Lesson 14

Nonhomogeneous Equations: Method of Undetermined Coefficients (3.5)

Given a nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t),$$ a function $y = \phi(t)$ is a solution if and only if $L[\phi] = \phi'' + p\phi' + q\phi = g(t)$.

Thm 3.5.1 If $Y_1(t)$ and $Y_2(t)$ are solutions to the nonhomogeneous equation above, then $Y_1(t) - Y_2(t)$ is a solution to the homogeneous equation
$$y'' + py' + qy = 0.$$

Proof. $L[Y_1 - Y_2] = L[Y_1] - L[Y_2]$
$= g(t) - g(t) = 0$.

This tells us that if $\{y_1, y_2\}$ is a fundamental set for the homogeneous equation, then
$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$ for some c_1 and c_2.

Thm 3.5.2 If $y_1(t)$ and $y_2(t)$ form a fundamental set for $y'' + py' + qy = 0$ and if $Y(t)$ is a specific solution to $y'' + py' + qy = g(t)$, then the general solution for $y'' + py' + qy = g(t)$ is
$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

We often call $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ the complementary solution and $Y(t)$ a particular solution

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Proof. $L[c_1y_1 + c_2y_2 + Y] = c_1L[y_1] + c_2L[y_2] + L[Y]$
$= 0 + 0 + g(t)$

And if ϕ is a solution, then
$\phi - Y = c_1y_1 + c_2y_2$, so $\phi = c_1y_2 + c_2y_2 + Y$
for some coefficients c_1 and c_2. $\blacksquare$

Hence, in order to solve a nonhomogeneous equation, we find the complementary solution, find a particular solution, and we add them.

How do we find a particular solution? One way is using the Method of Undetermined Coefficients.

Given $ay'' + by' + cy = g(t)$, we look at each term of $g(t)$ individually and guess the form of a particular solution based on that.

- If $e^{\alpha t}$ is a term, guess $Ae^{\alpha t}$ is a term of $Y(t)$.
- If $\cos(\alpha t)$ or $\sin(\alpha t)$ is a term, guess $A\cos(\alpha t) + B\sin(\alpha t)$ is a term of $Y(t)$
- If $a_nt^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$ is a term, guess $A_nt^n + A_{n-1}t^{n-1} + \dots + A_1t + A_0$ is a term of $Y(t)$.
- If a product of the above forms appears as a term, guess the product of the appropriate guesses is a term of $Y(t)$.

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Ex 1. If $g(t)$ is as given below, what is your initial guess for $Y(t)$?

(a) $g(t) = e^{2t} + \sin(3t)$

terms: e^{2t} $\quad \sin(3t)$

guess: Ae^{2t} $\quad B\sin(3t) + C\cos(3t)$

initial guess: $Y(t) = Ae^{2t} + B\sin(3t) + C\cos(3t)$

(b) $g(t) = 4t^3 + t^2\sin(2t)$

terms: $4t^3$ $\quad t^2\sin(2t)$

guess: $A_3t^3 + A_2t^2 + A_1t + A_0$ $\quad (B_2t^2 + B_1t + B_0)\cos(2t) + (C_2t^2 + C_1t + C_0)\sin(2t)$

initial guess: $Y(t) = A_3t^3 + A_2t^2 + A_1t + A_0 + (B_2t^2 + B_1t + B_0)\cos(2t) + (C_2t^2 + C_1t + C_0)\sin(2t)$

(c) $g(t) = te^t + te^{3t}\cos t$

terms: te^t $\quad te^{3t}\cos t$

guess: $(A_1t + A_0)e^t$ $\quad (B_1t + B_0)e^{3t}\cos t + (C_1t + C_0)e^{3t}\sin t$

initial guess: $Y(t) = (A_1t + A_0)e^t + (B_1t + B_0)e^{3t}\cos t + (C_1t + C_0)e^{3t}\sin t$

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This technique is not good enough. What if one of the terms of $g(t)$ is part of the complementary solution? Taking inspiration from repeated roots, multiply that term's guess by t.

Ex 2. Given the complementary solution $y_c(t)$ and $g(t)$, what is your guess for $Y(t)$?

