulations, the authors conjecture that a fourfold symmetric s-wave component is generated around the vortex and that its magnitude is of order $\mathbf{O}(\frac{1}{\beta})$ as $T \uparrow T_c^d$. The authors of [1] propose a more refined picture, that asymptotically $\psi_s \approx \frac{\mu}{\beta}\left(\Pi_x^2 - \Pi_y^2\right)\psi_d \approx \frac{\mu}{\beta}\left(\Pi_{0x}^2 - \Pi_{0y}^2\right)d_0$ as $\beta \to \infty$ where $\Pi_0 = \nabla - iA_0$. In [6] the solution is expanded as a series in $\frac{1}{\beta}$ and this relation is derived formally. In [10] the authors analyze the zeroes of the function $\left(\Pi_{0x}^2 - \Pi_{0y}^2\right)d_0$. It follows from their work that if $\kappa > 2$ then

$$\left.\begin{array}{l} \left|\left(\Pi_{0x}^2 - \Pi_{0y}^2\right)d_0(r,\theta)\right| = \left|\left(\Pi_{0x}^2 - \Pi_{0y}^2\right)d_0(r,\theta+\alpha\pi)\right| \text{ for all } (r,\theta) \\ \text{if and only if } 2\alpha \in \mathbb{Z}. \end{array}\right\} \tag{1.7}$$

This last result is the only prior analytic evidence supporting fourfold symmetry that we know of.

In this work, we consider the full set of GL equations (1.4)-(1.6) and find a locally unique solution, prove that it has fourfold symmetry, and establish the above approximation rigorously. Setting $\eta = \frac{1}{\beta}$, $s = \beta\psi_s$, and $d = \psi_d$ we consider the following equations on all of $\mathbb{R}^2$ for small $\eta > 0$.

$$-\Pi^2 d - \eta\mu(\Pi_x^2 - \Pi_y^2)s - \kappa^2(1-|d|^2)d = 0 \tag{1.8}$$

$$-K\eta\Pi^2 s - \mu(\Pi_x^2 - \Pi_y^2)d + s + \gamma\eta^3|s|^2 s = 0 \tag{1.9}$$

$$curlcurlA + \frac{1}{2}i(d^*\Pi d - d\Pi^* d^*) + \frac{1}{2}\eta^2 Ki(s^*\Pi s - s\Pi^* s^*) \tag{1.10}$$

$$+\frac{\eta\mu}{2}\begin{pmatrix} -is\Pi_x^* d^* + id^*\Pi_x s + c.c \\ is\Pi_y^* d^* - id^*\Pi_y s + c.c \end{pmatrix} = 0$$

where $\Pi = \nabla - iA$, $\quad A = A_0 + A_1$, $\quad d = d_0 + d_1$, $(d_1, A_1) \in (\mathbf{H}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{R}^2))$ and $s \in \mathbf{H}^2(\mathbb{R}^2, \mathbb{C})$. Our main theorem follows.

Theorem 1.1. *Given $\gamma, \kappa > 0$ and $\mu \in \mathbb{R}$ there exist constants $K_1(\gamma, \kappa, \mu)$, $\eta_1(K, \gamma, \kappa, \mu)$, $\delta_1(K, \gamma, \kappa, \mu) > 0$ such that if $K \geq K_1$, and $0 < \eta < \eta_1$ then there exists a unique solution $((d_1, A_1), \psi_s) \in (\mathbf{H}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{R}^2)) \cap \mathbf{K}^\perp \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{C})$ to (1.8) - (1.10), for which $\|(d_1, A_1)\|_{\mathbf{H}^2} \leq \delta_1$.*

Here $\mathbf{K} = ker\mathbf{F}_1$ where $\mathbf{F}_1$ is the linearized operator of $\mathbf{F}_0$ at (d_0, A_0). Moreover, we can expand $\tilde{w}_1 = (d_1, A_1) = \eta\mathfrak{w}_1 + \mathfrak{w}_2$ and show that $\| \mathfrak{w}_2 \|_{\mathbf{H}^2} = \mathbf{O}(\eta^2)$. We further show that $s = \mu(\Pi_{0x}^2 - \Pi_{0y}^2)d_0 + \mathbf{O}(\eta)$ in $\mathbf{L}^2(\mathbb{R}^2)$, or equivalently $\psi_s = \eta\mu(\Pi_{0x}^2 - \Pi_{0y}^2)d_0 + \mathbf{O}(\eta^2)$, which coincides with what Affleck, Franz, and Amin stated in [1]. We then prove that the solution satisfies the underlying invariance for the problem.

Theorem 1.2. *Let $(d, A, \eta s)(\eta)$ be the solution from Theorem 1.1. Then*

$$\begin{aligned} d(x,y) &= id(y,-x), \\ (A^1(x,y), A^2(x,y)) &= (-A^2(y,-x), A^1(y,-x)), \\ s(x,y) &= -is(y,-x) \end{aligned}$$

for all $(x,y) \in \mathbb{R}^2$.

We next examine $(\Pi_{0x}^2 - \Pi_{0y}^2)d_0$ further and show that (1.7) is valid for all $\kappa \neq \frac{1}{\sqrt{2}}$. We then combine this with our expansion in η for s to show that radial symmetry is broken exactly as in Theorem 1.2 by proving a nonlinear version of (1.7).

Theorem 1.3. *Let $\kappa \neq \frac{1}{\sqrt{2}}$, $\mu \neq 0$, and $(d, A, \eta s)(\eta)$ be the solution from Theorem 1.1. There exists $\eta_2 > 0$ so that if $0 < \eta < \eta_2$ then*

$$|s(r,\theta)| = |s(r, \theta + \alpha\pi)| \text{ for all } (r,\theta) \quad \text{if and only if } \ 2\alpha \in \mathbb{Z}.$$

Lastly we remark that the following existence theorem which is analogous to Theorem 1.1, requiring that the coupling constant μ be sufficiently small rather than K be large, is true and can be proved in the same manner. The assertions from Theorems 1.2 and 1.3 hold for these solutions as well.

Theorem 1.4. *Given $K, \gamma, \kappa > 0$ there exist constants $\mu_0(K, \gamma, \kappa)$, $\eta_1(K, \gamma, \kappa, \mu)$, $\delta_1(K, \gamma, \kappa, \mu) > 0$ such that if $|\mu| \le \mu_0$, and $0 < \eta < \eta_1$ then there exists a unique solution $((d_1, A_1), \psi_s) \in (\mathbf{H}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{R}^2)) \cap \mathbf{K}^\perp \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{C})$ to* (1.8) - (1.10)*, for which $\|(d_1, A_1)\|_{\mathbf{H}^2} \le \delta_1$.*

2 Existence and uniqueness of the solution

In this section and Section 3 we prove Theorem 1.1 assuming $\mu = \gamma = \kappa = 1$ for convenience and note that our analysis works for any $\mu \in \mathbb{R}$ and $\gamma, \kappa > 0$. Thus we consider the following equations on all of $\mathbb{R}^2$.

$$-\Pi^2 d - \eta(\Pi_x^2 - \Pi_y^2)s - (1 - |d|^2)d = 0 \tag{2.1}$$

$$-K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s = 0 \tag{2.2}$$

$$\begin{aligned} &curlcurlA + \frac{1}{2}i(d^*\Pi d - d\Pi^* d^*) + \frac{1}{2}\eta^2 Ki(s^*\Pi s - s\Pi^* s^*) \\ &+\frac{\eta}{2}\begin{pmatrix} -is\Pi_x^* d^* + id^*\Pi_x s + c.c \\ is\Pi_y^* d^* - id^*\Pi_y s + c.c \end{pmatrix} = 0 \end{aligned} \tag{2.3}$$

where $\Pi = \nabla - iA$, $\quad A = A_0 + A_1$, $\quad d = d_0 + d_1$ and $(d_1, A_1) \in (\mathbf{H}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{R}^2))$ and $s \in \mathbf{H}^2(\mathbb{R}^2, \mathbb{C})$.

We will use $w_0 = (d_0, A_0)$, $w_1 = (d_1, A_1)$ and let

$$\begin{aligned} \mathbf{F}_\eta(d_1, A_1, s) &= \mathbf{F}_\eta(d_0 + d_1, A_0 + A_1, s) \\ &\equiv \mathbf{F}_\eta(w_1, s) \\ &\equiv \mathbf{F}_0(d, A) + \mathbf{H}_\eta(d_1, A_1, s) \end{aligned}$$

where

$$\mathbf{F}_0(d, A) \equiv \begin{pmatrix} -\Pi^2 d - (1 - |d|^2)d \\ curlcurlA + \frac{1}{2}i(d^*\Pi d - d\Pi^* d^*) \end{pmatrix}, \tag{2.4}$$

$$\mathbf{H}_\eta(d_1, A_1, s) \equiv \begin{pmatrix} -\eta(\Pi_x^2 - \Pi_y^2)s \\ \frac{1}{2}\eta^2 Ki(s^*\Pi s - s\Pi^* s^*) + \frac{\eta}{2}\begin{pmatrix} -is\Pi_x^* d^* + id^*\Pi_x s + c.c \\ is\Pi_y^* d^* - id^*\Pi_y s + c.c \end{pmatrix} \end{pmatrix} \tag{2.5}$$

and where s is the solution of (2.2) corresponding to (d_1, A_1). For given $(d_1, A_1) \in \mathbf{H}^2$, the equation (2.2) is the first variation with respect to s for the energy (2.18). This energy is strictly convex with respect to s and as such there exists a unique solution $s \in \mathbf{H}^1$ and it is easy to show that $s \in \mathbf{H}^2$ since $(d_1, A_1) \in \mathbf{H}^2$.
Now split $\mathbf{F}_0$ as a linear part $\mathbf{F}_1$ and nonlinear part $\mathbf{F}_2$.
Let $\mathbf{F}_0(d_0 + d_1, A_0 + A_1) \equiv \mathbf{F}_1(d_1, A_1) + \mathbf{F}_2(d_1, A_1)$ where

$$\begin{aligned} &\mathbf{F}_1(d_1, A_1) \equiv \\ &\begin{pmatrix} \left[-\Pi_{A_0}^2 + \left(2|d_0|^2 - 1\right)\right] d_1 + d_0^2 d_1^* + i\left[2\Pi_{A_0} d_0 + d_0\nabla\right] \cdot A_1 \\ Im\Big(\left[\Pi_{A_0}^* d_0^* - d_0^*\Pi_{A_0}\right] d_1\Big) + \left(-\Delta + \nabla\nabla + |d_0|^2\right) \cdot A_1 \end{pmatrix}. \end{aligned} \tag{2.6}$$

and

$$\mathbf{F}_2(w_1) \equiv \mathbf{F}_2(d_1, A_1) \equiv \tag{2.7}$$
$$\begin{pmatrix} 2d_0|d_1|^2 + d_0^* d_1^2 + |d_1|^2 d_1 + |A_1|^2(d_0+d_1) + i(\nabla\cdot A_1)d_1 + 2iA_1\cdot\nabla d_1 + 2A_0\cdot A_1 d_1 \\ -Im\left(d_1^*(\nabla - iA_0)d_1\right) + A_1\left(2\mathrm{Re}(d_0^* d_1) + |d_1|^2\right)\end{pmatrix}.$$

It is easy to see from (1.1) and (1.3) that for any given smooth function φ,

$$\varepsilon_1(\psi, A) = \varepsilon_1(\psi e^{i\varphi}, A+\nabla\varphi) \text{ and } \varepsilon_2(\psi_d, \psi_s, A) = \varepsilon_2(\psi_d e^{i\varphi}, \psi_s e^{i\varphi}, A+\nabla\varphi)$$

which is to say that the densities are *gauge invariant*. The equation (1.2) is invariant under coordinate translations, and because of the invariance of ε_1 the equation is invariant under gauge transformations as well. Gustafson and Sigal study the linearized operator $\mathbf{F}_1$ in [8]. They use these invariants to characterize the kernel of $\mathbf{F}_1$.

Proposition 1. *[8],[16]*

$$\mathbf{K} \equiv Ker\mathbf{F}_1 =$$
$$span\left\{ \begin{pmatrix} id_0\varphi \\ \nabla\varphi \end{pmatrix}, \begin{pmatrix} \partial_x d_0 - iA_0^1 d_0 \\ \begin{pmatrix} 0 \\ \partial_x A_0^2 - \partial_y A_0^1 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \partial_y d_0 - iA_0^2 d_0 \\ \begin{pmatrix} \partial_y A_0^1 - \partial_x A_0^2 \\ 0 \end{pmatrix} \end{pmatrix} \Big| \varphi \in \mathbf{H}^3 \right\},$$

and $(\mathbf{F}_1 w, w) \geq \tau_0(w,w)$ *on* $\mathbf{K}^\perp \cap \mathbf{H}^2$ *for some* $\tau_0 > 0$.
Here $(w,w) = \| f \|_{\mathbf{L}^2}^2 + \| \mathbf{g} \|_{\mathbf{L}^2}^2$ *for*

$$w = (f, \mathbf{g}) \in \mathbf{L}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{L}^2(\mathbb{R}^2, \mathbb{R}^2)$$

and $\mathbf{K}^\perp$ *is the subspace orthogonal to* $\mathbf{K} \subset \mathbf{L}^2 \times \mathbf{L}^2$. *The operator* $\mathbf{F}_1$ *is an invertible map from* $\mathbf{K}^\perp \cap \mathbf{H}^2$ *onto* $\mathbf{K}^\perp$. *Moreover by elliptic estimates* $\| w \|_{\mathbf{H}^2} \leq \tau_1 \| \mathbf{F}_1(w) \|_{\mathbf{L}^2}$ *for some* $\tau_1 < \infty$.

2.1 Existence of projected solutions

In this subsection,we will show

$$P\left(\mathbf{F}_\eta(d_0+d_1, A_0+A_1, s)\right) = 0 \tag{2.8}$$
$$-K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s = 0 \tag{2.9}$$

have a unique solution $(d_1, A_1) \in \left(\mathbf{H}^2(\mathbb{R}^2,\mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2,\mathbb{R}^2)\right) \cap \mathbf{K}^\perp$ and $s \in \mathbf{H}^2(\mathbb{R}^2,\mathbb{C})$ using a fixed point theorem where P is the orthogonal projection onto $\mathbf{K}^\perp$.
We need some preliminaries.

Proposition 2. *(Interpolation; Gagliardo and Nirenberg) [13]*
Let $u \in \mathbf{L}^q \bigcap \mathbf{L}^{\tilde{q}}$ *in* $\mathbb{R}^n$ *and its derivatives of order* m,

$$\mathbf{D}^m u \in \mathbf{L}^r, 1 \leq q, r \leq \infty, 0 \leq j < m, \text{ for some } \tilde{q} < \infty.$$

Then $\left\|\mathbf{D}^j u\right\|_p \le C \left\|\mathbf{D}^m u\right\|_r^a \left\|u\right\|_q^{1-a}$ *provided*

$$1 \le p \le \infty, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \quad \textit{and} \quad \frac{j}{m} \le a \le 1$$

(the constant C depending only on n, m, j, q, r, a), with the following exception. If $1 < r < \infty$, and $m - j - \frac{n}{r}$ is a non negative integer then it is required that $\frac{j}{m} \le a < 1$.

Let $\mathbf{M} = \mathbf{B}_{\mathbf{H}^2}(0, \delta_1) \cap \mathbf{K}^{\perp}$ and $0 < \eta \le \eta_1$ where δ_1, $\eta_1 > 0$ will be fixed later. For a solution of (2.2) corresponding to a given $(d_1, A_1) \in \mathbf{M}$, we get the following estimate.

