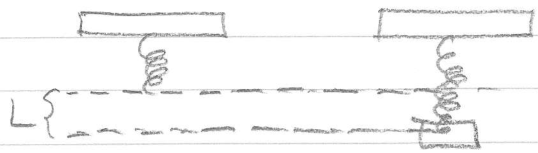
Lesson 16

Mechanical Vibrations (3.7)

Imagine a spring hanging from the ceiling.
Now imagine a mass being attached to the spring and stretching it some. We assume $\downarrow$ is the positive direction



$\downarrow +$

By Hooke's Law, the spring force is proportional to the length L that the mass stretches the spring.
At equilibrium, then, we have

$$F = ma$$

$$mg - kL = 0$$ where k is the spring constant.

When the mass is in motion, we assume that gravity, the spring force, and a dampening force (assume proportional to the velocity of the mass) act on the mass.

Let: $u(t)$ be the displacement of the mass at time t.

$$F = ma$$

$$\underbrace{\text{gravity}}_{mg} - \underbrace{\text{dampening force}}_{-\gamma u'(t)} - \underbrace{\text{spring force}}_{-k(L+u)} = mu''(t)$$

(- sign needed to always work opposite to velocity)

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We then obtain the equation

$$mu''(t) + \gamma u'(t) + kL + ku(t) - mg = 0$$

Recall that $mg - kL = 0$, so

$$\boxed{mu'' + \gamma u' + ku = 0}$$

is an equation modeling the displacement u of a mass m on a spring with spring constant k affected by a damper with coefficient γ.

Ex 1. A 500 gram mass stretches a spring 5 cm. Suppose the mass is stretched an additional 2 cm downward from its initial position and released. If a damper with coefficient 20 dyn·s/cm acts on the mass, set up an IVP modelling this situation.

Make sure units are consistent!

$1 \text{ dyne} = 1 \ \frac{g \cdot cm}{s^2}$

$g = 9.8 \ m/s^2 = \frac{9.8 \ m}{s^2} \cdot \frac{100 \ cm}{1 \ m} = 980 \ cm/s^2$

$mg - kL = 0$

$(500 \ g)(980 \ cm/s^2) - k(5 \ cm) = 0$

$k = 98{,}000 \ \frac{g}{s^2}$

$\gamma = 20 \ \frac{dyn \cdot s}{cm}, \quad m = 500 \ g$

$500u'' + 20u' + 98{,}000u = 0$

Initially stretched 2 cm downward (positive direction), so $u(0) = 2$

"Released" implies initial velocity is 0, so $u'(0) = 0$

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$$\boxed{500u'' + 20u' + 98{,}000u = 0, \quad u(0) = 2, \quad u'(0) = 0}$$

$u(t)$ is measured in cm, t in seconds.

<u>Undamped Free Vibrations.</u>

If we don't have any damping, our diff eq looks like $mu'' + ku = 0$.

This leads to a solution of the form

$$A\cos(\alpha t) + B\sin(\alpha t)$$

For physical processes, it is often nice to describe oscillation in terms of amplitude, frequency, and period. To do this, we would like to express our function as

$$u = R\cos(\omega_0 t - \delta)$$

ω_0 is the frequency, R is the amplitude, δ is the phase, and $T = \frac{2\pi}{\omega_0}$ is the period.

Using the angle difference formula,

$$R\cos(\omega_0 t - \delta) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t)$$

So we get $A = R\cos(\delta)$, $B = R\sin(\delta)$

Squaring both, adding, and taking the square root...

$$\sqrt{A^2 + B^2} = \sqrt{R^2\cos^2(\delta) + R^2\sin^2(\delta)} = \sqrt{R^2(\cos^2\delta + \sin^2\delta)}$$
$$= \sqrt{R^2} = R \quad (R > 0).$$

So $\boxed{R = \sqrt{A^2 + B^2}}$

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Dividing, we get

$$\frac{B}{A} = \frac{R\sin(\delta)}{R\cos(\delta)} = \tan(\delta),$$

so $\boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$ (be careful of quadrant here!)

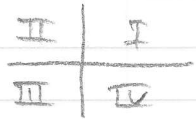
Ex 2. Write $-\cos(3t) + \sqrt{3}\sin(3t)$ in $R\cos(\omega_0 t - \delta)$ form.

Know: $A = -1$, $B = \sqrt{3}$, $\omega_0 = 3$

$$R = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$\delta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3})$$

But quadrant considerations...



$\tan^{-1}$ only gives values in quadrants I and IV

Since A is negative, $\cos\delta$ is negative.

Since B is positive, $\sin\delta$ is positive.

Therefore, δ is in quadrant II.

Thus, $\delta = \tan^{-1}(-\sqrt{3}) + \pi$

≈ 2.0944

so $\boxed{u \approx 2\cos(3t - 2.0944)}$

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If we consider damping again, we don't have a frequency or period, but we can get something close.

If our solution happens to be of the form $u = Ae^{\lambda t}\cos(\mu t) + Be^{\lambda t}\sin(\mu t)$, then we can write $u = e^{\lambda t}(A\cos(\mu t) + B\sin(\mu t))$

$$u = e^{\lambda t} R\cos(\mu t - \delta)$$

$Re^{\lambda t}$ is a sort of quasi-amplitude

μ is the <u>quasi-frequency</u>

$T_d = \frac{2\pi}{\mu}$ is the <u>quasi-period</u>.

If our solution is not of this form, then the solution to $mu'' + \gamma u' + ku = 0$ has characteristic polynomial with 1 repeated root of 2 distinct real roots.

$$mr^2 + \gamma r + k = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

If $\gamma^2 = 4mk$ (so we have a repeated root), the system is said to be <u>critically damped</u>.

If $\gamma^2 > 4mk$ (distinct real roots), the system is said to be <u>overdamped</u>.

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Ex 3. A mass of 700 grams stretches a spring 8 cm. Find the value of γ for which the system is critically damped.

Write $m = 0.7$ kg, $L = 0.08$ m

$$mg - kL = 0$$
$$(0.7)(9.8) - k(0.08) = 0$$
$$k = 85.75 \ \frac{kg}{s^2}$$

$$0.7u'' + \gamma u' + 85.75u = 0$$

Want $\gamma^2 = 4(0.7)(85.75)$

$$\gamma^2 = 240.1$$
$$\gamma \approx 15.4952$$

$\gamma u'$ is a force, so the units of $\gamma u'$ is Newtons.

u' is velocity, so the units of u' is $\frac{m}{s}$

$$(\text{units of } \gamma) \cdot \frac{m}{s} = N$$
$$\text{units of } \gamma = \frac{N \cdot s}{m}$$

So $\boxed{\gamma \approx 15.4952 \ \frac{N \cdot s}{m}}$