(a) $y_c(t) = c_1 e^t + c_2 e^{2t}, \quad g(t) = e^t + \cos(3t)$

terms:	e^t	$\cos(3t)$
guess 1:	Ae^t	$B\cos(3t) + C\sin(3t)$
in y_c?:	yes!	no!
guess 2:	Ate^t	same as guess 1

$$Y(t) = Ate^t + B\cos(3t) + C\sin(3t)$$

(b) $y_c(t) = c_1\cos t + c_2 \sin t, \quad g(t) = \sin t + t^2 \cos 3t$

terms:	$\sin t$	$t^2 \cos 3t$
guess 1:	$A\cos t + B\sin t$	$(C_2t^2 + C_1t + C_0)\cos 3t + (D_2t^2 + D_1t + D_0)\sin 3t$
in y_c?:	yes	no
guess 2:	$At\cos t + Bt\sin t$	Same as guess 1

$$Y(t) = At\cos t + Bt\sin t + (C_2t^2 + C_1t + C_0)\cos 3t + (D_2t^2 + D_1t + D_0)\sin 3t$$

Now that we can guess the form of $Y(t)$, we can determine the coefficients by plugging $Y(t)$ into the diff eq.

Method of Undetermined Coefficients

Given $ay'' + by' + cy = g(t)$

1. Find the complementary solution $y_c(t)$ by solving the homogeneous equation $ay'' + by' + cy = 0$.
2. Analyze the terms of $g(t)$ to guess the form of the particular solution $Y(t)$.
3. Plug $Y(t)$ into $ay'' + by' + cy = g(t)$, set up a system of equations, and solve for each coefficient.

Ex 3. Find the general solution to
$$y'' + y' - 2y = 3e^t + te^{-2t}$$

$$y_c(t) = c_1 e^t + c_2 e^{-2t}$$

terms:	$3e^t$	te^{-2t}
guess 1:	Ae^t	$(Bt + C)e^{-2t}$
in y_c?:	yes	yes
guess 2:	Ate^t	$(Bt^2 + Ct)e^{-2t}$

$$Y(t) = Ate^t + (Bt^2 + Ct)e^{-2t}$$
$$Y'(t) = Ate^t + Ae^t + (Bt^2 + Ct)(-2e^{-2t}) + (2Bt + C)e^{-2t}$$
$$= (At + A)e^t + (-2Bt^2 - 2Ct + 2Bt + C)e^{-2t}$$
$$Y''(t) = (At + A)e^t + Ae^t + (-2Bt^2 - 2Ct + 2Bt + C)(-2e^{-2t})$$
$$+ (-4Bt - 2C + 2B)e^{-2t}$$
$$= (At + 2A)e^t + (4Bt^2 + 4Ct - 4Bt - 2C - 4Bt - 2C + 2B)e^{-2t}$$
$$= (At + 2A)e^t + [4Bt^2 + (4C - 8B)t + (-4C + 2B)]e^{-2t}$$

Plug in to $y''+y'-2y=3e^t+te^{-2t}$

$$(At+2A)e^t+[4Bt^2+(4C-8B)t+(-4C+2B)]e^{-2t}$$
$$+(At+A)e^t+[-2Bt^2+(-2C+2B)t+C]e^{-2t}$$
$$-2[Ate^t+(Bt^2+Ct)e^{-2t}] \quad = 3e^t+te^{-2t}$$

$$\underline{3A}e^t+[\underline{-6B}t+(-3C+2B)]e^{-2t}=\underline{3}e^t+\underline{\ }te^{-2t}$$

$$\left.\begin{aligned}3A&=3\\-6B&=1\\-3C+2B&=0\end{aligned}\right\} \Rightarrow \begin{aligned}A&=1\\B&=-\tfrac{1}{6}\\3C&=-\tfrac{1}{3} \Rightarrow C=-\tfrac{1}{9}\end{aligned}$$

So $Y(t)=te^t+\left(-\frac{1}{6}t^2-\frac{1}{9}t\right)e^{-2t}$

Thus, the general solution is

$$\boxed{y(t)=c_1e^t+c_2e^{-2t}+te^t-\frac{1}{6}t^2e^{-2t}-\frac{1}{9}te^{-2t}}$$