Lemma 2.1. *If $(d_1, A_1) \in \mathbf{M}$ and s is the solution of (2.2) corresponding to (d_1, A_1), then*

$$\begin{aligned} & K^2\eta^2 \left\| \Pi^2 s \right\|_{\mathbf{L}^2}^2 + 2\int K\eta|\Pi s|^2 + \|s\|_{\mathbf{L}^2}^2 + 2\int \eta^3|s|^4 \\ & \le \left\| (\Pi_x^2 - \Pi_y^2)(d_0 + d_1) \right\|_{\mathbf{L}^2}^2 \le C. \end{aligned}$$

(We will use a constant C for any constant independent of η_1, δ_1, K and η.)

Proof. By (2.9)

$$\begin{aligned} & \left\| (-K\eta)\, \Pi^2 s + s + \eta^3|s|^2 s \right\|_{\mathbf{L}^2}^2 \\ = \; & K^2\eta^2 \left\| \Pi^2 s \right\|_{\mathbf{L}^2}^2 + \|s\|_{\mathbf{L}^2}^2 + \left\|\eta^3 s^3\right\|_{\mathbf{L}^2}^2 + 2\int \eta^3|s|^4 \\ & + \underbrace{\int (-K\eta)\Big(\left(\Pi^2 s\right) s^* + \left(\Pi^2 s\right)^* s\Big)}_{(I)} + \underbrace{\int (-K\eta^4)\Big(\left(\Pi^2 s\right)|s|^2 s^* + \left(\Pi^2 s\right)^* |s|^2 s\Big)}_{(II)}. \end{aligned}$$

Now $(I) = 2\int K\eta|\Pi s|^2$ and

$$\begin{aligned} (II) &= K\eta^4 \int \Pi s \Pi^* \left(s s^* s^*\right) + c.c. \\ &= K\eta^4 \int \Pi s \Big(2\left(\nabla s^* + iAs^*\right)|s|^2 + \left(\nabla s - iAs\right)\left(s^*\right)^2\Big) + c.c. \\ &= K\eta^4 \int 2|\Pi s|^2|s|^2 + \left(\Pi s\right)^2\left(s^*\right)^2 + c.c. \ge 0. \end{aligned}$$

Since we may assume δ_1, $\eta_1 \le 1$,

$$\begin{aligned} & \left\| \Pi_x^2(d_0 + d_1) \right\|_{\mathbf{L}^2} \\ & = \left\| \partial_{xx}\left(d_0 + d_1\right) - 2iA^1\partial_x(d_0 + d_1) - i\partial_x A^1(d_0 + d_1) - A^1A^1(d_0 + d_1) \right\|_{\mathbf{L}^2} \\ & \le C\left(1 + \left\| d_1 \right\|_{\mathbf{H}^2} + \left\| \partial_x A^1 d_1 \right\|_{\mathbf{L}^2} + \left\| \partial_x d_1 A^1 \right\|_{\mathbf{L}^2} + \left\| A^1A^1 d_1 \right\|_{\mathbf{L}^2}\right) \\ & \le C + C\left\| \partial_x A^1 \right\|_{\mathbf{L}^4} \left\| d_1 \right\|_{\mathbf{L}^4} + C\left\| \partial_x d_1 \right\|_{\mathbf{L}^4} \left\| A^1 \right\|_{\mathbf{L}^4} \\ & \le C + C\left\| A^1 \right\|_{\mathbf{H}^2} \left\| d_1 \right\|_{\mathbf{H}^2} \le C. \end{aligned}$$

Here and after we use Proposition 2 for various values. For example, the third inequality follows from Proposition 2 for $p = 4, j = 0, m = 1, q = r = n = 2$ and $a = \frac{1}{2}$.
So the claim follows. $\square$

Using

$$\int |\Pi s|^2 = \int \left(|\nabla s|^2 - iAs\nabla s^* + iAs^*\nabla s + A \cdot Ass^* \right) \geq \frac{1}{2} \int |\nabla s|^2 - C_1 \int |s|^2,$$

we can get the following inequality if we choose η_1 small enough ($e.g. K\eta_1 C_1 \leq \frac{1}{8}$).

$$\begin{aligned} & K^2\eta^2 \left\| \Pi^2 s \right\|_{\mathbf{L}^2}^2 + \int K\eta |\nabla s|^2 + \frac{1}{2} \|s\|_{\mathbf{L}^2}^2 + \int \eta^3 |s|^4 \\ & \leq \left\| (\Pi_x^2 - \Pi_y^2)(d_0 + d_1) \right\|_{\mathbf{L}^2}^2 \leq C. \end{aligned} \tag{2.10}$$

Lemma 2.2. *If s satisfies* (2.9), *then*

$$\eta \| \Delta s \|_{\mathbf{L}^2} \leq \frac{C}{K} + C\eta_1.$$

Proof.

$$\begin{aligned} \| \Delta s \|_{\mathbf{L}^2} &\leq \left\| \Pi^2 s \right\|_{\mathbf{L}^2} + \| 2A \cdot \nabla s \|_{\mathbf{L}^2} + \| (\nabla \cdot A)\, s \|_{\mathbf{L}^2} + \| A \cdot As \|_{\mathbf{L}^2} \\ \left\| \Pi^2 s \right\|_{\mathbf{L}^2} &\leq \frac{C}{\eta K} \\ \| 2A \cdot \nabla s \|_{\mathbf{L}^2} &\leq C_1 \| \nabla s \|_{\mathbf{L}^2} \leq C_1 \Big(\frac{\|\Delta s\|_{\mathbf{L}^2}}{2C_1} + C_2 \|s\|_{\mathbf{L}^2} \Big). \end{aligned}$$

The second inequality is due to Lemma 2.1.
Moreover, using Proposition 2 we have

$$\begin{aligned} \| (\nabla \cdot A)s \|_{\mathbf{L}^2} &\leq C \|s\|_{\mathbf{L}^2} + \| (\nabla \cdot A_1)s \|_{\mathbf{L}^2} \\ &\leq C \|s\|_{\mathbf{L}^2} + \| \nabla \cdot A_1 \|_{\mathbf{L}^4} \|s\|_{\mathbf{L}^4} \\ &\leq C \|s\|_{\mathbf{L}^2} + C\delta_1 \Big(\| \Delta s \|_{\mathbf{L}^2}^{\frac{1}{4}} \|s\|_{\mathbf{L}^2}^{\frac{3}{4}} \Big) \\ &\leq C \|s\|_{\mathbf{L}^2} + C_3\delta_1 \|\Delta s\|_{\mathbf{L}^2} + C\delta_1 \|s\|_{\mathbf{L}^2}. \end{aligned}$$

Also $\|A \cdot As\|_{\mathbf{L}^2} \leq C \|s\|_{\mathbf{L}^2}$ by Proposition 2. Therefore, if $C_3\delta_1 \leq \frac{1}{4}$ then

$$\eta \|\Delta s\|_{\mathbf{L}^2} \leq \frac{C}{K} + C\eta \|s\|_{\mathbf{L}^2} \leq \frac{C}{K} + C\eta_1. \qquad \square$$

Note. If we choose η_1 small and K big enough, we can make $\eta \|s\|_{\mathbf{H}^2} \leq \delta_2$ (for any $\delta_2 > 0$) which will be fixed later. Without loss of generality we may put $K = \frac{C_0}{\delta_2}$ for some fixed constant C_0.

Lemma 2.3. *For given* $w_1 = (d_1, A_1), \quad w_2 = (d_2, A_2) \in \mathbf{M}$*, let* $s_1,\ s_2$ *be the solutions of the following equations*

$$-K\eta\Pi_1^2 s_1 - \left(\Pi_{1x}^2 - \Pi_{1y}^2\right)(d_0 + d_1) + s_1 + \eta^3|s_1|^2 s_1 = 0 \qquad (2.11)$$
$$-K\eta\Pi_2^2 s_2 - \left(\Pi_{2x}^2 - \Pi_{2y}^2\right)(d_0 + d_2) + s_2 + \eta^3|s_2|^2 s_2 = 0 \qquad (2.12)$$

where $\Pi_1 = \nabla - i(A_0 + A_1) \quad \Pi_2 = \nabla - i(A_0 + A_2)$, $\Pi_{1x} = \nabla_x - i(A_0 + A_1)^1$ *and* A_0^1 *means first component of* A_0 *and* A_0^2 *means second component of* A_0*. Then*

$$\begin{aligned}\eta\|s_1 - s_2\|_{\mathbf{H}^2} &\le \left(\frac{C}{K} + \eta_1 C\right)\|(d_1, A_1) - (d_2, A_2)\|_{\mathbf{H}^2} \\ &\le C\delta_2\|w_1 - w_2\|_{\mathbf{H}^2}.\end{aligned}$$

Proof. From (2.11) and (2.12)

$$\begin{aligned}&-K\eta\Pi_1^2(s_1 - s_2) + s_1 - s_2 + \eta^3|s_1 - s_2|^2(s_1 - s_2) \\ &= K\eta\left(\Pi_1^2 - \Pi_2^2\right)s_2 + \left(\Pi_{1x}^2 - \Pi_{1y}^2\right)d_1 - \left(\Pi_{2x}^2 - \Pi_{2y}^2\right)d_2 \\ &\quad + \left(\Pi_{1x}^2 - \Pi_{1y}^2\right)d_0 - \left(\Pi_{2x}^2 - \Pi_{2y}^2\right)d_0 \\ &\quad -\eta^3(s_1 s_2^* + s_1^* s_2)(s_1 - s_2) - \eta^3 s_1 s_2(s_1^* - s_2^*).\end{aligned}$$

Similar to Lemma 2.1

$$\begin{aligned}&\frac{1}{8}\left(K^2\eta^2\left\|\Pi_1^2(s_1 - s_2)\right\|_{\mathbf{L}^2}^2 + \int K\eta|\nabla(s_1 - s_2)|^2 + \|s_1 - s_2\|_{\mathbf{L}^2}^2 + \int \eta^3|s_1 - s_2|^4\right) \\ &\le \left\|\left(\Pi_{1x}^2 - \Pi_{1y}^2\right)d_1 - \left(\Pi_{2x}^2 - \Pi_{2y}^2\right)d_2\right\|_{\mathbf{L}^2}^2 + \left\|K\eta\left(\Pi_1^2 - \Pi_2^2\right)s_2\right\|_{\mathbf{L}^2}^2 \\ &\quad + \left\|\left(\Pi_{1x}^2 - \Pi_{1y}^2\right)d_0 - \left(\Pi_{2x}^2 - \Pi_{2y}^2\right)d_0\right\|_{\mathbf{L}^2}^2 \\ &+ \|\eta^3(s_1 s_2^* + s_1^* s_2)(s_1 - s_2) + \eta^3 s_1 s_2(s_1^* - s_2^*)\|_{\mathbf{L}^2}^2.\end{aligned}$$

Since

$$\begin{aligned}&\|\eta^3(s_1 s_2^* + s_1^* s_2)(s_1 - s_2) + \eta^3 s_1 s_2(s_1^* - s_2^*)\|_{\mathbf{L}^2} \\ \le\ &C_1\eta\delta_2^2\|s_1 - s_2\|_{\mathbf{L}^2},\end{aligned}$$

$$\begin{aligned}&\frac{1}{16}\left(K\eta\left\|\Pi_1^2(s_1 - s_2)\right\|_{\mathbf{L}^2} + \|s_1 - s_2\|_{\mathbf{L}^2}\right) \\ &\le \left\|\Pi_{1x}^2 d_1 - \Pi_{2x}^2 d_2\right\|_{\mathbf{L}^2} + \left\|\Pi_{1y}^2 d_1 - \Pi_{2y}^2 d_2\right\|_{\mathbf{L}^2} + \left\|K\eta\left(\Pi_1^2 - \Pi_2^2\right)s_2\right\|_{\mathbf{L}^2} \\ &\quad + \left\|\Pi_{1x}^2 d_0 - \Pi_{2x}^2 d_0\right\|_{\mathbf{L}^2} + \left\|\Pi_{1y}^2 d_0 - \Pi_{2y}^2 d_0\right\|_{\mathbf{L}^2}.\end{aligned}$$

The first term

$$
\begin{aligned}
&\left\| \Pi_{1x}^2 d_1 - \Pi_{2x}^2 d_2 \right\|_{\mathbf{L}^2} \\
&\leq \Big\| \partial_{xx} d_1 - 2i\left(A_0^1 + A_1^1\right)\partial_x d_1 - i\partial_x\left(A_0^1 + A_1^1\right) d_1 - \left(A_0^1 + A_1^1\right)\left(A_0^1 + A_1^1\right) d_1 \\
&\quad -\partial_{xx} d_2 - 2i\left(A_0^1 + A_2^1\right)\partial_x d_2 - i\partial_x\left(A_0^1 + A_2^1\right) d_2 - \left(A_0^1 + A_2^1\right)\left(A_0^1 + A_2^1\right) d_2 \Big\|_{\mathbf{L}^2} \\
&\leq \left\| \partial_{xx} d_1 - \partial_{xx} d_2 \right\|_{\mathbf{L}^2} + \left\| 2iA_0^1\left(\partial_x d_1 - \partial_x d_2\right) \right\|_{\mathbf{L}^2} + \left\| i\partial_x A_0^1\left(d_1 - d_2\right) \right\|_{\mathbf{L}^2} \\
&\quad + \left\| A_0^1 A_0^1\left(d_1 - d_2\right) \right\|_{\mathbf{L}^2} + \left\| 2A_0^1\left(A_1^1 d_1 - A_2^1 d_2\right) \right\|_{\mathbf{L}^2} + \left\| A_1^1 A_1^1 d_1 - A_2^1 A_2^1 d_2 \right\|_{\mathbf{L}^2} \\
&\quad +2\left\| A_1^1\partial_x d_1 - A_2^1\partial_x d_2 \right\|_{\mathbf{L}^2} + \left\| \partial_x A_1^1 d_1 - \partial_x A_2^1 d_2 \right\|_{\mathbf{L}^2}.
\end{aligned}
$$

First four terms are easily bounded by $C\Big\|w_1 - w_2\Big\|_{\mathbf{H}^2}$.

$$
\begin{aligned}
2\left\| A_0^1\left(A_1^1 d_1 - A_2^1 d_2\right) \right\|_{\mathbf{L}^2} &\leq C\left\| A_1^1 d_1 - A_2^1 d_2 \right\|_{\mathbf{L}^2} \\
&= C\left\| A_1^1 d_1 - A_1^1 d_2 + A_1^1 d_2 - A_2^1 d_2 \right\|_{\mathbf{L}^2} \\
&\leq C\Big(\left\|A_1^1\right\|_\infty \left\|d_1 - d_2\right\|_{\mathbf{L}^2} + \left\|d_2\right\|_\infty \left\|A_1^1 - A_2^1\right\|_{\mathbf{L}^2} \Big) \\
&\leq C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2}.
\end{aligned}
$$

