Lesson 12

Complex Roots of the Characteristic Equation (3.3)

We saw in lesson 10 that if r_1 and r_2 are the roots of ar^2+br+c, then the general solution of $ay''+by'+cy=0$ is

$$y(t)=c_1 e^{r_1 t}+c_2 e^{r_2 t}.$$

What if these roots are complex conjugates

$$r_1=\lambda+i\mu,\quad r_2=\lambda-i\mu.$$

Then we have $c_1 e^{(\lambda+i\mu)t}+c_2 e^{(\lambda-i\mu)t}$

What does this even mean?

Leonhard Euler noticed a very interesting relationship when messing around with Maclaurin series.

Recall: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

What if we plug in $x=it$?

$$e^{it}=1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+\frac{(it)^5}{5!}+\frac{(it)^6}{6!}+\frac{(it)^7}{7!}+\dots$$

$$=1+it-\frac{t^2}{2!}-i\frac{t^3}{3!}+\frac{t^4}{4!}+i\frac{t^5}{5!}-\frac{t^6}{6!}-i\frac{t^7}{7!}+\dots$$

$$=\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\dots\right)+i\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right)$$

$$=\cos t+i\sin t$$

So $e^{it}=\cos t+i\sin t$

We can use this formula to compute numbers raised to complex powers.

Lesson 12

Ex 1. Compute $e^{i\pi}$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i\cdot 0 = \boxed{-1}$$

Fact: Adding 1 to both sides gives

$$e^{i\pi} + 1 = 0$$

which is regarded by many mathematicians as the most beautiful equation in mathematics.

Ex 2. Compute 2^{3-4i}

$$2^{3-4i} = 2^3 \cdot 2^{-4i}$$
$$= 8 \cdot 2^{-4i}$$

Now, $2 = e^{\ln 2}$ so

$$2^{3-4i} = 8\cdot\left(e^{\ln 2}\right)^{-4i} = 8\cdot e^{-4\ln 2\, i}$$
$$= 8\left(\cos(-4\ln 2) + i\sin(-4\ln 2)\right)$$
$$= \boxed{8\cos(4\ln 2) - 8i\sin(4\ln 2)}$$

(using $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$)

Ex 3. Compute $e^{(\lambda+\mu i)t}$

$$e^{(\lambda+\mu i)t} = e^{\lambda t}\cdot e^{\mu i t} = \boxed{e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)}$$
$$= \boxed{e^{\lambda t}\cos(\mu t) + i e^{\lambda t}\sin(\mu t)}$$

Using this fact, we can now make sense of the solution:

$$y(t) = k_1 e^{(\lambda+i\mu)t} + k_2 e^{(\lambda-i\mu)t}$$

Lesson 12

So $y(t) = k_1 e^{\lambda t}\cos\mu t + k_1 i e^{\lambda t}\sin\mu t$
$+ k_2 e^{\lambda t}\underbrace{\cos(-\mu t)}_{\cos(\mu t)} + k_2 i e^{\lambda t}\underbrace{\sin(-\mu t)}_{-\sin(\mu t)}$

So $y(t) = \underbrace{(k_1+k_2)}_{c_1} e^{\lambda t}\cos\mu t + i e^{\lambda t}\underbrace{(k_1-k_2)}_{c_2}\sin\mu t$

$y(t) = c_1 e^{\lambda t}\cos\mu t + i c_2 e^{\lambda t}\sin\mu t$

Can we get this solution in terms of real numbers only?

Thm 3.2.6 If $u(t) + i v(t)$ is a solution to $y'' + p(t)y' + q(t)y = 0$, then $u(t)$ and $v(t)$ are solutions as well.

Proof. Use $L[\phi] = \phi'' + p(t)\phi' + q(t)\phi$

$0 = L[u+iv] = L[u] + i L[v]$

Real part is $L[u]$ which must equal 0

Imaginary part is $L[v]$ which must equal 0.

