An Introduction to Proofs in Mathematics

The subject of mathematics is often regarded as a pinnacle in the achievement of human reasoning. The reason that mathematics is so highly regarded in the realm of human achievement and that mathematics is so useful in the sciences is that virtually every fact in mathematics is proven through deductive reasoning.

Throughout this course, you will be asked to “prove” or “show” certain facts. As such, you should know the basics of mathematical proof, which are explained in this document. You will by no means be an expert at proofs or mathematical reasoning by the end of the course, but hopefully you will be able to learn some of the basics of how mathematical proofs work. Additionally, one of the best ways to get experience in proof writing is to read proofs. Therefore, I highly suggest that you do *not* skip the proofs when you are reading the textbook.

1. Starting with definitions

When faced with the task of proving something, knowing how to start is usually half the battle. How do you start a proof? One of the most common starting places is to use what you already know. Many times, the only thing we will know is a definition. As such, we will often prove something using the definition. For illustrative purposes, let’s introduce a definition.

<u>Definition 1</u>. A real number r is <u>positive</u> if $r > 0$ and is <u>negative</u> if $r < 0$. 0 is considered to be neither positive nor negative.

<u>Theorem 1</u>. If r and s are positive numbers, then $r + s$ is positive.

Since we are dealing with mathematics, in order for the above statement to be considered a theorem, it *must* be proven. So let’s prove the theorem. Notice that the only fact that we currently have about positive numbers is the definition of a positive number. As such, we can only use the definition in our proof.

<u>Proof of Theorem 1</u>. Let r and s be positive. Then, by Definition 1, $0 < r$ and $0 < s$. As such, $0 + 0 < r + s$, so $0 < r + s$. Therefore, by Definition 1, $r + s$ is positive. This concludes the proof.

Notice that in this proof, **generality** is used. We use generic positive real numbers r and s as opposed to specific real numbers. If we used $r = 2$ and $s = 3$, for example, this would *not* constitute a proof of the theorem. It would only prove that $2 + 3$ is a positive real number. As such, we should always strive for generality (instead of using specific things) to do proofs.

2. If and only if (iff) statements

An "if and only if" (sometimes abbreviated as "iff") statement is a very powerful type of statement and as such, requires special attention.

An "if and only if" statement will be something along the lines of "A is true if and only if B is true." This gives us a lot of information. If such a theorem is proven, we will know the following:

If A is true, then B is true (if we can show that A is true, we automatically have that B is true).

If B is true, then A is true (if we can show that B is true, we automatically have that A is true).

If A is false, then B is false (if we can show that A is false, we automatically have that B is false).

If B is false, then A is false (if we can show that B is false, we automatically have that A is false).

Although there are a few ways to go about proving an "if and only if" statement, for our class, it is enough to use the following method. We can prove "A is true if and only if B is true" by proving "if A is true, then B is true" *and* by proving "if B is true, then A is true."

Let's introduce an "if and only if" theorem.

<u>Theorem 2</u>. r is a negative number if and only if $r = -1 \cdot r'$ for some positive number r'.

To prove this, we must show that if r is a negative number, then $r = -1 \cdot r'$ for some positive number r' *and* we must show that if $r = -1 \cdot r'$ for some positive number r', then r is negative.

Proof of Theorem 2. Let r be a negative number. By Definition 1, $r < 0$. If we multiply both sides of this inequality with -1, we get $-r > 0$. Thus, $-r$ is a positive number by Definition 1. Furthermore, $r = -1 \cdot -r$. As such, if $r' = -r$, then $r = -1 \cdot r'$ for some positive number r'.

Let $r = -1 \cdot r'$ for some positive number r'. Since r' is positive, Definition 1 gives us that $r' > 0$. Multiplying both sides of this inequality by -1 gives us that $-1 \cdot r' < 0$. Since $r = -1 \cdot r'$, we have that $r < 0$. Thus, r is negative by Definition 1.

This concludes the proof.

3. Proving things with more than just definitions

Now that we've proven some theorems, we can use the theorems to prove other results. We don't have to rely only on definitions.

Theorem 3. If r and s are negative numbers, then $r + s$ is negative.