Similarly,

$$
\Big\|A_1^1 A_1^1 d_1 - A_2^1 A_2^1 d_2\Big\|_{\mathbf{L}^2} \leq C\Big\|w_1 - w_2\Big\|_{\mathbf{H}^2} \quad \text{and}
$$

$$
\begin{aligned}
&2\left\| A_1^1\partial_x d_1 - A_2^1\partial_x d_2 \right\|_{\mathbf{L}^2} \\
&= 2\left\| A_1^1\partial_x d_1 - A_2^1\partial_x d_1 + A_2^1\partial_x d_1 - A_2^1\partial_x d_2 \right\|_{\mathbf{L}^2} \\
&\leq 2\left\| A_1^1 - A_2^1 \right\|_{\mathbf{L}^4} \left\| \partial_x d_1 \right\|_{\mathbf{L}^4} + \left\| A_2^1 \right\|_\infty \left\| \partial_x d_1 - \partial_x d_2 \right\|_{\mathbf{L}^2} \\
&\leq C\left\| D^2\left(A_1^1 - A_2^1\right) \right\|_{\mathbf{L}^2}^{\frac{1}{4}} \left\| \left(A_1^1 - A_2^1\right) \right\|_{\mathbf{L}^2}^{\frac{3}{4}} \left\| D^2 d_1 \right\|_{\mathbf{L}^2}^{\frac{3}{4}} \left\| d_1 \right\|_{\mathbf{L}^2}^{\frac{1}{4}} + C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2} \\
&\leq C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2}.
\end{aligned}
$$

Furthermore, $\left\| \partial_x A_1^1 d_1 - \partial_x A_2^1 d_2 \right\|_{\mathbf{L}^2}$ is estimated in the same manner.
Therefore

$$
\left\| \Pi_{1x}^2 d_1 - \Pi_{2x}^2 d_2 \right\|_{\mathbf{L}^2} \leq C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2} \quad \text{and} \quad \left\| \Pi_{1y}^2 d_1 - \Pi_{2y}^2 d_2 \right\|_{\mathbf{L}^2} \leq C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2}.
$$

$$
\begin{aligned}
&\left\| K\eta\left(\Pi_1^2 - \Pi_2^2\right) s_2 \right\|_{\mathbf{L}^2} \\
&\leq \left\| K\eta\left(-i\nabla(A_0 + A_1)s_2 + i\nabla(A_0 + A_2)s_2\right) \right\|_{\mathbf{L}^2} (\equiv \mathbf{I}_1) \\
&\quad +2K\eta\left\| \left(A_0 + A_1\right)\nabla s_2 - \left(A_0 + A_2\right)\nabla s_2 \right\|_{\mathbf{L}^2} (\equiv \mathbf{I}_2) \\
&\quad +K\eta\left\| \left(A_0 + A_1\right)\cdot\left(A_0 + A_1\right) s_2 - \left(A_0 + A_2\right)\cdot\left(A_0 + A_2\right) s_2 \right\|_{\mathbf{L}^2} (\equiv \mathbf{I}_3)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_1 &\leq K\left\|\eta s_2\right\|_\infty \left\| \nabla A_1 - \nabla A_2 \right\|_{\mathbf{L}^2} \\
&\leq CK\delta_2\left\| \nabla A_1 - \nabla A_2 \right\|_{\mathbf{L}^2} \leq C\left\| w_1 - w_2 \right\|_{\mathbf{H}^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_2 &\le 2K\eta \, \| (A_1 - A_2) \nabla s_2 \|_{\mathbf{L}^2} \\
&\le 2K \, \| A_1 - A_2 \|_4 \, \| \eta \nabla s_2 \|_4 \\
&\le CK \left\| D^2 (A_1 - A_2) \right\|_{\mathbf{L}^2}^{\frac14} \| A_1 - A_2 \|_{\mathbf{L}^2}^{\frac34} \left\| \eta D^2 s_2 \right\|_{\mathbf{L}^2}^{\frac34} \| \eta s_2 \|_{\mathbf{L}^2}^{\frac14} \\
&\le CK\delta_2 \, \| w_1 - w_2 \|_{\mathbf{H}^2} \le C \, \| w_1 - w_2 \|_{\mathbf{H}^2} \\
\mathbf{I}_3 &= K\eta \, \| 2A_0 (A_1 - A_2) s_2 + (A_1 + A_2) \cdot (A_1 - A_2) s_2 \|_{\mathbf{L}^2} \\
&\le CK \, \| \eta s_2 \|_\infty \| A_1 - A_2 \|_{\mathbf{L}^2} \\
&\quad + CK \, \| \eta s_2 \|_\infty \Big(\|A_1\|_\infty + \|A_2\|_\infty \Big) \| A_1 - A_2 \|_{\mathbf{L}^2} \\
&\le CK\delta_2 \, \| w_1 - w_2 \|_{\mathbf{H}^2} \le C \, \| w_1 - w_2 \|_{\mathbf{H}^2} .
\end{aligned}
$$

Therefore

$$
K\eta \left\| \Pi_1^2 (s_1 - s_2) \right\|_{\mathbf{L}^2} + \| s_1 - s_2 \|_{\mathbf{L}^2} \le C \, \| w_1 - w_2 \|_{\mathbf{H}^2} . \tag{2.13}
$$

As in the proof of lemma 2.2,

$$
\eta \, \| \Delta(s_1 - s_2) \|_{\mathbf{L}^2} \le \frac{C \, \| w_1 - w_2 \|_{\mathbf{H}^2}}{K} + C\eta_1 \, \| (s_1 - s_2) \|_{\mathbf{L}^2} .
$$

So $\eta \, \| s_1 - s_2 \|_{\mathbf{H}^2} \le \left(\frac{C}{K} + \eta_1 C \right) \| w_1 - w_2 \|_{\mathbf{H}^2}$. Therefore, the claim follows. □

Note $\quad \eta \, \| s_1 - s_2 \|_{\mathbf{H}^2} \le C\delta_2 \, \| w_1 - w_2 \|_{\mathbf{H}^2}$.

Lemma 2.4. *If s_1, s_2 satisfy (2.2) corresponding to w_1, w_2 respectively, then*

$$
\| \mathbf{H}_\eta (d_1, A_1, s_1) - \mathbf{H}_\eta (d_2, A_2, s_2) \|_{\mathbf{L}^2} \le C\delta_2 \, \| w_1 - w_2 \|_{\mathbf{H}^2} .
$$

Proof. Recall

$$
\mathbf{H}_\eta(d_1, A_1, s) \equiv \begin{pmatrix} -\eta(\Pi_x^2 - \Pi_y^2)s \\ \frac12 (\eta^2 K i (s^* \Pi s - s \Pi^* s^*)) + \frac{\eta}{2} \begin{pmatrix} -is\Pi_x^* d^* + id^* \Pi_x s + c.c \\ is\Pi_y^* d^* - id^* \Pi_y s + c.c \end{pmatrix} \end{pmatrix} .
$$

We will prove the inequality for each term separately.

$$
\begin{aligned}
& \|\eta \Pi_{1x}^2 s_1 - \eta \Pi_{2x}^2 s_2\|_{\mathbf{L}^2} \\
= & \Big\| \eta \, (\nabla - i(A_0 + A_1))^1 \, (\nabla - i(A_0 + A_1))^1 \, s_1 \\
& - \eta \, (\nabla - i(A_0 + A_2))^1 \, (\nabla - i(A_0 + A_2))^1 \, s_2 \Big\|_{\mathbf{L}^2} \\
\le & \, \| \eta \partial_{xx} (s_1 - s_2) \|_{\mathbf{L}^2} + \eta \left\| 2i(A_0 + A_1)^1 \cdot \partial_x s_1 - 2i(A_0 + A_2)^1 \cdot \partial_x s_2 \right\|_{\mathbf{L}^2} \\
& \quad + \eta \left\| \partial_x (A_0 + A_1)^1 s_1 - \partial_x (A_0 + A_2)^1 s_2 \right\|_{\mathbf{L}^2} \\
& \quad + \eta \left\| (A_0 + A_1)^1 (A_0 + A_1)^1 s_1 - (A_0 + A_2)^1 (A_0 + A_2)^1 s_2 \right\|_{\mathbf{L}^2} \\
& = (1) + (2) + (3) + (4)
\end{aligned}
$$

By Lemma 2.3

$$\begin{aligned}
(1) &\leq \eta \,\|\, s_1 - s_2 \,\|_{\mathbf{H}^2} \leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} \\
(2) &\leq C\eta \,\|\, \partial_x s_1 - \partial_x s_2 \,\|_{\mathbf{L}^2} + C\eta \,\|\, A_1 - A_2 \,\|_{\mathbf{L4}} \,\|\, \partial_x s_1 \,\|_{\mathbf{L4}} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2}
\end{aligned}$$

$$\begin{aligned}
(3) &\leq \left\|\, \eta \left(\partial_x A_0^1\right)(s_1 - s_2) \,\right\|_{\mathbf{L}^2} + \left\|\, \eta \left(\partial_x A_1^1 s_1 - \partial_x A_2^1 s_2\right) \,\right\|_{\mathbf{L}^2} \\
&\leq C \,\|\, \eta(s_1 - s_2) \,\|_{\mathbf{H}^2} + \left\|\, \eta \left(\partial_x A_1^1 - \partial_x A_2^1\right) s_1 + \eta \partial_x A_2^1 (s_1 - s_2) \,\right\|_{\mathbf{L}^2} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} + \|\, \eta s_1 \,\|_\infty \,\|\, \partial_x (A_1 - A_2) \,\|_{\mathbf{L}^2} + \|\, \partial_x A_2 \,\|_{\mathbf{L}^4} \,\|\, \eta(s_1 - s_2) \,\|_{\mathbf{L}^4} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} + C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} + C \,\|\, A_2 \,\|_{\mathbf{H}^2} \,\|\, \eta(s_1 - s_2) \,\|_{\mathbf{H}^2} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2}
\end{aligned}$$

$$\begin{aligned}
(4) &\leq \eta \left\|\, A_0^1 A_0^1 s_1 - A_0^1 A_0^1 s_2 \,\right\|_{\mathbf{L}^2} + 2\eta \left\|\, A_1^1 A_0^1 s_1 - A_2^1 A_0^1 s_2 \,\right\|_{\mathbf{L}^2} \\
&\quad + \eta \left\|\, A_1^1 A_1^1 s_1 - A_2^1 A_2^1 s_2 \,\right\|_{\mathbf{L}^2} \\
&\leq C\eta \,\|\, s_1 - s_2 \,\|_{\mathbf{H}^2} + C\eta \left\|\, A_1^1 s_1 - A_2^1 s_1 + A_2^1 (s_1 - s_2) \,\right\|_{\mathbf{L}^2} \\
&\quad + \eta \left\|\, A_1^1 A_1^1 s_1 - A_2^1 A_2^1 s_1 + A_2^1 A_2^1 (s_1 - s_2) \,\right\|_{\mathbf{L}^2} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} + 2C \,\|\, \eta s_1 \,\|_\infty \,\|\, A_1 - A_2 \,\|_{\mathbf{L}^2} + 2C\delta_2 \,\|\, \eta(s_1 - s_2) \,\|_{\mathbf{H}^2} \\
&\quad + \|\, \eta s_1 \,\|_\infty \left\|\, A_1^1 A_1^1 - A_2^1 A_2^1 \,\right\|_{\mathbf{L}^2} + \eta \,\|\, A_2 \,\|_\infty^2 \,\|\, s_1 - s_2 \,\|_{\mathbf{L}^2} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2}.
\end{aligned}$$

Similarly $\left\|\, \eta \Pi_{1y}^2 s_1 - \eta \Pi_{2y}^2 s_2 \,\right\|_{\mathbf{L}^2} \leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2}$.

$$\begin{aligned}
&\left\|\, \eta^2 K s_1^* \left(\nabla - i(A_0 + A_1)\right) s_1 - \eta^2 K s_2^* \left(\nabla - i(A_0 + A_2)\right) s_2 \,\right\|_{\mathbf{L}^2} \\
&\leq \underbrace{\left\|\, \eta^2 K s_1^* \nabla s_1 - \eta^2 K s_2^* \nabla s_2 \,\right\|_{\mathbf{L}^2}}_{(1)} \\
&\quad + \underbrace{\left\|\, \eta^2 K s_1^* (A_0 + A_1) s_1 - \eta^2 K s_2^* (A_0 + A_2) s_2 \,\right\|_{\mathbf{L}^2}}_{(2)}.
\end{aligned}$$

$$\begin{aligned}
(1) &\leq \eta^2 K \,\|\, s_1^* \nabla s_1 - s_2^* \nabla s_1 + s_2^* \left(\nabla s_1 - \nabla s_2\right) \,\|_{\mathbf{L}^2} \\
&\leq K \,\|\, \eta \left(s_1^* - s_2^*\right) \,\|_{\mathbf{L}^4} \,\|\, \eta \nabla s_1 \,\|_{\mathbf{L}^4} + \|\, \eta s_2 \,\|_\infty \, \eta K \,\|\, \nabla s_1 - \nabla s_2 \,\|_{\mathbf{L}^2} \\
&\leq C\delta_2 K \delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} + C\delta_2 K \delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} \\
&\leq C\delta_2 \,\|\, w_1 - w_2 \,\|_{\mathbf{H}^2} \quad \text{since } \delta_2 K = C_0, \text{ a fixed constant.}
\end{aligned}$$

$$(2) \leq \underbrace{\left\|\, \eta^2 K A_0 s_1 s_1^* - \eta^2 K A_0 s_2 s_2^* \,\right\|_{\mathbf{L}^2}}_{I} + \underbrace{\left\|\, \eta^2 K A_1 s_1 s_1^* - \eta^2 K A_2 s_2 s_2^* \,\right\|_{\mathbf{L}^2}}_{II}.$$

$$
\begin{aligned}
I &\le CK \left\| \eta^2 \left(s_1 s_1^* - s_1 s_2^* + s_1 s_2^* - s_2 s_2^*\right) \right\|_{\mathbf{L}^2} \\
&\le CK \| \eta s_1 \|_\infty \| \eta \left(s_1^* - s_2^*\right) \|_{\mathbf{L}^2} + CK \| \eta s_2 \|_\infty \| \eta \left(s_1 - s_2\right) \|_{\mathbf{L}^2} \\
&\le CK\delta_2^2 \| w_1 - w_2 \|_{\mathbf{H}^2} \le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} .
\end{aligned}
$$

$$
\begin{aligned}
II &\le K \left\| \eta^2 \left(A_1 s_1 s_1^* - A_2 s_1 s_1^* + A_2 s_1 s_1^* - A_2 s_1 s_2^* + A_2 (s_1 - s_2) s_2^*\right) \right\|_{\mathbf{L}^2} \\
&\le K \Big(\| \eta s_1 \|_\infty^2 \| A_1 - A_2 \|_{\mathbf{L}^2} + \| A_2 \|_\infty \| \eta s_1 \|_\infty \eta \| s_1 - s_2 \|_{\mathbf{L}^2} \\
&\quad + \| A_2 \|_\infty \| \eta s_2 \|_\infty \| \eta (s_1 - s_2) \|_{\mathbf{L}^2} \Big) \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} .
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \eta s_1 \left(\nabla + i(A_0 + A_1)\right)^1 \left(d_0^* + d_1^*\right) - \eta s_2 \left(\nabla + i(A_0 + A_2)\right)^1 \left(d_0^* + d_2^*\right) \right\|_{\mathbf{L}^2} \\
&\le \underbrace{\| \eta \partial_x d_0^* (s_1 - s_2) \|_{\mathbf{L}^2}}_{(1)} + \underbrace{\| \eta s_1 \partial_x d_1^* - \eta s_2 \partial_x d_2^* \|_{\mathbf{L}^2}}_{(2)} \\
&\quad + \underbrace{\left\| \eta s_1 \left(A_0 + A_1\right)^1 \left(d_0^* + d_1^*\right) - \eta s_2 \left(A_0 + A_2\right)^1 \left(d_0^* + d_2^*\right) \right\|_{\mathbf{L}^2}}_{(3)} .
\end{aligned}
$$