Thus, u and v are solutions. ▨

So we know $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$ are solutions. Can all solutions be written as a linear combination of these?

We check the Wronskian.

Lesson 12

$$\begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ -\mu e^{\lambda t}\sin\mu t + \lambda e^{\lambda t}\cos\mu t & \mu e^{\lambda t}\cos\mu t + \lambda e^{\lambda t}\sin\mu t \end{vmatrix}$$

$$= \mu e^{2\lambda t}\cos^2\mu t + \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t)$$
$$- (-\mu e^{2\lambda t}\sin^2\mu t + \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t)$$

$$= \mu e^{2\lambda t}(\cos^2\mu t + \sin^2\mu t) = \mu e^{2\lambda t}$$

This Wronskian is nonzero whenever $\mu \neq 0$ (but if $\mu = 0$, the we would have repeated real roots, which we will address in lesson 13).

So, the Wronskian tells us that if $\lambda \pm i\mu$ are roots of the characteristic polynomial $ar^2 + br + c$ then the general solution to

$$ay'' + by' + cy = 0$$

is $\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)}$

Ex 4. Solve the IVP

$$y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$$

$$r^2 + r + 1.25 = 0$$

$$r = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2}$$

$$r = -\tfrac{1}{2} \pm i$$

$$y(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$$

$$3 = y(0) = c_1(1)(1) + c_2(1)(0)$$

so $c_1 = 3$

Lesson 12

$y(t) = 3e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$

$y'(t) = -3e^{-t/2}\sin t - \frac{3}{2}e^{-t/2}\cos t + c_2 e^{-t/2}\cos t - \frac{c_2}{2}e^{-t/2}\sin t$

$1 = y'(0) = -3(1)(0) - \frac{3}{2}(1)(1) + c_2(1)(1) - \frac{c_2}{2}(1)(0)$

$1 = -\frac{3}{2} + c_2$ so $c_2 = \frac{5}{2}$

$$\boxed{y(t) = 3e^{-t/2}\cos t + \frac{5}{2}e^{-t/2}\sin t}$$

Graph this.
Notice it has decaying oscillation
(which makes sense since $\lim_{t\to\infty} y(t) = 0$
and $\sin t$ and $\cos t$ cause oscillation)

Ex 5. Solve the IVP

$y'' + 16y = 0, \quad y(0) = 1, \quad y'(0) = 1$

$r^2 + 16 = 0$

$r = 0 \pm 4i$

$y(t) = c_1 e^{0t}\cos(4t) + c_2 e^{0t}\sin(4t)$

$= c_1\cos(4t) + c_2\sin(4t)$

$1 = y(0) = c_1(1) + c_2(0)$, so $c_1 = 1$

$y'(t) = -4\sin(4t) + 4c_2\cos(4t)$

$1 = y'(0) = 0 + 4c_2$, so $c_2 = \frac{1}{4}$

$$\boxed{y(t) = \cos(4t) + \frac{1}{4}\sin(4t)}$$

Graph this.
Notice it has steady oscillation.

Lesson 12

Ex 6. Solve the IVP
$y'' - 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1$

$r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$

$r = 1 \pm i$

$y(t) = c_1 e^t \cos t + c_2 e^t \sin t$

$2 = y(0) = c_1(1)(1) + c_2(1)(0)$, so $c_1 = 2$

$y'(t) = -2e^t \sin t + 2e^t \cos t + c_2 e^t \cos t + c_2 e^t \sin t$

$1 = y'(0) = -2(1)(0) + 2(1)(1) + c_2(1)(1) + c_2(1)(0)$

$1 = -2 + c_2$, so $c_2 = 3$

$\boxed{y(t) = 2e^t \cos t + 3e^t \sin t}$

Graph this.
Notice it has growing oscillation
(which makes sense since $e^t \to \infty$ as $t \to \infty$
and has $\cos t$ and $\sin t$ controlling oscillation)