Lesson 10

Homogeneous Equations with Constant Coefficients (3.1)

Chapter 3 is about second order linear diff eqs. These are of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

Such an equation is called homogeneous if $g(t) \equiv 0$ (i.e., $g(t)$ is the zero function - for all t)
If $g(t) \not\equiv 0$, then $g(t)$ is called the nonhomogeneous part of the equation.

In this section, we deal with homogeneous equations in which the coefficient functions are constants. i.e., of the form

$$ay'' + by' + cy = 0$$

We motivate how to solve such equations with the example $y'' - y = 0$.

It's not hard to see that $w_1(t) = e^t$ is a solution.
Also, $w_2(t) = e^{-t}$ is a solution.
And, for any constants c_1 and c_2,
$c_1 e^t + c_2 e^{-t}$ is a solution!

It seems natural that solutions of

$$ay'' + by' + cy = 0$$

Will then be of the form e^{rt} and if there is more than one, we get linear combinations.

Assuming $y(t) = e^{rt}$ is a solution of
$ay'' + by' + cy = 0$, we see
$y'(t) = re^{rt}$, $y''(t) = r^2 e^{rt}$
$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$
$e^{rt}(ar^2 + br + c) = 0$
This is only possible when $ar^2 + br + c = 0$.

Given a diff eq of the form
$ay'' + by' + cy = 0$,
the equation $ar^2 + br + c = 0$
is called the characteristic equation of the diff eq.

The roots of the characteristic polynomial satisfy the diff. eq.

From algebra, we know the roots can be
(i) distinct real numbers (this lesson)
(ii) complex conjugates (lesson 12)
(iii) a single repeated real number (lesson 13)

Thus, if given $ay'' + by' + cy = 0$,
Find the roots of the characteristic polynomial.
If they are real numbers $r_1 \neq r_2$, then the general solution is of the form
$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
where c_1 and c_2 are constants.

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Ex 1. Find the general solution to

$$y'' + y' - 12y = 0$$

characteristic equation: $r^2 + r - 12 = 0$

$(r+4)(r-3) = 0$

So $r_1 = -4$, $r_2 = 3$

$$\boxed{y(t) = c_1 e^{-4t} + c_2 e^{3t}}$$

For second order equations, we require two initial conditions to find a particular solution.

Ex 2. Solve the IVP

$$y'' + y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 3$$

$r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0$

$r = 1$ or $r = -2$

$y(t) = c_1 e^{t} + c_2 e^{-2t}$

$y'(t) = c_1 e^{t} - 2c_2 e^{-2t}$

$0 = y(0) = c_1 + c_2$

$3 = y'(0) = c_1 - 2c_2$

Have system of equations $\begin{cases} c_1 + c_2 = 0 \\ c_1 - 2c_2 = 3 \end{cases}$

$$\begin{array}{r} c_1 + c_2 = 0 \\ -c_1 + 2c_2 = -3 \\ \hline 3c_2 = -3 \\ c_2 = -1 \end{array}$$

$c_1 + (-1) = 0$

$c_1 = 1$

$$\boxed{y(t) = e^{t} - e^{-2t}}$$

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Ex 3. Determine the behavior of $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ based on the values of r_1 and r_2.

Notice that if r is negative, $\lim_{t \to \infty} ce^{rt} = 0$
If r is positive, $\lim_{t \to \infty} ce^{rt} = \pm \infty$ (depending on the sign of c)

If both r_1, r_2 are negative,
$\lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0$

If both r_1 and r_2 are positive,
$\lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty$
depending on the signs of c_1, c_2 and whether $r_1 < r_2$ or $r_2 < r_1$.

If r_1 is positive and r_2 is nonpositive,
$\lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = \pm \infty$
depending on the sign of c_1
Can have a max or min in this case!

If $r_1 = 0$ and r_2 is negative,
$\lim_{t \to \infty} (c_1 e^{0} + c_2 e^{r_2 t}) = c_1$

These assume $c_1 \neq 0$ and $c_2 \neq 0$!

Ex 4. Solve the IVP

$$y''-y'-2y=0,\ y(0)=\alpha,\ y'(0)=2.$$

Determine the value of α so the solution approaches zero as $t\to\infty$.

$r^2-r-2=0 \Rightarrow (r-2)(r+1)=0 \Rightarrow r=2$ or $r=-1$

$y(t)=c_1e^{2t}+c_2e^{-t}$, $y'(t)=2c_1e^{2t}-c_2e^{-t}$

$$\begin{cases}\alpha = c_1+c_2\\ 2=2c_1-c_2\end{cases}$$

$2+\alpha=3c_1$ so $c_1=\frac{2+\alpha}{3}$

$\alpha=\frac{2+\alpha}{3}+c_2$ so $c_2=\frac{2-2\alpha}{3}$

$$y(t)=\underbrace{\left(\frac{2+\alpha}{3}\right)e^{2t}}+\left(\frac{2-2\alpha}{3}\right)e^{-t}$$

Since $\to$ is positive and tends to ∞ as $t\to\infty$

Need $\frac{2+\alpha}{3}=0$, so $\boxed{\alpha=-2}$

Ex 5. Find the maximum value to the solution of $2y''-3y'+y=0$, $y(0)=2$, $y'(0)=\frac{1}{2}$

$2r^2-3r+1=0 \Rightarrow (2r-1)(r-1)=0 \Rightarrow r=\frac{1}{2}$ or $r=1$

$y(t)=c_1e^{t}+c_2e^{t/2}$

$$\begin{cases}2=c_1+c_2\\ \frac{1}{2}=c_1+\frac{c_2}{2}\end{cases} \to \begin{cases}2=c_1+c_2\\ -\frac{1}{2}=-c_1-\frac{c_2}{2}\end{cases}$$

$\frac{3}{2}=\frac{c_2}{2} \Rightarrow c_2=3$

so $c_1=-1$

$y(t)=-e^t+3e^{t/2}$, $y'(t)=-e^t+\frac{3}{2}e^{t/2}$

(continued)

$$0 \overset{\text{set}}{=} -e^{t} + \frac{3}{2}e^{t/2}$$

Multiply by $e^{-t/2}$

$$0 = -e^{t/2} + \frac{3}{2}$$

$$-\frac{3}{2} = -e^{t/2}$$

$$\ln\left(\frac{3}{2}\right) = t/2$$

$$t = 2\ln\left(\frac{3}{2}\right) = \ln\left(\frac{9}{4}\right)$$

$$y\left(\ln\left(\frac{9}{4}\right)\right) = -e^{\ln\left(\frac{9}{4}\right)} + 3e^{\ln\left(\frac{9}{4}\right)/2}$$

$$= -\left(\frac{9}{4}\right) + 3\left(\frac{3}{2}\right)$$

$$= -\frac{9}{4} + \frac{9}{2}$$

$$= -\frac{9}{4} + \frac{18}{4}$$

$$= \boxed{\frac{9}{4}}$$