$$
\begin{aligned}
(1) &\le C \| \eta (s_1 - s_2) \|_{\mathbf{H}^2} \le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \\
(2) &= \| \eta s_1 \partial_x d_1^* - \eta s_1 \partial_x d_2^* + \eta (s_1 - s_2) \partial_x d_2^* \|_{\mathbf{L}^2} \\
&\le \| \eta s_1 \|_\infty \| \partial_x d_1 - \partial_x d_2 \|_{\mathbf{L}^2} + \| \eta (s_1 - s_2) \|_\infty \| \partial_x d_2 \|_{\mathbf{L}^2} \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \\
(3) &\le \left\| \eta (s_1 - s_2) A_0^1 d_0^* \right\|_{\mathbf{L}^2} + \left\| \eta s_1 A_1^1 d_0^* - \eta s_2 A_2^1 d_0^* \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \eta s_1 A_0^1 d_1^* - \eta s_2 A_0^1 d_2^* \right\|_{\mathbf{L}^2} + \left\| \eta s_1 A_1^1 d_1^* - \eta s_2 A_2^1 d_2^* \right\|_{\mathbf{L}^2} \\
&= I + II + III + IV.
\end{aligned}
$$

$$
\begin{aligned}
I &\le C \| \eta (s_1 - s_2) \|_{\mathbf{H}^2} \le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \\
II &\le C \left\| \eta s_1 A_1^1 - \eta s_1 A_2^1 + (\eta s_1 - \eta s_1) A_2^1 \right\|_{\mathbf{L}^2} \\
&\le C \| \eta s_1 \|_\infty \| A_1 - A_2 \|_{\mathbf{L}^2} + C \| A_2 \|_\infty \| \eta (s_1 - s_2) \|_{\mathbf{L}^2} \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \\
III &\le C \| \eta s_1 d_1^* - \eta s_2 d_1^* + \eta s_2 \left(d_1^* - d_2^*\right) \|_{\mathbf{L}^2} \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \\
IV &\le \left\| \eta s_1 A_1^1 d_1^* - \eta s_2 A_1^1 d_1^* + \eta s_2 d_1^* \left(A_1^1 - A_2^1\right) + \eta s_2 A_2^1 \left(d_1^* - d_2^*\right) \right\|_{\mathbf{L}^2} \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} + \| \eta s_2 \|_\infty \| d_1 \|_\infty \| A_1 - A_2 \|_{\mathbf{L}^2} \\
&\quad + \| \eta s_2 \|_\infty \| A_2 \|_\infty \| d_1 - d_2 \|_{\mathbf{L}^2} \\
&\le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} .
\end{aligned}
$$

Furthermore $\frac{\eta}{2} i (s \Pi_y^* d^* - i d^* \Pi_y s)$ can be estimated in the same way. Therefore

$$
\| \mathbf{H}_\eta (d_1, A_1, s_1) - \mathbf{H}_\eta (d_2, A_2, s_2) \|_{\mathbf{L}^2} \le C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} .
$$

$\square$

Lemma 2.5.

$$\| \mathbf{F}_2(w_1) - \mathbf{F}_2(w_2) \|_{\mathbf{L}^2} \leq C\delta_1 \| w_1 - w_2 \|_{\mathbf{H}^2} \quad \textit{for} \quad w_1, w_2 \in \mathbf{M}.$$

Proof.

$$\begin{aligned} \left\| 2d_0|d_1|^2 - 2d_0|d_2|^2 \right\|_{\mathbf{L}^2} &\leq C \| d_1^* d_1 - d_2^* d_1 + d_2^*(d_1 - d_2) \|_{\mathbf{L}^2} \\ &\leq C \| d_1 \|_\infty \| d_1 - d_2 \|_{\mathbf{L}^2} + C \| d_2 \|_\infty \| d_1 - d_2 \|_{\mathbf{L}^2} \\ &\leq C\delta_1 \| w_1 - w_2 \|_{\mathbf{H}^2}. \end{aligned}$$

Similar estimates hold for $\left\| d_0^* d_1^2 - d_0^* d_2^2 \right\|_{\mathbf{L}^2}$, $\left\| |A_1|^2 d_0 - |A_2|^2 d_0 \right\|_{\mathbf{L}^2}$ and $\| 2A_0 \cdot A_1 d_1 - 2A_0 \cdot A_2 d_2 \|_{\mathbf{L}^2}$.

$$\begin{aligned} &\| i(\nabla \cdot A_1)d_1 - i(\nabla \cdot A_2)d_2 \|_{\mathbf{L}^2} \\ &\leq \| (\nabla \cdot A_1)d_1 - (\nabla \cdot A_2)d_1 + (\nabla \cdot A_2)(d_1 - d_2) \|_{\mathbf{L}^2} \\ &\leq \| d_1 \|_\infty \| \nabla(A_1 - A_2) \|_{\mathbf{L}^2} + \| \nabla \cdot A_2 \|_{\mathbf{L}^4} \| d_1 - d_2 \|_{\mathbf{L}^4} \\ &\leq C\delta_1 \| A_1 - A_2 \|_{\mathbf{H}^2} + C\delta_1 \| d_1 - d_2 \|_{\mathbf{H}^2} \leq C\delta_1 \| w_1 - w_2 \|_{\mathbf{H}^2}. \end{aligned}$$

$$\begin{aligned} \left\| |d_1|^2 d_1 - |d_2|^2 d_2 \right\|_{\mathbf{L}^2} &\leq \left\| |d_1|^2 d_1 - |d_2|^2 d_1 + |d_2|^2 (d_1 - d_2) \right\|_{\mathbf{L}^2} \\ &\leq \| d_1 \|_\infty \left\| |d_1|^2 - |d_2|^2 \right\|_{\mathbf{L}^2} + \| d_2 \|_\infty^2 \| d_1 - d_2 \|_{\mathbf{L}^2} \\ &\leq C\delta_1 \| d_1 - d_2 \|_{\mathbf{H}^2}. \end{aligned}$$

The remaining differences are estimated in an identical manner. Therefore the claim follows. $\square$

Lemma 2.6. *If* $w_1 = (d_1, A_1) \in \mathbf{M}$ *then*

$$(1) \;\; \| \mathbf{F}_2(w_1) \|_{\mathbf{L}^2} \leq C\delta_1^2 \quad \textit{and} \quad (2) \;\; \| \mathbf{H}_\eta(w_1, s) \|_{\mathbf{L}^2} \leq C\delta_2.$$

Proof.

$$\begin{aligned} (1) \qquad &\left\| 2d_0|d_1|^2 \right\|_{\mathbf{L}^2} \leq C \| d_1 \|_\infty^2 \leq C\delta_1^2 \\ &\| i(\nabla \cdot A_1)d_1 \|_{\mathbf{L}^2} \leq C \| d_1 \|_\infty \| \nabla \cdot A_1 \|_{\mathbf{L}^2} \leq C\delta_1^2 \\ &\left\| |d_1|^2 d_1 \right\|_{\mathbf{L}^2} \leq \| d_1 \|_\infty^3 \leq C\delta_1^2. \end{aligned}$$

All other terms are similar to these.

$$(2) \;\; \left\| \eta \left(\Pi_x^2 - \Pi_y^2 \right) s \right\|_{\mathbf{L}^2} \leq \underbrace{\left\| \eta \Pi_x^2 s \right\|_{\mathbf{L}^2}}_{I} + \underbrace{\left\| \eta \Pi_y^2 s \right\|_{\mathbf{L}^2}}_{II}.$$

$$\begin{aligned} I &\leq \eta \| \partial_{xx} s \|_{\mathbf{L}^2} + \eta \left\| 2A^1 \partial_x s \right\|_{\mathbf{L}^2} + \eta \left\| \partial_x A^1 s \right\|_{\mathbf{L}^2} + \eta \left\| A^1 A^1 s \right\|_{\mathbf{L}^2} \\ &\leq C\delta_2 + C\eta \| A \|_{\mathbf{H}^2} \| s \|_{\mathbf{H}^2} + C\eta \| A \|_{\mathbf{H}^2}^2 \| s \|_{\mathbf{H}^2} \leq C\delta_2. \end{aligned}$$

Similarly, $II \leq C\delta_2$.

$$\begin{aligned} \left\| \eta^2 K s^* \Pi s \right\|_{\mathbf{L}^2} &\leq \eta^2 K \| s^* (\nabla - iA) s \|_{\mathbf{L}^2} \\ &\leq \eta^2 K \| s^* \nabla s \|_{\mathbf{L}^2} + C\eta^2 K \| s^* s \|_{\mathbf{L}^2} \\ &\leq C\eta^2 K \| s \|_{\mathbf{H}^2}^2 \leq C\delta_2. \end{aligned}$$

Similarly, $\| \eta s \Pi_x^* d^* \|_{\mathbf{L}^2} \leq C\delta_2$ and $\| \eta d \Pi_x s \|_{\mathbf{L}^2} \leq C\delta_2$. So the claim follows. □

Theorem 2.1. *There exist K_1, $\eta_1(K)$, $\delta_1(K)$, > 0 such that if $K \geq K_1$, $0 < \eta < \eta_1$ then there exists a unique solution $((d_1, A_1), s) \in (\mathbf{H}^2(\mathbb{R}^2, \mathbb{C}) \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{R}^2)) \cap \mathbf{K}^\perp \times \mathbf{H}^2(\mathbb{R}^2, \mathbb{C})$ to (2.8) and (2.9), for which $\|(d_1, A_1)\|_{\mathbf{H}^2} \leq \delta_1$.*

Proof. We argue as in [16].
If $w \in \mathbf{M}$ and s is the solution of (2.9) corresponding to w, then

$$\begin{aligned} \mathbf{F}_\eta(w_0 + w, s) &= \mathbf{F}_1(w) + \mathbf{F}_2(w) + \mathbf{H}_\eta(w, s) \quad \text{and} \\ P\mathbf{F}_\eta(w_0 + w, s) &= P\mathbf{F}_1(w) + P\mathbf{F}_2(w) + P\mathbf{H}_\eta(w, s). \end{aligned}$$

Since $\mathbf{L}_0 \equiv (P\mathbf{F}_1)\Big|_{\mathbf{K}^\perp \cap \mathbf{H}^2}$ is invertible and $\|w\|_{\mathbf{H}^2} \leq \tau_1 \| \mathbf{L}_0(w) \|_{\mathbf{L}^2}$,

$$\mathbf{L}_0^{-1} (P\mathbf{F}_\eta(w_0 + w, s)) = w + \mathbf{L}_0^{-1} (P\mathbf{F}_2(w) + P\mathbf{H}_\eta(w)).$$

Now let

$$S_\eta(w) = -\mathbf{L}_0^{-1} (P\mathbf{F}_2(w) + P\mathbf{H}_\eta(w, s))$$

then

$$\begin{aligned} \| S_\eta(w) \|_{\mathbf{H}^2} &\leq \tau_1 \| P\mathbf{F}_2(w) + P\mathbf{H}_\eta(w, s) \|_{\mathbf{L}^2} \\ &\leq \tau_1 \Big(\| P\mathbf{F}_2(w) \|_{\mathbf{L}^2} + \| P\mathbf{H}_\eta(w, s) \|_{\mathbf{L}^2} \Big) \\ &\leq \tau_1 \left(C_1 \delta_1^2 + C_2 \delta_2 \right). \end{aligned}$$

We choose δ_1 sufficiently small so that $\tau_1 C_1 \delta_1 \leq \frac{1}{2}$. Then take K big enough and η_1 small enough so that $\tau_1 C_2 \delta_2 \leq \frac{1}{2}\delta_1$. We then have $\| S_\eta(w) \|_{\mathbf{H}^2} \leq \delta_1$. Thus $S_\eta(w) \in \mathbf{M}$ if $w \in \mathbf{M}$. Furthermore

$$\begin{aligned} \| S_\eta(w_1) - S_\eta(w_2) \|_{\mathbf{H}^2} &\leq \tau_1 \| \mathbf{F}_2(w_1) - \mathbf{F}_2(w_2) \|_{\mathbf{L}^2} + \tau_1 \| \mathbf{H}_\eta(w_1, s_1) - \mathbf{H}_\eta(w_2, s_2) \|_{\mathbf{L}^2} \\ &\leq \tau_1 \Big(C\delta_1 \| w_1 - w_2 \|_{\mathbf{H}^2} + C\delta_2 \| w_1 - w_2 \|_{\mathbf{H}^2} \Big) \\ &\leq \frac{1}{2} \| w_1 - w_2 \|_{\mathbf{H}^2} \end{aligned}$$

if $\tau_1 C(\delta_1 + \delta_2) \leq \frac{1}{2}$, where s_1 and s_2 are the solutions of (2.9) corresponding to w_1 and w_2. Therefore S_η is a contraction map and S_η has a unique fixed point w_η in $\mathbf{M}$. Then (w_η, s_η) solves the equations (2.8), (2.9) where s_η is the solution (2.9) corresponding to w_η. Therefore this proves the theorem. □

2.2 $w_1 = \mathbf{O}(\eta)$

In this section,we show that if (w_1, s) is the solution found in Section 2.1, then $\| w_1 \|_{\mathbf{H}^2} \leq CK\eta$. Let $w_1 = (d_1, A_1) \in \mathbf{M}$, $s = s_0 + s_1$ be the unique solution of

$$\begin{array}{l} P\mathbf{F}_\eta(d_0 + d_1, A_0 + A_1, s) = 0 \\ -K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s = 0 \end{array}$$

where $d = d_0 + d_1$, $A = A_0 + A_1$, $s_0 = \left(\Pi_{0x}^2 - \Pi_{0y}^2\right) d_0$. Then s_0 is smooth and is in $\mathbf{H}^2$.
We need some properties of $\mathbf{F}_2(w_1)$ and $\mathbf{H}_\eta(w_1, s)$.

Lemma 2.7. *(1)* $\| \mathbf{F}_2(w_1) \|_{\mathbf{L}^2} \leq C\delta_1 \| w_1 \|_{\mathbf{H}^2}$
(2) $\| \mathbf{H}_\eta(w_1, s_0 + s_1) \|_{\mathbf{L}^2} \leq C\eta \left(\left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \right) + C\eta \| s_0 \|_{\mathbf{H}^2}$.

