Lesson 11

Solutions of 2nd Order Linear Homogeneous Equations & The Wronskian (3.2)

Recall from linear algebra that a function $L: V \to W$ between vector spaces is called linear if $L(v_1+v_2) = L(v_1) + L(v_2)$ and $L(cv_1) = c\,L(v_1)$.

A linear operator is a linear map for which our vector spaces have functions as their vectors.

Given a differential equation

$$y'' + py' + qy = 0,$$

we develop the linear operator

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Notice that $L[\phi]$ is a function!

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

ϕ is a solution to $y'' + py' + qy = 0$ if and only if $L[\phi] = 0$.

We now show that L is linear. Let ϕ and ψ be functions.

$$\begin{aligned} L[\phi+\psi] &= (\phi+\psi)'' + p(\phi+\psi)' + q(\phi+\psi) \\ &= \phi'' + \psi'' + p\phi' + p\psi' + q\phi + q\psi \\ &= \phi'' + p\phi' + q\phi + \psi'' + p\psi' + q\psi \\ &= L[\phi] + L[\psi] \end{aligned}$$

Let c be a constant

$$\begin{aligned} L[c\phi] &= (c\phi)'' + p(c\phi)' + q(c\phi) \\ &= c\phi'' + cp\phi' + cq\phi \\ &= c(\phi'' + p\phi' + q\phi) = c\,L[\phi] \end{aligned}$$

(This works because the diff eq is linear)

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The terminology of a linear operator is very tricky at first, but it will be extraordinarily useful in Chapter 6, and to prove the following theorem!

<u>Theorem 3.2.2 (Principle of Superposition)</u>. If y_1 and y_2 are solutions to the linear equation

$$y'' + p(t)y' + q(t)y = 0$$

then $c_1y_1 + c_2y_2$ is a solution as well for any constants c_1 and c_2.

<u>Proof</u>. Use the linear operator $L[\phi] = \phi'' + p\phi' + q\phi$.
Then
$$\begin{aligned} L[c_1y_1 + c_2y_2] &= L[c_1y_1] + L[c_2y_2] \\ &= c_1L[y_1] + c_2L[y_2] \\ &= c_1(0) + c_2(0) = 0. \quad \boxtimes \end{aligned}$$

<u>Ex 1</u>. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions to $y'' - y = 0$. Are $y_3(t) = \sinh t$ and $y_4(t) = \cosh t$ also solutions?

$y_1' = e^t$, $y_1'' = e^t$ so $(e^t) - (e^t) = 0$ ✓
$y_2' = -e^{-t}$, $y_2'' = e^{-t}$ so $(e^{-t}) - (e^{-t}) = 0$ ✓

By the Principle of Superposition, $c_1e^t + c_2e^{-t}$ is a solution for any c_1 and c_2.

$\sinh t = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ $\quad (c_1 = \frac{1}{2},\ c_2 = -\frac{1}{2})$
$\cosh t = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$ $\quad (c_1 = \frac{1}{2},\ c_2 = \frac{1}{2})$

So yes, these are also solutions!

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So we know that if two solutions exist, then any linear combination is also a solution. Can we establish that solutions exist? Yes!

Theorem 3.2.1 (Existence and Uniqueness Thm)
Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

If p, q, and g are continuous on an interval I, then there exists a unique, twice-differentiable function $y = \phi(t)$ satisfying the IVP.

Ex 2. On what interval is there guaranteed to be a unique twice-differentiable solution to

$$(t+1)y'' + \ln(1-t)y' + y = \sqrt{t}, \quad y(\tfrac{1}{2}) = 1, \quad y'(\tfrac{1}{2}) = 2$$

$$y'' + \frac{\ln(1-t)}{(t+1)}y' + \frac{1}{t+1}y = \frac{\sqrt{t}}{t+1}$$

discontinuous when $t+1 = 0 \Rightarrow t = -1$

when $1-t \leq 0 \Rightarrow t \geq 1$

when $t < 0$

so continuous on $\boxed{0 \leq t < 1}$

We've addressed existence of solutions and how to form solutions from others. But how do we know we've found all solutions? How do we know, for example, that every solution to $y'' - y = 0$ can be written as $c_1 e^t + c_2 e^{-t}$ for some constants c_1 and c_2? To do this, we introduce the Wronskian.

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Given differentiable functions f and g, the <u>Wronskian</u> of f and g is the determinant
$$\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

<u>Theorem 3.2.4</u> Suppose that y_1 and y_2 are solutions to $L[y] = y'' + p(t)y' + q(t)y = 0$. Then the family of solutions $y = c_1 y_1 + c_2 y_2$ includes <u>every</u> solution if and only if the Wronskian of y_1 and y_2 is not zero for some point t_0.

(The proof comes from linear algebra - the solutions of $L[y]=0$ are a vector space of dimension 2, and the Wronskian not being zero guarantees that $\{f, g\}$ is linearly independent, so it is a basis, so it spans the solution set)

If a set of solutions $\{y_1, y_2\}$ has Wronskian not equal to zero, we call it a <u>fundamental set of solutions</u> since $c_1 y_1 + c_2 y_2$ is the general solution, containing every solution.

Fundamental sets of solutions are not necessarily unique.

<u>Ex 3</u>. Show that $\{e^t, e^{-t}\}$ and $\{\sinh t, \cosh t\}$ are both fundamental sets of solutions to
$$y'' - y = 0.$$

Already seen that all are solutions, so we just need to check the Wronskians.

$$\begin{vmatrix} e^t & e^{-t} \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(e^{-t}) \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -e^te^{-t} - e^te^{-t}$$

$$= -1 - 1 = -2$$

$-2 \neq 0$, so $\{e^t, e^{-t}\}$ is a fundamental set.

$$\begin{vmatrix} \sinh t & \cosh t \\ \frac{d}{dt}(\sinh t) & \frac{d}{dt}(\cosh t) \end{vmatrix} = \begin{vmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{vmatrix}$$

$$= \sinh^2 t - \cosh^2 t = -1$$

$-1 \neq 0$ so $\{\sinh t, \cosh t\}$ is a fundamental set.

Ex 4. Show that $\{e^{-t}$ and $2e^{-t}\}$ is not a fundamental set of solutions to $y'' - y = 0$.

$$\begin{vmatrix} e^{-t} & 2e^{-t} \\ -e^{-t} & -2e^{-t} \end{vmatrix} = -2e^{-2t} + 2e^{-2t} = 0$$

Wronskian is zero !

Ex 5. Suppose $y_1 = x$ and $y_2 = xe^x$ are solutions to $L[y] = 0$. Is $\{y_1, y_2\}$ a fundamental set?

$$\begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x^2e^x + xe^x - xe^x = x^2e^x$$

The Wronskian is not zero for some value of x (e.g. $x = 1$), so it is a fundamental set.

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<u>Ex 6</u>. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

$$3e^{4t} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g$$

$$e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

(First order linear ODE!)

$$g' - 2g = 3e^{2t}$$

$$\mu(t) = \exp\left(\int -2\,dt\right) = e^{-2t}$$

$$\frac{d}{dt}\left[e^{-2t}g\right] = 3$$

$$e^{-2t}g = 3t + C$$

$$\boxed{\begin{array}{c} g(t) = 3te^{2t} + Ce^{2t} \\ \text{for some constant } C \end{array}}$$