Lesson 17

Forced Vibrations (3.8)

In lesson 16, we dealt with mass-spring systems in which only gravity, the spring force, and a damping force acted on the mass. In this lesson, we assume an additional force is acting on the mass, given by the equation $g(t)$. Such a function is called a forcing function.

Again, using $F = ma$, we get

$$\underbrace{mg}_{\text{gravity}} - \underbrace{\gamma u'}_{\text{damping force}} - \underbrace{k(L+u)}_{\text{spring force}} + \underbrace{g(t)}_{\text{additional force}} = mu''$$

We end up with (recall $mg - kL = 0$)

$$mu'' + \gamma u' + ku = g(t)$$

So we get a nonhomogeneous equation.

Generally, we will focus on when $g(t)$ is a periodic function ($g(t) = \cos(\alpha t)$ or $\sin(\alpha t)$), so the Method of Undetermined Coefficients will be extremely useful.

When doing these problems, always be very careful about units! Specify what your function represents!

Lesson 17

Ex 1. A mass of 2 kg stretches a spring 8 cm. The mass is acted on by an external force of $3\cos(2t)$ N and has a damping force which has magnitude 3N when the mass has a speed of 2 cm/s. If the mass is initially at rest in its equilibrium position, set up an IVP for the position of the mass at time t.

$1\text{ N} = 1 \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$. Notice that a Newton implies we are working with kg, m, and s, so convert all units accordingly

$m = 2\text{ kg}$, $L = 8\text{ cm} = 0.08\text{ m}$

speed of $2 \frac{\text{cm}}{\text{s}} = 2 \frac{\text{cm}}{\text{s}} \cdot \frac{1\text{ m}}{100\text{ cm}} = 0.02\text{ m/s}$

$$mg - kL = 0$$
$$(2\text{ kg})(9.8\text{ m/s}^2) - k(0.08\text{ m}) = 0$$
$$k = 245 \text{ kg/s}^2$$

Damping force is $\gamma u'$, where u' is velocity. If the damping force is 3N when velocity is 0.02 m/s, then $\gamma(0.02\text{ m/s}) = 3\text{N}$

$$\gamma = 150 \frac{\text{N}\cdot\text{s}}{\text{m}}$$

$$\boxed{2u'' + 150u' + 245u = 3\cos(2t),\ u(0) = 0,\ u'(0) = 0}$$

u is measured in meters, t in seconds

Lesson 17

If a periodic external force is applied to a vibrating system, there is an important concept called resonance.

If the frequency ω of the external force is equal to ~~(or approximately equal to)~~ the natural frequency ω_0 of the vibrating system, you have resonance. Resonance has a huge impact on the system (even when the external force is relatively small). This could be beneficial or disastrous, depending on context.

For example, in a seismograph, resonance can allow us to detect minor earthquakes which we could not feel ourselves. Resonance is used as a way to tune instruments. If two instruments have similar (but not equal) frequency, a "beat" in the sound develops, and resonance occurs when in tune.
If a bridge or building has a natural frequency which is similar to the frequency of an external force, it can cause the collapse and destruction of the structure.

Ex 2. A mass weighing 3 lb stretches a spring 1 in. Assume there is no damping on the mass. An external force of $2\cos(\omega t)$ lb is applied. Determine the value of ω for which resonance occurs.

First, unit of force is lb. Standard unit for distance is <u>feet</u>.

$mg = 3\,\text{lb}$, $g = 32\ \text{ft/s}^2$, so $m = \frac{3}{32}\ \frac{\text{lb}\cdot\text{s}^2}{\text{ft}} = \frac{3}{32}$ slugs

$L = 1\ \text{in} = \frac{1}{12}\ \text{ft}$.

$$mg - kL = 0$$
$$(3\,\text{lb}) - k\left(\tfrac{1}{12}\ \text{ft}\right) = 0$$
$$k = 36\ \text{lb/ft}$$

$$\frac{3}{32}u'' + 36u = 2\cos(\omega t)$$

Natural frequency of the system is found through the complementary (homogeneous) solution:

$$\frac{3}{32}r^2 + 36 = 0$$
$$r^2 = -384$$
$$r = \pm i\sqrt{384} = \pm 8\sqrt{6}\,i$$
$$y_c(t) = c_1\cos(8\sqrt{6}\,t) + c_2\sin(8\sqrt{6}\,t)$$
$$= R\cos(8\sqrt{6}\,t - \delta)$$
$$\omega_0 = 8\sqrt{6}$$

so resonance occurs when external force has frequency $\boxed{\omega = 8\sqrt{6}}$