Proof. (1) By simple estimation, the claim follows.
(2) Using

$$\eta \left\| \left(\Pi_x^2 - \Pi_y^2\right) s \right\|_{\mathbf{L}^2} \leq \left\| \eta\Pi_x^2 s \right\|_{\mathbf{L}^2} + \eta \left\| \Pi_y^2 s \right\|_{\mathbf{L}^2},$$

$$\begin{aligned} &\left\| \Pi_x^2(s_0 + s_1) \right\|_{\mathbf{L}^2} \\ &= \left\| \partial_{xx}(s_0 + s_1) - 2iA^1\partial_x(s_0 + s_1) - i(s_0 + s_1)\partial_x A^1 - (A^1A^1)(s_0 + s_1) \right\|_{\mathbf{L}^2} \\ &\leq \| \partial_{xx}s_0 \|_{\mathbf{L}^2} + \| \partial_{xx}s_1 \|_{\mathbf{L}^2} + C \| \partial_x(s_0 + s_1) \|_{\mathbf{L}^2} \\ &\quad + \| s_0 + s_1 \|_{\mathbf{L}^4} \left\| \partial_x A^1 \right\|_{\mathbf{L}^4} + C \| s_0 + s_1 \|_{\mathbf{L}^2} . \end{aligned}$$

Thus

$$\left\| \eta \left(\Pi_x^2 - \Pi_y^2\right) s \right\|_{\mathbf{L}^2} \leq C\eta \left(\left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \right) + C\eta \| s_0 \|_{\mathbf{H}^2}, \tag{2.14}$$

Since $\Pi_0 d_0 \in \mathbf{L}^2$,

$$\begin{aligned} \| s\Pi_x^* d^* \|_{\mathbf{L}^2} &\leq \|s\|_\infty \| \Pi_x^* d \|_{\mathbf{L}^4} \leq C\|s\|_{\mathbf{H}^2} \\ &\leq C \left(\left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \right) + C \| s_0 \|_{\mathbf{H}^2}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \eta^2 K \| s^*(\nabla - iA)s \|_{\mathbf{L}^2} &\leq \eta^2 K \left(\|s\|_\infty \|\nabla s\|_{\mathbf{L}^2} + C\|s\|_\infty \|s\|_{\mathbf{L}^2} \right) \\ &\leq C\eta \left(\|s_1\|_{\mathbf{L}^2} + \left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \|s_0\|_{\mathbf{H}^2} \right). \end{aligned} \tag{2.16}$$

Note. $\|\eta s\|_\infty \leq C\|\eta s\|_{\mathbf{H}^2} \leq C\delta_2$ and $\delta_2 K = C_0$ a constant. By (2.14), (2.15) and (2.16), the claim follows. ☐

Lemma 2.8. $K\eta \left\| \Pi^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \leq CK\eta + C\|w_1\|_{\mathbf{H}^2}$

Proof. Set $s = s_0 + s_1$ in (2.2). Thus s_1 satisfies

$$\begin{aligned} - \;& K\eta\Pi^2 s_1 + s_1 + \eta^3 |s_1|^2 s_1 \\ = \;& \underbrace{\left(\Pi_x^2 - \Pi_y^2\right) d_1}_{(1)} + \underbrace{K\eta\Pi^2 s_0}_{(2)} + \underbrace{\left(\left(\Pi_x^2 - \Pi_{0x}^2\right) - \left(\Pi_y^2 - \Pi_{0y}^2\right) \right) d_0}_{(3)} \\ & \underbrace{-\eta^3 \left(|s_0|^2 s_0 + 2|s_0|^2 s_1 + s_0^2 s_1^* + s_0^* s_1^2 + 2|s_1|^2 s_0 \right)}_{(4)} . \end{aligned}$$

Since $\| \eta s \|_{\mathbf{H}^2} \leq \delta_2$ and $s_0 \in \mathbf{H}^2$, we may assume $\| \eta s_1 \|_{\mathbf{H}^2} \leq 2\delta_2$. (Choose small η_1 if necessary.) As in (2.10),

$$\begin{aligned} & K^2\eta^2 \left\| \Pi^2 s_1 \right\|^2_{\mathbf{L}^2} + \int K\eta |\nabla s_1|^2 + \|s_1\|^2_{\mathbf{L}^2} + \int \eta^3 |s_1|^4 \\ & \leq C_1 \left(\| (1) \|^2_{\mathbf{L}^2} + \| (2) \|^2_{\mathbf{L}^2} + \| (3) \|^2_{\mathbf{L}^2} + \| (4) \|^2_{\mathbf{L}^2} \right). \end{aligned}$$

$$\begin{aligned} & C_1 \| (4) \|^2_{\mathbf{L}^2} \leq 4C_1\eta^2 (\left\| |s_0|^2 s_0 \right\|^2_{\mathbf{L}^2} + \left\| 2|s_0|^2 s_1 \right\|^2_{\mathbf{L}^2} + \left\| |s_0|^2 {s_1}^* \right\|^2_{\mathbf{L}^2} \\ & + \left\| {s_0}^* {s_1}^2 + 2|s_1|^2 s_0 \right\|^2_{\mathbf{L}^2}) \\ \leq \quad & C\eta^2 + C_2\eta^2 \| s_1 \|^2_{\mathbf{L}^2} + C_3\eta^2 \int |s_1|^4. \end{aligned}$$

Therefore if we choose η_1 small enough such that $C_2\eta_1^2 \leq \frac{1}{8}$ and $C_3\eta_1 \leq \frac{1}{8}$, then

$$K\eta \left\| \Pi^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \leq C\eta + C \left(\| (1) \|_{\mathbf{L}^2} + \| (2) \|_{\mathbf{L}^2} + \| (3) \|_{\mathbf{L}^2} \right) \text{ and}$$

$$\begin{aligned} & \left\| \Pi_x^2 d_1 \right\|_{\mathbf{L}^2} = \left\| \partial_{xx} d_1 - 2iA^1 \partial_x d_1 - d_1 \partial_x A^1 - A^1 \cdot A^1 d_1 \right\|_{\mathbf{L}^2} \\ & \leq C \| d_1 \|_{\mathbf{H}^2} + 2 \| A \|_\infty \| \partial_x d_1 \|_{\mathbf{L}^2} + \left\| d_1 \left(\partial_x A_0^1 + \partial_x A_1^1 \right) \right\|_{\mathbf{L}^2} + \| A \|_\infty \| d_1 \|_{\mathbf{L}^2}. \end{aligned}$$

Using Proposition 2, we can estimate the term

$$\left\| d_1 \partial_x A_1^1 \right\|_{\mathbf{L}^2} \quad \leq \quad \| d_1 \|_{\mathbf{L}^4} \left\| \partial_x A_1^1 \right\|_{\mathbf{L}^4} \leq C \| d_1 \|_{\mathbf{H}^2} \| A_1 \|_{\mathbf{H}^2} \leq C \| w_1 \|_{\mathbf{H}^2}.$$

All other terms related to (1) can be easily bounded by $C \| w_1 \|_{\mathbf{H}^2}$.
So $\|(1)\|_{\mathbf{L}^2} \leq C\|w_1\|_{\mathbf{H}^2}$.

$$\begin{aligned} \| (2) \|_{\mathbf{L}^2} \quad & \leq \quad K\eta \left\| \Pi^2 s_0 \right\|_{\mathbf{L}^2} \\ & = \quad K\eta \| \Delta s_0 - 2iA \cdot \nabla s_0 - is_0 \nabla \cdot A - A \cdot A s_0 \|_{\mathbf{L}^2} \\ & \leq \quad K\eta \left(\| \Delta s_0 \|_{\mathbf{L}^2} + C \| \nabla s_0 \|_{\mathbf{L}^2} + \| s_0 \|_{\mathbf{L}^4} \| \nabla \cdot A_1 \|_{\mathbf{L}^4} \right) \\ & \leq \quad CK\eta. \end{aligned}$$

In (3), the term

$$\begin{aligned} \left\| \left(\Pi_x^2 - \Pi_{0x}^2 \right) d_0 \right\|_{\mathbf{L}^2} \quad & = \quad \left\| -iA_1^1 \left(\partial_x - iA_0^1 \right) d_0 - i \left(\partial_x - iA_0^1 - iA_1^1 \right) \left(A_1^1 d_0 \right) \right\|_{\mathbf{L}^2} \\ & \leq \quad C \| A_1 \|_{\mathbf{L}^2} + C \| DA_1 \|_{\mathbf{L}^2} + C \| A_1 \|_{\mathbf{L}^2} \\ & \leq \quad C \| w_1 \|_{\mathbf{H}^2}. \end{aligned}$$

So $\|(3)\|_{\mathbf{L}^2} \leq C\|w_1\|_{\mathbf{H}^2}$. □

Theorem 2.2. *Let w_1, $s = s_0 + s_1$ be the unique solution of (2.8) - (2.9) found in Section 2.1. Then there exists a $\eta_2 = \eta_2(K) > 0$ such that $\| w_1 \|_{\mathbf{H}^2} \leq C\eta$ for any $0 < \eta < \eta_2 \leq \eta_1$.*

Proof. Since $P\mathbf{F}_1(w_1) = -P(\mathbf{F}_2(w_1) + \mathbf{H}_\eta(w_1, s))$,

$$\begin{aligned}
\| w_1 \|_{\mathbf{H}^2} &\le \tau_1 \| P\mathbf{F}_1(w_1) \|_{\mathbf{L}^2} \\
&\le \tau_1 \left(\| \mathbf{F}_2(w_1) \|_{\mathbf{L}^2} + \| \mathbf{H}_\eta(w_1, s) \|_{\mathbf{L}^2} \right) \\
&\le C_1\delta_1 \| w_1 \|_{\mathbf{H}^2} + C\eta \left(\left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \right) + C\eta.
\end{aligned}$$

Taking δ_1 sufficiently small (e.g. $C_1\delta_1 \le \frac{1}{2}$) if necessary, we get

$$\| w_1 \|_{\mathbf{H}^2} \le C\eta \left(\left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \| s_1 \|_{\mathbf{L}^2} \right) + C\eta. \tag{2.17}$$

Like in Lemma 2.2,

$$K^2\eta^2 \left\| D^2 s_1 \right\|_{\mathbf{L}^2} + \|s_1\|^2_{\mathbf{L}^2} \le CK^2\eta^2 \left\| \Pi^2 s_1 \right\|^2_{\mathbf{L}^2} + C_2K^2\eta^2\|s_1\|^2_{\mathbf{L}^2} + \|s_1\|^2_{\mathbf{L}^2}.$$

If η_2 is small enough (e.g. $C_2K^2\eta_2^2 \le 1$),

$$\begin{aligned}
K^2\eta^2 \left\| D^2 s_1 \right\|^2_{\mathbf{L}^2} + \|s_1\|^2_{\mathbf{L}^2} &\le CK^2\eta^2 \left\| \Pi^2 s_1 \right\|^2_{\mathbf{L}^2} + 2\|s_1\|^2_{\mathbf{L}^2} \\
&\le C_1 \| w_1 \|^2_{\mathbf{H}^2} + CK^2\eta^2 \\
&\le C_1\eta^2 \left(\left\| D^2 s_1 \right\|^2_{\mathbf{L}^2} + \|s_1\|^2_{\mathbf{L}^2} \right) + CK^2\eta^2.
\end{aligned}$$

The second inequality follows from Lemma 2.8. Therefore replacing K by a larger one if necessary, we get $\left\| D^2 s_1 \right\|^2_{\mathbf{L}^2} \le C$ for small η_2. So $\| w_1 \|_{\mathbf{H}^2} \le C\eta$ by (2.17). □

2.3 Projected solutions are solutions

In this section, we show that if (w_1, s) is the solution found in Section 2.1, then (w_1, s) is the solution of (2.1) - (2.3).

Theorem 2.3. *There exists a $r > 0$ which is independent of η so that if $w_1 = (d_1, A_1)$, s satisfy*

$$\begin{cases} P\mathbf{F}_\eta(w_0 + w_1, s) = 0 \\ -K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s = 0 \end{cases}$$

and $\|w_1\|_{\mathbf{H}^2} \le r$, then (w_1, s) is the solution of (2.1) - (2.3).

To prove this theorem, we need following lemma.
Let

$$\begin{aligned}
&\mathbf{G}_\eta\,(d_0 + d_1, A_0 + A_1, s) \equiv \mathbf{G}_\eta\,(d, A, s) \qquad (2.18) \\
\equiv\ &\int_{\mathbb{R}^2} |curl A|^2 + |\Pi d|^2 + \frac{1}{2}\left(1 - |d|^2\right)^2 + \eta|s|^2 + \frac{\eta^4}{2}|s|^4 + K\eta^2\,|\Pi s|^2 \\
&\quad + \eta\left\{\Pi_x s \Pi_x^* d^* - \Pi_y s \Pi_y^* d^* + c.c\right\} dz.
\end{aligned}$$

We can get the following lemma using that the integral is invariant under translation.

Lemma 2.9. *For $d_1,\ A_1,\ s \in \mathbf{H}^2(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} \mathbf{F}_\eta(d_1, A_1, s) \cdot \begin{pmatrix} \frac{\partial d}{\partial x} - iA_0^1 d \\ \frac{\partial A}{\partial x} - \nabla A_0^1 \end{pmatrix}$$
$$+\Big(-K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s\Big) \cdot \Big(\frac{\partial s}{\partial x} - iA_0^1 s\Big) = 0.$$

Proof. Notation. $d(h) \equiv d(z+he_1),\ A(h) \equiv A(z+he_1),\ s(h) \equiv s(z+he_1)$, $d = d(z),\ A = A(z),\ s = s(z)$.
Since $\mathbf{G}_\eta\left(d(h), A(h), s(h)\right) = \mathbf{G}_\eta(d, A, s)$,

$$\lim_{h\to 0} \frac{\mathbf{G}_\eta\left(d(h), A(h), s(h)\right) - \mathbf{G}_\eta(d, A, s)}{h} = 0.$$

Using difference quotients, gauge invariance and the fact that the terms $curl A$, s, $\left(1 - |d|^2\right)$, Πs, $\Pi d \in \mathbf{H}^1$, we get the proof of the lemma (see [8],[16]). □

Remark. (1) Similar to Lemma 2.9,

$$\int_{\mathbb{R}^2} \mathbf{G}_{1\eta}(d, A, s) \cdot \begin{pmatrix} \frac{\partial d}{\partial y} - iA_0^2 d \\ \frac{\partial A}{\partial y} - \nabla A_0^2 \\ \frac{\partial s}{\partial y} - iA_0^2 s \end{pmatrix} = 0 \tag{2.19}$$

where

$$\mathbf{G}_{1\eta}(d_1, A_1, s) = \begin{pmatrix} \mathbf{F}_\eta(d, A, s) \\ -K\eta\Pi^2 s - (\Pi_x^2 - \Pi_y^2)d + s + \eta^3|s|^2 s \end{pmatrix}.$$

(2) Since $\mathbf{G}_\eta$ is gauge invariant, for any $\varphi \in \mathbf{H}^2(\mathbb{R}^2, \mathbb{R})$,

$$\mathbf{G}_\eta\left(de^{i\varphi}, A + \nabla\varphi, se^{i\varphi}\right) = \mathbf{G}_\eta(d, A, s).$$

Using

$$\frac{\partial \mathbf{G}_\eta}{\partial h}\left(de^{ih\varphi}, A + h\nabla\varphi, se^{ih\varphi}\right)\Big|_{h=0} = 0$$

we can get

$$\int_{\mathbb{R}^2} \mathbf{G}_{1\eta}(d, A, s) \cdot \begin{pmatrix} id\varphi \\ \nabla\varphi \\ is\varphi \end{pmatrix} = 0. \tag{2.20}$$

proof of Theorem 2.3. Suppose there is no such r. Then for each i, there exist $(\tilde{d}_i, \tilde{A}_i, \tilde{s}_i)$ satisfying the equations (2.8) - (2.9) and $\eta_i > 0$ such that $\left\| (\tilde{d}_i, \tilde{A}_i) \right\|_{\mathbf{H}^2} \leq \frac{1}{i}$ and $\mathbf{F}_{\eta_i}(\tilde{d}_i, \tilde{A}_i, \tilde{s}_i) \in Ker\mathbf{F}_1$
i.e. there exist $C_{1i},\ C_{2i} \in \mathbb{R}$ and $\varphi_i \in \mathbf{H}^1(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\mathbf{F}_{\eta_i}(\tilde{d}_i, \tilde{A}_i, \tilde{s}_i)$$
$$= C_{1i} \underbrace{\begin{pmatrix} \frac{\partial d_0}{\partial x} - iA_0^1 d_0 \\ \frac{\partial A_0}{\partial x} - \nabla A_0^1 \end{pmatrix}}_{\alpha_1} + C_{2i} \underbrace{\begin{pmatrix} \frac{\partial d_0}{\partial y} - iA_0^2 d_0 \\ \frac{\partial A_0}{\partial y} - \nabla A_0^2 \end{pmatrix}}_{\alpha_2} + \left(-\varphi_i v_0, \varphi_i u_0, (\varphi_i)_x, (\varphi_i)_y\right) \neq 0$$

where $\tilde{s}_i$ is the solution of (2.2) corresponding to $(\tilde{d}_i, \tilde{A}_i)$. After normalization we may assume

$$\| C_{1i}\alpha_1 + C_{2i}\alpha_2 + (-\varphi_i v_0, \varphi_i u_0, (\varphi_i)_x, (\varphi_i)_y) \|_2 = 1. \tag{2.21}$$