We could prove Theorem 3 directly from Definition 1 (and it would work), but we can also make use of Theorems 1 and 2 in our proof, which we will do now.

Proof of Theorem 3. Suppose r and s are negative numbers. Then, by Theorem 2, $r = -1 \cdot r'$ and $s = -1 \cdot s'$ for some positive numbers r' and s'.

$$r + s = (-1 \cdot r') + (-1 \cdot s') = -1 \cdot (r' + s')$$

Since r' and s' are positive numbers, Theorem 1 gives us that $r' + s'$ is a positive number. Since $r + s = -1 \cdot (r' + s')$ for some positive number $r' + s'$, Theorem 2 tells us that $r + s$ is negative. This concludes the proof.

As we can see, there are sometimes multiple ways to prove a theorem. As long as you use definitions and information that has already been proven, your proof should be fine.

4. Disproving false statements

There will be several times throughout the semester where you are asked to disprove something, so we address that now. A mathematical statement is false if even one counterexample exists. As such, we can disprove a statement (proving that a statement is false) by counterexample. Even though you cannot prove something is true using specifics, you can (and should) disprove a false statement using specifics.

Statement 1. If $r + s$ is positive, then r and s are both positive.

This statement is *false*, and we can show it is false by counterexample.

Disproof of Statement 1. Let $r = -1$ and $s = 2$. Then $r + s = -1 + 2 = 1$. 1 is positive, but $-1 = r$ is not positive. This disproves statement 1.

5. Watch out! Some common logical fallacies that trip up beginners

Proof by example. As stated in section 1, we absolutely cannot prove a statement is true by example. While it is acceptable to disprove something with a counterexample, we simply must use generality when proving that something is true.

Converse error. It is wonderful to prove a statement such as "If A is true, then B is true." However, one should be very careful in his or her application of such a statement. Many beginners may want to assume the converse of this statement is true. i.e., if we know that B is true, you may want to conclude that A must be true too. This is a logical fallacy known as the converse error. Even if the statement "If A is true, then B is true" is a true statement, we do *not* know that "If B is true, then A is true" is a true statement.

An example of this can be seen by looking at Theorem 1 and Statement 1. Theorem 1 is a statement of the form "If A is true, then B is true" where A is "r and s are positive numbers" and B is "$r + s$ is positive." Statement 1 is the converse of Theorem 1. In other words, using the same A and B, Statement 1 is of the form "If B is true, then A is true." Notice that we proved Theorem 1 (meaning that Theorem 1 is true), but we *disproved* Statement 1 (meaning that Statement 1 is false).

Another common example of the converse error can be seen in Calculus II. When dealing with infinite series, students learn the "Test for Divergence," which states that:

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0.$$

If we encounter a series $\sum_{n=1}^{\infty} b_n$ where $\lim_{n\to\infty} b_n = 0$, however, we *cannot* conclude that $\sum_{n=1}^{\infty} b_n$ converges. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a counterexample to the converse of the Test for Divergence.

Circular reasoning. This is assuming that what you want to show is true. If you *assume* that something is true, of course you can show that it is true based off of your assumption; however, this is *not* a valid proof technique as you cannot assume that something is true when proving that something is true.

This fallacy can be quite subtle. The way in which it may creep up is by using statements to prove each other. Above, we proved Theorem 3 by using Theorems 1 and 2. If we were to have proven either Theorem 1 or Theorem 2 using Theorem 3, this would constitute as circular reasoning. (If we had assumed that Theorem 3 was true in the proof of Theorem 1 or Theorem 2 and then used that to show that Theorem 3 was true, we would have assumed Theorem 3 was true in order to show that Theorem 3 is true.)

An easy way to avoid this is by only using theorems that you have already proven to prove a new theorem.

Note: If this document is a little bit challenging, don't worry too much. I do not expect perfection because most of you are fairly new to proofs, and you will learn more as time goes on, especially if you take more advanced math classes (such as Abstract Algebra or Real Analysis). Continue reading the proofs in the book and pay attention to any proofs in class. If you have trouble with proofs, feel free to visit me during office hours.