Since α_1 and α_2 are perpendicular to $(-\varphi_i v_0, \varphi_i u_0, (\varphi_i)_x, (\varphi_i)_y)$ and linearly independent, there exists a constant M independent of i such that $|C_{1i}|+|C_{2i}|+\| \varphi_i \|_{\mathbf{H}^1} \leq M$.
By Lemma 2.9, remark (2.19), (2.20) and density of $\mathbf{H}^2$ in $\mathbf{H}^1$,

$$\int_{\mathbb{R}^2} (C_{1i}\alpha_1 + C_{2i}\alpha_2 + (-\varphi_i v_0, \varphi_i u_0, (\varphi_i)_x, (\varphi_i)_y)) \cdot\Bigg(C_{1i}\begin{pmatrix} \frac{\partial d}{\partial x} - iA_0^1 d \\ \frac{\partial A}{\partial x} - \nabla A_0^1 \end{pmatrix} + C_{2i}\begin{pmatrix} \frac{\partial d}{\partial y} - iA_0^2 d \\ \frac{\partial A}{\partial y} - \nabla A_0^2 \end{pmatrix} + (-\varphi_i(v_0+v_{1i}), \varphi_i(u_0+u_{1i}), (\varphi_i)_x, (\varphi_i)_y) \Bigg) = 0.$$

Since A_{1i}, $d_{1i} \longrightarrow 0$ in $\mathbf{H}^2$,

$$\int_{\mathbb{R}^2} |C_{1i}\alpha_1 + C_{2i}\alpha_2 + (-\varphi_i v_0, \varphi_i u_0, (\varphi_i)_x, (\varphi_i)_y)|^2 \longrightarrow 0 \quad \text{as} \quad i \to \infty. \tag{2.22}$$

This contradicts (2.21). Therefore the claim follows. □

By Theorem 2.2, if we choose η_2 small enough such that $\|w_1\|_{\mathbf{H}^2} \leq r$, then we find that (w_1, s) satisfies (2.1) - (2.3).

3 Expansion of the solutions

In this section, we show that the solution $\tilde{w}$ found in Section 2 can be expanded $\tilde{w} = \eta\mathfrak{w}_1 + \mathfrak{w}_2$ and $\| \mathfrak{w}_2 \|_{\mathbf{H}^2} = \mathbf{O}(\eta^2)$. Note that we can prove $\tilde{w} = \mathbf{O}(\eta)$ as the same method in Section 2.2.
Let $\tilde{w}$, s be the unique solution of

$$\begin{cases} \mathbf{F}_\eta(\tilde{w}, s) \equiv \mathbf{F}_1(\tilde{w}) + \mathbf{F}_2(\tilde{w}) + \mathbf{H}_\eta(\tilde{w}, s) = 0 \\ -K\eta\Pi^2 s - \left(\Pi_x^2 - \Pi_y^2\right) d + s + \eta^3|s|^2 s = 0 \end{cases}$$

where $w_0 = \begin{pmatrix} d_0 \\ A_0 \end{pmatrix}$. If we formally expand $\tilde{w} = \eta\mathfrak{w}_1 + \mathfrak{w}_2$ and $s = s_0 + \eta s_1 + s_2$, where $s_0 = \left(\Pi_{x0}^2 - \Pi_{y0}^2\right) d_0$, then (after comparing the coefficients of the constant and η term) we are led to define $\mathfrak{w}_1 = \begin{pmatrix} \mathfrak{d}_1 \\ \mathfrak{A}_1 \end{pmatrix} \in \mathbf{K}^\perp \cap \mathbf{H}^2$ as the solution to

$$\mathbf{F}_1 \begin{pmatrix} \mathfrak{d}_1 \\ \mathfrak{A}_1 \end{pmatrix} = -\begin{pmatrix} -\left(\Pi_{0x}^2 - \Pi_{0y}^2\right) s_0 \\ \frac{1}{2}\begin{pmatrix} -is_0\Pi_{0x}^* d_0^* + id_0^*\Pi_{0x} s_0 + c.c. \\ is_0\Pi_{0y}^* d_0^* - id_0^*\Pi_{0y} s_0 + c.c. \end{pmatrix} \end{pmatrix}. \tag{3.1}$$

In order for this to be well defined, we need to verify that the right side is in $\mathbf{K}^{\perp}$.

Lemma 3.1.

$$-\begin{pmatrix} -\left(\Pi_{0x}^2 - \Pi_{0y}^2\right) s_0 \\ \frac{1}{2}\begin{pmatrix} -is_0\Pi_{0x}^* d_0^* + id_0^*\Pi_{0x}s_0 + c.c. \\ is_0\Pi_{0y}^* d_0^* - id_0^*\Pi_{0y}s_0 + c.c. \end{pmatrix} \end{pmatrix} \in \mathbf{K}^{\perp}.$$

Proof. Only in this proof, we use the notation $A_0 = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$.
Let

$$\alpha = \begin{pmatrix} \xi_1 \\ B_1 \end{pmatrix} \equiv - \begin{pmatrix} -(\Pi_{0x}^2 - \Pi_{0y}^2)s_0 \\ \frac{1}{2}\begin{pmatrix} -is_0\Pi_{0x}^* d_0^* + id_0^*\Pi_{0x}s_0 + c.c \\ is_0\Pi_{0y}^* d_0^* - id_0^*\Pi_{0y}s_0 + c.c \end{pmatrix} \end{pmatrix}.$$

It is enough to show that α satisfy

$$\begin{aligned} &(a) \qquad Im(d_0^*\xi_1) = \nabla \cdot B_1 \\ &(b) \qquad \left(\alpha, \begin{pmatrix} \Pi_{0x}d_0 \\ \begin{pmatrix} 0 \\ \partial_x A^2 - \partial_y A^1 \end{pmatrix} \end{pmatrix}\right) = 0 \\ &(c) \qquad \left(\alpha, \begin{pmatrix} \Pi_{0y}d_0 \\ -\begin{pmatrix} \partial_x A^2 - \partial_y A^1 \\ 0 \end{pmatrix} \end{pmatrix}\right) = 0. \end{aligned}$$

For (a), we will show $\left(\alpha, \begin{matrix} i\gamma d_0 \\ \nabla\gamma \end{matrix}\right) = 0$ for any $\gamma \in C_0^{\infty}(\mathbb{R}^2, \mathbb{R})$.
Then by integration by parts, (a) follows.

For $\gamma \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$,

$$
\begin{aligned}
&-\int (\Pi_{0x}^2 - \Pi_{0y}^2) s_0 (-i\gamma d_0^*) + c.c \\
&= i\int s_0 (\Pi_{0x}^{*2} - \Pi_{0y}^{*2})(\gamma d_0^*) + c.c \\
&= i\int s_0 \Big(\partial_{xx} - \partial_{yy} + 2iA^1\partial_x - 2iA^2\partial_y + 2i\partial_x A^1 - \big((A^1)^2 - (A^2)^2\big)\Big)(\gamma d_0^*) + c.c \\
&= i\int s_0 (\partial_{xx}\gamma d_0^* + 2\partial_x\gamma\partial_x d_0^* - 2\partial_y\gamma\partial_y d_0^* + 2iA^1\partial_x\gamma d_0^* - 2iA^2\partial_y\gamma d_0^*) + c.c \\
&\quad + \underbrace{\left(i\int s_0\gamma(\Pi_{0x}^2 - \Pi_{0y}^2) d_0^*) + c.c\right)}_{\equiv 0 \quad \text{since} \quad (\Pi_{0x}^{*2} - \Pi_{0y}^{*2}) d_0^* = s_0^*} \\
&= i\int (-\partial_x\gamma\partial_x s_0 d_0^* - \partial_x\gamma s_0 \partial_x d_0^* + 2s_0\partial_x\gamma\partial_x d_0^* + 2iA^1 d_0^*\partial_x\gamma s_0) + c.c. \\
&\quad + i\int (\partial_y\gamma)(\partial_y s_0 d_0^* + s_0\partial_y d_0^* - 2s_0\partial_y d_0^* - 2iA^2 d_0^* s_0) + c.c. \\
&= i\int (\partial_x\gamma)(-\partial_x s_0 d_0^* + s_0\partial_x d_0^* + iA^1 d_0^* s_0 + iA^1 d_0^* s_0) + c.c. \\
&\quad i\int (\partial_y\gamma)(\partial_y s_0 d_0^* - s_0\partial_x d_0^* - iA^2 d_0^* s_0 - iA^2 d_0^* s_0) + c.c \\
&= -i\int (-s_0\Pi_{0x}^* d_0^* + d_0^*\Pi_{0x} s_0)\partial_x\gamma - i\int (s_0\Pi_{0y}^* d_0^* - d_0^*\Pi_{0y} s_0)\partial_y\gamma + c.c. \ .
\end{aligned}
$$

So (a) follows.
(b) Using the integration by parts, we can easily get

$$
\int \Pi_{0y} u \Pi_{0x}^* d_0^* - \int \Pi_{0x} u \Pi_{0y} d_0^* = i\int u d_0^* (\partial_x A^2 - \partial_y A^1) \tag{3.2}
$$

for any $u \in \mathbf{H}^1(\mathbb{R}^2, \mathbb{C})$. Also, for any $v \in \mathbf{L}^2(\mathbb{R}^2, \mathbb{C})$,

$$
\int \Pi_{0x}\Pi_{0y} s_0 v - \int \Pi_{0y}\Pi_{0x} s_0 v = -i\int s_0 (\partial_x A^2 - \partial_y A^1) v. \tag{3.3}
$$

Therefore,

$$
\begin{aligned}
& -\int \Pi_{0x}^2 s_0 \Pi_{0x}^* d_0^* + \int \Pi_{0y}^2 s_0 \Pi_{0x}^* d_0^* \\
& = \int (\Pi_{0x} s_0 \Pi_{0x}^{*2} d_0^*) + \int \Pi_{0x}(\Pi_{0y} s_0)\Pi_{0y}^* d_0^* + i\int \Pi_{0y} s_0 d_0^* (\partial_x A^2 - \partial_y A^1) \\
& = \int (\Pi_{0x} s_0 \Pi_{0x}^{*2} d_0^*) + \int \Pi_{0y}\Pi_{0x} s_0 \Pi_{0y}^* d_0^* \\
& \quad + i\int \Pi_{0y} s_0 d_0^* (\partial_x A^2 - \partial_y A^1) - i\int s_0 \Pi_{0y}^* d_0^* (\partial_x A^2 - \partial_y A^1) \\
& = \int \Pi_{0x} s_0 (\Pi_{0x}^{*2} - \Pi_{0y}^{*2}) d_0 + i\int \Pi_{0y} s_0 d_0^* (\partial_x A^2 - \partial_y A^1) \\
& \quad - i\int s_0 (\partial_x A^2 - \partial_y A^1) \Pi_{0y}^* d_0^*.
\end{aligned}
$$

Since

$$
\int \Pi_{0x} s_0 s_0^* + \int \Pi_{0x}^* s_0^* s_0 = -\int s_0 \Pi_{0x}^* s_0^* + \int \Pi_x^* s_0^* s_0 = 0,
$$

we get (b). In a similar way to (b), we can get (c). Therefore, the claim follows. □

By Lemma 3.1, $\mathfrak{w}_1 \in \mathbf{K}^{\perp} \cap \mathbf{H}^2$ in (3.1) is well-defined.
Define

$$
\begin{aligned}
s_1 \equiv\ & K\left(\nabla - iA_0\right)^2 s_0 + \left(\Pi_{0x}^2 - \Pi_{0y}^2\right)\mathfrak{d}_1 + (\nabla - iA_0)_x(-i\mathfrak{A}_1)_x d_0 \\
+\ & (-i\mathfrak{A}_1)_x \cdot (\nabla - iA_0)_x d_0 - (\nabla - iA_0)_y(-i\mathfrak{A}_1)_y d_0 + (-i\mathfrak{A}_1)_y(\nabla - iA_0)_y d_0.
\end{aligned}
$$

By the elliptic regularity theory, $\|\mathfrak{w}_1\|_{\mathbf{H}^4} \le C$ and thus $\|s_1\|_{\mathbf{H}^2} \le CK$.
We now prove the expansion rigorously by showing that the remainder $\|\mathfrak{w}_2\|_{\mathbf{H}^2} = \mathbf{O}(\eta^2)$ using a similar argument as in Section 2.2.

$$
\begin{aligned}
& -K\eta\Pi^2 s_2 + s_2 + \eta^3 |s_2| s_2 \\
& = \left(\Pi_x^2 - \Pi_y^2\right)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) + K\eta\Pi^2(\tilde{s}_1) - \tilde{s}_1 \\
& -\eta^3(|\tilde{s}_1|^2\tilde{s}_1 + 2|\tilde{s}_1|^2 s_2^* + \tilde{s}_1^2 s_2^* + \tilde{s}_1^* s_2^2 + 2|s_2|^2\tilde{s}_1)
\end{aligned}
\tag{3.4}
$$

where $\tilde{s}_1 = s_0 + \eta s_1$. Expanding the right side we get

$$
\begin{aligned}
& \left(\Pi_x^2 - \Pi_y^2\right)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) + K\eta\Pi^2(\tilde{s}_1) - \tilde{s}_1 \\
& -\eta^3(|\tilde{s}_1|^2\tilde{s}_1 + 2|\tilde{s}_1|^2 s_2^* + \tilde{s}_1^2 s_2^* + \tilde{s}_1^* s_2^2 + 2|s_2|^2\tilde{s}_1)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-i)\mathfrak{A}_2^1(\partial_x - iA_0^1 - i\eta\mathfrak{A}_1^1 - i\mathfrak{A}_2^1)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) \\
& -i\eta\mathfrak{A}_1^1(-i\eta\mathfrak{A}_1^1 - i\mathfrak{A}_2^1)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) + (-i\eta\mathfrak{A}_1^1)(\partial_x - iA_0^1)(\eta\mathfrak{d}_1 + \mathfrak{d}_2) \\
& +(\partial_x - iA_0^1)(\partial_x - iA_0^1)\mathfrak{d}_2 - (\partial_y - iA_0^2)(\partial_y - iA_0^2)\mathfrak{d}_2 \\
& +iA_2^2(\partial_y - iA_0^2 - i\eta A_1^2 - iA_2^2)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) \\
& +i\eta A_1^2(-i\eta A_1^2 - iA_2^2)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2) + (+i\eta A_1^2)(\partial_y - iA_0^2)(\eta\mathfrak{d}_1 + \mathfrak{d}_2) \\
& +K\eta(-i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)\cdot(\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_0 \\
& +K\eta(\nabla - iA_0)(-i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_0 \\
& +K\eta^2(\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)\cdot(\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_1 \\
& -\eta^3\left(\eta|s_0|^2 s_1 + \eta s_0^2 s_1^* + 2\eta^2|s_1|^2 s_0 + \eta^2 s_1^2 s_0^* + \eta^3|s_1|^2 s_1\right) \\
& -\eta^3(2|\tilde{s}_1|^2 s_2^* + \tilde{s}_1^2 s_2^* + \tilde{s}_1^* s_2^2 + 2|s_2|^2\tilde{s}_1).
\end{aligned}
$$

We may assume $\|\tilde{s}_1\|_{\mathbf{H}^2} \le C$. Using this, we get

$$
\left\| -\eta^3\left(\eta|s_0|^2 s_1 + \eta s_0^2 s_1^* + 2\eta^2|s_1|^2 s_0 + \eta^2 s_1^2 s_0^* + \eta^3|s_1|^2 s_1\right)\right\|_{\mathbf{L}^2} \le C\eta^3.
$$

Also

$$
\begin{aligned}
& 2\left\| -\eta^3(+2|\tilde{s}_1|^2 s_2^* + \tilde{s}_1^2 s_2^* + \tilde{s}_1^* s_2^2 + 2|s_2|^2\tilde{s}_1)\right\|_{\mathbf{L}^2}^2 \\
& \le C_1\eta^6 \,\| s_2 \|_{\mathbf{L}^2}^2 + C_2\eta^6 \int |s_2|^4.
\end{aligned}
$$

Therefore if we choose η_4 small enough then we get the following lemma.

Lemma 3.2. *There exists a $\eta_4 = \eta_4(K, \delta_1) > 0$ such that*

$$
\begin{aligned}
& K^2\eta^2 \left\| \Pi^2 s_2 \right\|_{\mathbf{L}^2}^2 + \int K\eta|\nabla s_2|^2 + \|s_2\|_{\mathbf{L}^2}^2 + \int \eta^3|s_2|^4 \\
& \le C\|\mathfrak{w}_2\|_{\mathbf{H}^2}^2 + CK^4\eta^4 \qquad \text{for } 0 < \eta \le \eta_4\,.
\end{aligned}
$$

Proof. Since $\|\tilde{w}\|_{\mathbf{H}^2} \le \delta_1$ for small η, we may assume $\|\mathfrak{w}_2\|_{\mathbf{H}^2} \le 2\delta_1$.
We will estimate each term separately.

$$
\begin{aligned}
& \left\| -i\mathfrak{A}_2^1(\nabla_x - iA_0^1 - i\eta\mathfrak{A}_1^1 - i\mathfrak{A}_2^1)(d_0\eta\mathfrak{d}_1 + \mathfrak{d}_2)\right\|_{\mathbf{L}^2} \\
& \le \left\| (\nabla_x - iA_0^1 - i\eta\mathfrak{A}_1^1 - i\mathfrak{A}_2^1)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2)\right\|_\infty \|\mathfrak{A}_2^1\|_{\mathbf{L}^2} \le C\|\mathfrak{w}_2\|_{\mathbf{H}^2}.
\end{aligned}
$$

$$
\begin{aligned}
& \left\| -i\eta\mathfrak{A}_1^1(-i\eta\mathfrak{A}_1^1 - i\mathfrak{A}_2^1)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2)\right\|_{\mathbf{L}^2} \le \left\| \eta\mathfrak{A}_1^1(\eta\mathfrak{A}_1^1 + \mathfrak{A}_2^1)(d_0 + \eta\mathfrak{d}_1 + \mathfrak{d}_2)\right\|_{\mathbf{L}^2} \\
& \le C\eta^2\|\mathfrak{A}_1^1\|_{\mathbf{L}^2} + C\|\mathfrak{A}_2^1\|_{\mathbf{L}^2} \le C\eta^2 + C\|\mathfrak{w}_2\|_{\mathbf{H}^2}.
\end{aligned}
$$

$$
\left\| (-i\eta\mathfrak{A}_1^1)\left((\eta\nabla_x\mathfrak{d}_1 + \nabla_x\mathfrak{d}_2) + (-ia_0^1)(\eta\mathfrak{d}_1 + \mathfrak{d}_2)\right)\right\|_{\mathbf{L}^2} \le C\eta^2\|\mathfrak{w}_1\|_{\mathbf{H}^2} + C\|\mathfrak{w}_2\|_{\mathbf{H}^2}.
$$

Using $\left\| \mathfrak{A}_1^1\nabla_x\mathfrak{d}_1\right\|_{\mathbf{L}^2} \le C\|\mathfrak{A}_1^1\|_{\mathbf{H}^2}\|\mathfrak{d}_1\|_{\mathbf{H}^2}$,

$$
\begin{aligned}
& \| K\eta(-i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)\cdot(\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_0 \|_{\mathbf{L}^2} \\
& \le \| (\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_0 \|_\infty \, \| K\eta(-i\eta\mathfrak{A}_1 - i\mathfrak{A}_2) \|_{\mathbf{L}^2} \\
& \le CK\eta^2\|\mathfrak{A}_1\|_{\mathbf{H}^2} + CK\eta\|\mathfrak{A}_2\|_{\mathbf{H}^2} \le CK\eta^2\|\mathfrak{w}_1\|_{\mathbf{H}^2} + C\|\mathfrak{w}_2\|_{\mathbf{H}^2}
\end{aligned}
$$

if we choose η_4 small enough. Next

$$\begin{aligned}&\left\| K\eta^2(\nabla - iA_0 - \eta\mathfrak{A}_1 - i\mathfrak{A}_2)\cdot(\nabla - iA_0 - i\eta\mathfrak{A}_1 - i\mathfrak{A}_2)s_1 \right\|_{\mathbf{L}2}\\ &\le CK\eta^2\|s_1\|_{\mathbf{H}^2} \ \le CK^2\eta^2.\end{aligned}$$

Other terms can be estimated similarly. So the claim follows. □

Lemma 3.3. *1.* $\| \mathbf{F}_2(\eta\mathfrak{w}_1 + \mathfrak{w}_2) \|_{\mathbf{L}^2} \le C\eta^2 + C\delta_1\|\mathfrak{w}_2\|_{\mathbf{H}^2}$
2. $\left\| \eta\mathbf{F}_1\begin{pmatrix}\mathfrak{d}_1\\ \mathfrak{A}_1\end{pmatrix} + \mathbf{H}_\eta(\eta\mathfrak{w}_1 + \mathfrak{w}_2)\right\|_{\mathbf{L}^2} \le CK^2\eta^2 + C\eta\|\mathfrak{w}_2\|_{\mathbf{H}^2} + C\eta\left(\left\| D^2 s_2\right\|_{\mathbf{L}^2} + \| s_2 \|_{\mathbf{L}^2}\right)$
for small η_4.

The proof of Lemma 3.3 is similar to the proof of Lemma 2.7.

Theorem 3.1. *Let* $\tilde{w} = \eta\mathfrak{w}_1 + \mathfrak{w}_2$, $s = s_0 + \eta s_1 + s_2$ *be the unique solution of (2.1) - (2.3) found in Section 2.2. Then there exists a* $\eta_4 = \eta_4(K, \delta_1) > 0$ *so that* $\|\mathfrak{w}_2\|_{\mathbf{H}^2} + \|s_2\|_{\mathbf{L}^2} \le CK^2\eta^2$ *for any* $0 < \eta < \eta_4 \le \eta_1$.

Proof. The main idea is similar to that for Theorem 2.2. Since $\mathbf{F}_1(\mathfrak{w}_2) = -\left(\eta\mathbf{F}_1(\mathfrak{w}_1) + \mathbf{H}_\eta(\tilde{w}, s) + \mathbf{F}_2(\eta\mathfrak{w}_1 + \mathfrak{w}_2)\right)$,

$$\begin{aligned}\|\mathfrak{w}_2\|_{\mathbf{H}^2} &\le C\left(\| \mathbf{F}_2(\tilde{w}) \|_{\mathbf{L}^2} + \| \eta\mathbf{F}_1(\mathfrak{w}_1) + \mathbf{H}_\eta(\tilde{w}, s) \|_{\mathbf{L}^2}\right)\\ &\le C\delta_1\|\mathfrak{w}_2\|_{\mathbf{H}^2} + C\eta^2K^2 + C\eta\|\mathfrak{w}_2\|_{\mathbf{H}^2} + C\eta\left(\left\| D^2 s_2\right\|_{\mathbf{L}^2} + \| s_2 \|_{\mathbf{L}^2}\right).\end{aligned}$$

So if we substitute small δ_1 and choose η_4 small,

$$\|\mathfrak{w}_2\|_{\mathbf{H}^2} \le CK^2\eta^2 + C\eta\left(\|D^2 s_2\|_{\mathbf{L}^2} + \|s_2\|_{\mathbf{L}^2}\right). \tag{3.5}$$

Moreover for large K,

$$K^2\eta^2\left\| D^2 s_2\right\|^2_{\mathbf{L}^2} + \| s_2 \|^2_{\mathbf{L}^2} \le C_1 \| \mathfrak{w}_2 \|^2_{\mathbf{H}^2} + C_2K^4\eta^4 \quad \text{by Lemma 3.2}\ .$$

Therefore we can get $\eta^2\left\| D^2 s_2\right\|^2_{\mathbf{L}^2} + \| s_2 \|^2_{\mathbf{L}^2} \le CK^4\eta^4$ and then $\| \mathfrak{w}_2 \|_{\mathbf{H}^2} \le CK^2\eta^2$ by (3.5). □

4 Fourfold symmetry of the solutions

Lemma 4.1. *Let* $d = d_0 + d_1$, $A = A_0 + A_1 = \begin{pmatrix}A_0^1\\ A_0^2\end{pmatrix} + \begin{pmatrix}A_1^1\\ A_1^2\end{pmatrix}$, s *be the unique solution of* (1.8) - (1.10) *found in Theorem 1.1.*
Let

$$\begin{aligned}&\tilde{d}(x,y) = id(y,-x),\\ &\tilde{A}(x,y) = \begin{pmatrix}-A^2(y,-x)\\ A^1(y,-x)\end{pmatrix} = \begin{pmatrix}-A_0^2(y,-x)\\ A_0^1(y,-x)\end{pmatrix} + \begin{pmatrix}-A_1^2(y,-x)\\ A_1^1(y,-x)\end{pmatrix},\\ &\tilde{s}(x,y) = -is(y,-x).\end{aligned}$$

Then $\tilde{d}$, $\tilde{s}$, $\tilde{A}$ *satisfies the equations* (1.8) - (1.10).

Proof. Using $X = y$ and $Y = -x$,

$$\begin{aligned}\left(\partial_x - i\tilde{A}^1\right)\tilde{d} &= (-1)\partial_Y id(X,Y) - i(-1)A^2(X,Y)id(X,Y)\\ &= -i\left(\partial_Y d(X,Y) - iA^2(X,Y)d(X,Y)\right)\\ &= -i\Pi_Y d\Big|_{(X,Y)}\end{aligned}$$

$$\begin{aligned}(\partial_y - i\tilde{A}^2)\tilde{d} &= i\partial_X d(X,Y) - iA^1(X,Y)id(X,Y)\\ &= i(\partial_X d - iA^1 d)\\ &= i\Pi_X d\Big|_{(X,Y)}\end{aligned}$$

$$\begin{aligned}&(\partial_x - i\tilde{A}^1)(\partial_x - i\tilde{A}^1)\tilde{d}\\ &= -i\left((-1)\partial_{YY}d(X,Y) - i(-1)\partial_Y\left(A^2(X,Y)d(X,Y)\right)\right)\\ &\quad -i\left(-A^2(X,Y)\right)(-1)\left(\partial_Y d(X,Y) - iA^2(X,Y)d(X,Y)\right)\\ &= i\left(\partial_{YY}d - i\partial_Y(A^2 d) - iA^2\partial_Y d - A^2\cdot A^2 d(X,Y)\right)\\ &= i\Pi_Y^2 d\Big|_{(X,Y)}.\end{aligned}$$

Similarly, $(\partial_y - i\tilde{A}^2)\tilde{d} = i\Pi_X d\Big|_{(X,Y)}$, $(\partial_y - i\tilde{A}^2)\cdot(\partial_y - i\tilde{A}^2)\tilde{d} = i\Pi_X^2 d\Big|_{(X,Y)}$ and

$$\begin{aligned}curlcurl\tilde{A} &= \begin{pmatrix}\frac{\partial^2}{\partial x\partial y}\tilde{A}^2 - \frac{\partial^2}{\partial y^2}\tilde{A}^1\\ \frac{\partial^2}{\partial x\partial y}\tilde{A}^1 - \frac{\partial^2}{\partial x^2}\tilde{A}^2\end{pmatrix}\\ &= \begin{pmatrix}-\frac{\partial^2}{\partial X\partial Y}A^1(X,Y) - (-1)\frac{\partial^2}{\partial X^2}A^2(X,Y)\\ -\frac{\partial^2}{\partial X\partial Y}(-1)A^2(X,Y) - \frac{\partial^2}{\partial Y^2}A^1(X,Y)\end{pmatrix}\\ &= \begin{pmatrix}-\frac{\partial^2}{\partial X\partial Y}A^1(X,Y) + \frac{\partial^2}{\partial X^2}A^2(X,Y)\\ \frac{\partial^2}{\partial X\partial Y}A^2(X,Y) - \frac{\partial^2}{\partial Y^2}A^1(X,Y)\end{pmatrix}.\end{aligned}$$

Therefore

$$\begin{aligned}&-\left((\partial_x - i\tilde{A}^1)\cdot(\partial_x - i\tilde{A}^1)\tilde{d} + (\partial_y - i\tilde{A}^2)\cdot(\partial_y - i\tilde{A}^2)\tilde{d}\right)\\ &\quad -\eta\mu\left(\tilde{\Pi}_x^2 - \tilde{\Pi}_y^2\right)\tilde{s} - \kappa^2(1-|\tilde{d}|^2)\tilde{d}\\ &= -\left(i\Pi_Y^2 d(X,Y) + i\Pi_X^2 d(X,Y)\right) - \eta\mu\left((-i)\Pi_Y^2 - (-i)\Pi_X^2\right)s(X,Y)\\ &\quad -\kappa^2\left(1 - |d(X,Y)|^2\right)id(X,Y)\\ &= i\left(-\Pi^2 d - \eta\mu(\Pi_X^2 - \Pi_Y^2)s - \kappa^2(1-|d|^2)d\right)\Big|_{(X,Y)}\\ &= 0 \quad \text{and}\end{aligned}$$

$$
\begin{aligned}
&-K\eta\tilde{\Pi}^2\tilde{s}-\mu\left(\tilde{\Pi}_x^2-\tilde{\Pi}_y^2\right)\tilde{d}+\tilde{s}+\eta|\tilde{s}|^2\tilde{s}\\
&=-K\eta\Pi^2(-i)s(X,Y)-\mu\left(\Pi_Y^2-\Pi_X^2\right)id(X,Y)-is(X,Y)\\
&\quad+\eta^3\left|s(X,Y)\right|^2(-i)s(X,Y)\\
&=-i\left(-K\eta\Pi^2 s-\mu(\Pi_X^2-\Pi_Y^2)d+s+\eta^3|s|^2 s\right)\Big|_{(X,Y)}=0.
\end{aligned}
$$

Consider

$$
\begin{aligned}
&curlcurl\tilde{A}+\frac{1}{2}i\left(\tilde{d}^*\tilde{\Pi}\tilde{d}-\tilde{d}\tilde{\Pi}^*\tilde{d}^*\right)\\
&+\frac{1}{2}\left(\eta^2 Ki(\tilde{s}^*\tilde{\Pi}\tilde{s}-\tilde{s}\tilde{\Pi}^*\tilde{s}^*)\right)+\frac{\eta\mu}{2}\begin{pmatrix}-i\tilde{s}\tilde{\Pi}_x^*\tilde{d}^*+i\tilde{d}^*\tilde{\Pi}_x s+c.c.\\ i\tilde{s}\tilde{\Pi}_y^*\tilde{d}^*-i\tilde{d}^*\tilde{\Pi}_y\tilde{s}+c.c.\end{pmatrix}.
\end{aligned}
\tag{4.1}
$$

The first component of (4.1) is

$$
\begin{aligned}
&\left(-\frac{\partial^2}{\partial X\partial Y}A^1(X,Y)+\frac{\partial^2}{\partial X^2}A^2(X,Y)\right)+\frac{1}{2}i\left((-i)d^*(-i)\Pi_Y d-idi\Pi_Y^* d^*\right)\\
&+\frac{1}{2}\eta^2 Ki\left(is^* i\Pi_Y s-(-i)s(-i)\Pi_Y^* s^*\right)+\frac{\eta\mu}{2}\left(-i(-i)si\Pi_Y^* d^*+i(-i)di\Pi_Y s+c.c.\right)\\
&=-\left(\frac{\partial^2}{\partial X\partial Y}A^1-\frac{\partial^2}{\partial X^2}A^2\right)-\frac{1}{2}i\left(d^*\Pi_Y d-d\Pi_Y^* d^*\right)\\
&\quad-\frac{1}{2}\eta^2 Ki\left(s^*\Pi_Y s-s\Pi_Y^* s^*\right)-\frac{\eta\mu}{2}\left(is\Pi_Y^* d^*-id\Pi_Y s+c.c.\right)=0.
\end{aligned}
$$

The second component of (4.1) is

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial X\partial Y}A^2(X,Y)-\frac{\partial^2}{\partial Y^2}A^1(X,Y)\right)+\frac{1}{2}i\left(d^*\Pi_X d-d\Pi_X^* d^*\right)+\frac{1}{2}\eta^2 Ki\big(s^*\Pi_X s\\
&-s\Pi_X^* s^*\big)+\frac{\eta\mu}{2}\left(i(-i)s\Pi_X^*(-i)d^*-i(-i)d^*\Pi_X(-i)s+c.c.\right)=0.
\end{aligned}
$$

Therefore, $\tilde{d}$, $\tilde{s}$, $\tilde{A}$ satisfy the equations. ☐

Lemma 4.2.

$$
\begin{aligned}
&(1)\; id_0(y,-x)=d_0(x,y)\quad \textit{and}\\
&(2)\; \begin{pmatrix}-A_0^2(y,-x)\\ A_0^1(y,-x)\end{pmatrix}=\begin{pmatrix}A_0^1(x,y)\\ A_0^2(x,y)\end{pmatrix}\qquad \textit{for any}\quad (x,y)\in\mathbf{R}^2
\end{aligned}
$$

Proof. (1) $id_0(y,-x)=if_1(r)e^{i\left(\theta-\frac{\pi}{2}\right)}=f_1(r)e^{i\theta}=d_0(x,y)$.
(2) $\begin{pmatrix}-A_0^2(y,-x)\\ A_0^1(y,-x)\end{pmatrix}=a_0(r)(y,-x)=A_0(x,y)$.
Note. $A_0=a_0(r)(y,-x)=\frac{a_1(r)}{r}\hat{x}^{\perp}$ and $d_0(x,y)=f_1(r)e^{i\theta}$.
Therefore, the claim follows. ☐

We may write $\tilde{d}=d_0+\tilde{d}_1$ and $\tilde{A}=A_0+\tilde{A}_1$ where

$$
\tilde{d}_1=id_1(y,-x)\,\text{and}\,\tilde{A}_1(x,y)=\begin{pmatrix}-A_1^2(y,-x)\\ A_1^1(y,-x)\end{pmatrix}.
$$

Lemma 4.3. $\tilde{d}_1(x,y) = id_1(y,-x)$, $\tilde{A}_1(x,y) = \begin{pmatrix} -A_1^2(y,-x) \\ A_1^1(y,-x) \end{pmatrix} \in \mathbf{K}^\perp$
i.e.

$$(i) Im(d_0^* \tilde{d}_1) = \nabla \cdot \tilde{A}_1$$
$$(ii)(\tilde{d}_1, \tilde{A}_1) \cdot (\partial_j d_0, \partial_j A_0) = 0 \, for \, j = 1, 2.$$

Proof. (i) Note. $Im(d_0^* d_1) = \nabla \cdot A_1$ i.e. $\frac{d_0^* d - d_0 d^*}{2i} = \partial_x A_1^1 + \partial_y A_1^2$
Let $-x = Y$, and $y = X$, then

$$\frac{d_0^*(-Y,X) i d_1(X,Y) - d_0(-Y,X)(-i) d_1^*(X,Y)}{2i} = \frac{d_0^*(X,Y) d_1(X,Y) - d_0(X,Y) d_1^*(X,Y)}{2i}$$

since

$$(-i) d_0(-Y,X) = -i f_1(r) e^{i\left(\theta + \frac{\pi}{2}\right)} = f_1(r) e^{i\theta} = d_0(X,Y)$$
$$\nabla \cdot \tilde{A}_1 = \partial_x \left(-A_1^2(y,-x)\right) + \partial_y \left(A_1^1(y,-x)\right)$$
$$= -\partial_Y \left(-A_1^2(X,Y)\right) + \partial_X \left(A_1^1(X,Y)\right) = (\nabla \cdot A_1)(X,Y).$$

Therefore, (i) follows.
(ii) We want to show

$$(a) \qquad \int \frac{\partial_x d_0(X,Y) \tilde{d}^* + \partial_x d_0^* \tilde{d}}{2} + \begin{pmatrix} \partial_x A_0^1 \\ \partial_x A_0^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{A}_1^1 \\ \tilde{A}_1^2 \end{pmatrix} = 0$$
$$(b) \qquad \int \frac{\partial_y d_0 \tilde{d}_1^* + \partial_y d_0^* \tilde{d}_1}{2} + \begin{pmatrix} \partial_y A_0^1 \\ \partial_y A_0^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{A}_1^1 \\ \tilde{A}_1^2 \end{pmatrix} = 0$$

when

$$\int \frac{\partial_x d_0 d_1^* + \partial_x d_0^* d_1}{2} + \begin{pmatrix} \partial_x A_0^1 \\ \partial_x A_0^2 \end{pmatrix} \cdot \begin{pmatrix} A_1^1 \\ A_1^2 \end{pmatrix} = 0 \quad \text{and}$$
$$\int \frac{\partial_y d_0 d_1^* + \partial_y d_0^* d_1}{2} + \begin{pmatrix} \partial_y A_0^1 \\ \partial_y A_0^2 \end{pmatrix} \cdot \begin{pmatrix} A_1^1 \\ A_1^2 \end{pmatrix} = 0.$$

Let $y = X$, $-x = Y$,then $\partial_x = -\partial_Y$,$\partial_y = \partial_X$ and

$$\begin{aligned}
(a) \quad = \quad & \int \frac{-\partial_Y d_0 \begin{pmatrix} -Y \\ X \end{pmatrix} (-i) d_1^*(X,Y) + (-1) \partial_Y d_0^*(-Y,X) i d_1(X,Y)}{2} \\
& - \begin{pmatrix} \partial_Y A_0^1(-Y,X) \\ \partial_Y A_0^2(-Y,X) \end{pmatrix} \cdot \begin{pmatrix} -A_1^2(X,Y) \\ A_1^1(X,Y) \end{pmatrix} \\
= \quad & \int \frac{-\partial_Y d_0(X,Y) d_1^*(X,Y) - \partial_Y d_0^*(X,Y) d_1(X,Y)}{2} \\
& - \begin{pmatrix} \partial_Y (-1) A_0^2(X,Y) \\ \partial_Y A_0^1(X,Y) \end{pmatrix} \cdot \begin{pmatrix} -A_1^2(X,Y) \\ A_1^1(X,Y) \end{pmatrix} = 0
\end{aligned}$$

since $(-i) d_0(-Y,X) = d_0(X,Y)$, $A_0^1(-Y,X) = -A_0^2(X,Y)$ and $A_0^2(-Y,X) = -A_0^1(X,Y)$.
We can prove (b) similarly. Therefore the claim follows. □

Proof of Theorem 1.2. Since the solution is uniquely determined in $\mathbf{B}_{\mathbf{H}^2}(0,\delta_1)\cap \mathbf{K}^{\perp}$, it follows that $(\tilde{d},\tilde{A},\tilde{s}) = (d,A,s)$ for η sufficiently small. Thus the three identities hold. □

Lemma 4.4. *Let (d_0, A_0) be a vortex solution. If $\kappa \neq \frac{1}{\sqrt{2}}$ then (1.7) holds. If $\kappa = \frac{1}{\sqrt{2}}$ then $|\left(\Pi_{0x}^2 - \Pi_{0y}^2\right) d_0|$ is radial.*

Proof. From [10] we have that

$$\Sigma = \left(\Pi_{0x}^2 - \Pi_{0y}^2\right) d_0(r,\theta) = \left(\cos(2\theta)h(r) + i\sin(2\theta)g(r)\right)e^{i\theta}$$

where

$$\left.\begin{aligned} h(r) &= \tfrac{2f_1'}{r} - \tfrac{2(1-a_1)^2 f_1}{r^2} - \kappa^2(f_1^2-1)f_1 \\ g(r) &= 2(1-a_1)\left(\tfrac{f_1'}{r} - \tfrac{f_1}{r^2}\right) - \tfrac{a_1' f_1}{r}. \end{aligned}\right\} \tag{4.2}$$

Note that $|\Sigma|^2 = (h(r)^2 - g(r)^2)\cos^2(2\theta) + g(r)^2$. Thus the lemma's assertions will follow once it is determined whether $h^2 \equiv g^2$ or not.

case i) $\kappa \neq \frac{1}{\sqrt{2}}$. The asymptotic forms for f_1, f_1', a_1, a_1' as $r \to \infty$ are derived in [14]. Inserting these into (4.2) we find that as $r \to \infty$

$$h(r) = \begin{cases} \kappa^{\frac{3}{2}} a r^{-\frac{1}{2}} e^{-\sqrt{2}\kappa r}\left[1 + o(1)\right] & \text{if } 0 < \kappa < \sqrt{2} \\ 2b^2 e^{-2r}\left[1+o(1)\right] & \text{if } \kappa = \sqrt{2} \\ b^2 r^{-1} e^{-2r}\left[\frac{(4-\kappa^2)}{(\kappa^2-2)} + o(1)\right] & \text{if } \kappa > \sqrt{2} \end{cases}$$
$$g(r) = -b r^{-\frac{1}{2}} e^{-r}\left[1+o(1)\right],$$

where a and b are positive constants. We see that if $\kappa \neq \frac{1}{\sqrt{2}}$ then $h^2 \not\equiv g^2$.

case ii) $\kappa = \frac{1}{\sqrt{2}}$. Equation (1.2) is called self–dual in this case and any solution for which $\mathcal{E}_1$ is finite also solves the first order Bogomolńyi system (see [11]). In the case of a vortex solution it reads

$$a_1' = \frac{r(1-f_1^2)}{2} \text{ and } f_1' = \frac{(1-a_1)f_1}{r}.$$

Inserting these expressions for a_1' and f_1' into (4.2) one sees that $h \equiv -g$. □

Proof of Theorem 1.3. Note that in polar coordinates, the identity for s in Theorem 1.2 reads $s(r,\theta;\eta) = -is(r,\theta - \frac{\pi}{2};\eta)$. This implies that the second assertion is sufficient for the first to hold. To prove that the second assertion is a necessary condition for the first, suppose that there exists a sequence $\eta_j \downarrow 0$ such that $|s(r,\theta;\eta_j)| = |s(r,\theta+\alpha_j\pi;\eta_j)|$ for all (r,θ) where $2\alpha_j \notin \mathbb{Z}$. Then it is easy to see that there exists a subsequence $\eta_{j'}$, sequences $k_{j'} \in \mathbb{Z}$, and $0 \leq \beta_{j'} < 2$, such that $\beta_{j'} = k_{j'}\alpha_{j'} \bmod 2$, for which $\beta_{j'} \to \beta_0$ where $2\beta_0 \notin \mathbb{Z}$. Since $\lim_{\eta \to 0} s(\cdot;\eta) = \mu\Sigma(\cdot)$ in $\mathbf{L}^2(\mathbb{R}^2)$ and $\mu \neq 0$ it follows that $|\Sigma(r,\theta)| = |\Sigma(r,\theta+\beta_0\pi)|$. Since $\kappa \neq \frac{1}{\sqrt{2}}$ this would contradict Lemma 4.4. □

References

[1] I. Affleck, M. Franz, M. H. S. Amin, Generalized London free energy for high-T_c vortex lattices , *Phys. Rev. B*, **Vol 55, No. 2**, pp. R704-R707, 1997.

[2] S. Alama, L. Bronsard, and T. Giorgi, Uniqueness of symmetric vortex solutions in the Ginzburg-Landau model of superconductivity , *J. Funct. Anal.* , **Vol 167, no. 2**, pp. 399–424, 1999.

[3] M. S. Berger and Y. Y. Chen, Symmetric vortices for the Ginzburg-Landau equations of superconductivity and the nonlinear desigularization phenomenon, *J. Funct. Anal.* , **82**, pp. 259-295, 1989.

[4] A. J. Berlinsky, A. L. Fetter, M. Franz, C. Kallin, and P. I. Soininen, Ginzburg-Landau theory of vortices in d-wave superconductors, *Phys. Rev. Letters*, **Vol 75, No. 11**, pp. 2200-2203, 1995.

[5] D. Chang, C-Y Mou, B. Rosenstein and C. L. Wu , Static and dynamical anisotropy effects in the mixed state of d-wave superconductors, *Phys. Rev. B*, **vol 57, No. 13**, pp. 7955-7969, 1998.

[6] Q. Du, Studies of a Ginzburg-Landau model for d-wave superconductors, *SIAM J. Appl. Math*, **Vol 59, No. 4**, pp. 1225-1250, 1999.

[7] M. Franz, C. Kallin, P. I. Soininen, A. J. Berlinsky and A. L. Fetter , Vortex state in a d-wave superconductor, *Phys. Rev. B*, **Vol. 53, No. 9**, pp. 5795-5814, 1996.

[8] S. Gustafson and I. M. Sigal, The stability of magnetic vortices, *Commun. Math. Phys.*, **212**, pp. 257-275, 2000.

[9] Q. Han and T-C Lin, Fourfold symmetric vortex solutions of the d-wave Ginzburg-Landau equation, *Nonlinearity*, **15**, pp. 257-269, 2002.

[10] R. Heeb, A. Otterlo, M. Sigrist, and G. Blatter, Vortices in d-wave superconductors, *Phys. Rev. B*, **54, No. 13**, pp. 9385-9398, 1996.

[11] A. Jaffe and C. Taubes ,*Vortices and Monopoles*, Birkhäuser , 1980.

[12] F. Lin and T-C Lin, Vortex state of d-wave superconductors in the Ginzburg-Landau energy, *SIAM J. Math. Anal*, **3**, pp. 493-503, 2000.

[13] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Sup. Pisa*, **13**, pp. 123-131, 1959.

[14] B. Plohr, The existence, regularity, and behavior and at infinity of isotropic solutions of classical gauge field theories, *J. Math. Phys.* , **22, No. 10**, pp. 2184–2190, 1981.

[15] E. Sandier and S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Birkhäuser , 2007.

[16] I. M. Sigal and F. Ting, Pinning of magnetic vortices by an external potential, *St. Petersburg Math. J.*, **Vol. 16, No. 1**, pp. 211-236, 2005.

[17] J-H Xu, Y. Ren, and C-S Ting, Structures of single vortex and vortex lattice in a d-wave superconductor, *Phys. Rev. B*, **Vol 53, No. 6**, pp. R2991-R2994, 1996.