Lesson 18

Higher Order Linear Equations (4.1, 4.2)

Since primes become cumbersome when there are many of them, we often denote the nth derivative of y as $y^{(n)}$, with a superscript (n) in parantheses.

Every nth order linear ODE can be expressed as

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \dots + p_1(t)\, y' + p_0(t)\, y = g(t)$$

For an initial value problem, we requrire n initial conditions: $y(t_0) = y_0,\ y'(t_0) = y'_0,\ \dots,\ y^{(n-1)}(t_0) = y^{(n-1)}_0$

<u>Thm 4.1.1 (Existence and Uniqueness)</u> If, as described above, $p_{n-1}, p_{n-2}, \dots, p_1, p_0$, and g are continuous on an open interval $I: \alpha < t < \beta$, then solutions to the diff eq exist. If we are given n initial conditions as above (with t_0 in I), then the solution to the IVP is unique.

<u>Ex 1</u>. Determine intervals in which solutions are sure to exist.

$$(t-8)\, y^{(5)} + t^2 y^{(3)} + \sin(t)\, y'' + 4y = \sqrt{t}$$

$$y^{(5)} + \frac{t^2}{t-8} y^{(3)} + \frac{\sin t}{t-8} y'' + \frac{4}{t-8} y = \frac{\sqrt{t}}{t-8}$$

discontinuities at $t = 8$ and $t < 0$

possibilities $\boxed{0 < t < 8 \text{ or } t > 8}$

Homogeneous Equations with Constant Coefficients

Similarly to 2nd order, if $y(t) = e^{rt}$ is a solution, we find r must satisfy a characteristic equation.

Given $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$, the characteristic equation is

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

- If r is a real root, then e^{rt} is a solution.
- If $\lambda \pm \mu i$ are complex conjugate roots, then $e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin(\mu t)$ are solutions.
- If a real root r is repeated k times, $e^{rt}, te^{rt}, t^2 e^{rt}, \dots, t^{k-1} e^{rt}$ are solutions
- If complex conjugate roots $\lambda \pm \mu i$ are repeated k times each, $e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$, $te^{\lambda t}\cos(\mu t)$, $te^{\lambda t}\sin(\mu t)$, $\dots$, $t^{k-1}e^{\lambda t}\cos(\mu t)$, $t^{k-1}e^{\lambda t}\sin(\mu t)$ are solutions.

Ex 2. Write the general solution if the characteristic polynomial has roots:
$r_1 = 2$, $r_2 = 3$, $r_3 = 3$, $r_4 = 3$, $r_5 = 2+3i$, $r_6 = 2+3i$, $r_7 = 2-3i$, $r_8 = 2-3i$

$$\boxed{\begin{aligned} y(t) = {} & c_1 e^{2t} + c_2 e^{3t} + c_3 te^{3t} + c_4 t^2 e^{3t} + c_5 e^{2t}\cos(3t) \\ & + c_6 e^{2t}\sin(3t) + c_7 te^{2t}\cos(3t) + c_8 te^{2t}\sin(3t) \end{aligned}}$$

Roots of Polynomials

How do we find roots of polynomials? Sometimes we can factor nicely:

- $r^3 - 3r^2 + 4r - 12 = 0$

  $r^2(r-3) + 4(r-3) = 0$

  $(r^2+4)(r-3) = 0$

  $r = 2i, \ r = -2i, \ r = 3$

- If we know one root α, we can divide the polynomial by $r - \alpha$. This leaves a polynomial one degree lower.

- If $a_n r^n + \dots + a_1 r + a_0 = 0$, and assume $r = \frac{p}{q}$ is a rational number root.

  It follows that p divides a_0 and q divides a_n.

  So our only choices are quotients of a factor of a_0 and a factor of a_n. (This only works for rational roots)

Ex 3. Find the general solution of

$$y^{(4)} + y''' - 5y'' + y' - 6y = 0$$

$$r^4 + r^3 - 5r^2 + r - 6 = 0$$

Factors of -6: $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of 1: ± 1

Possible rational roots: $\frac{\pm 1}{\pm 1}, \frac{\pm 2}{\pm 1}, \frac{\pm 3}{\pm 1}, \frac{\pm 6}{\pm 1}$

$$\pm 1, \pm 2, \pm 3, \pm 6$$

We can plug each of these in to check whether or not it is a root.

We get both $r_1 = 2$ and $r_2 = -3$ as roots.
No others work.
The polynomial is divisible by $(r-2)(r+3)$
$= r^2 + r - 6$

$$\begin{array}{r|l} & r^2 + 1 \\ \hline r^2+r-6 & r^4 + r^3 - 5r^2 + r - 6 \\ & -(r^4 + r^3 - 6r^2) \\ \hline & \qquad r^2 + r - 6 \\ & \quad -(r^2 + r - 6) \\ \hline & \qquad\qquad 0 \end{array}$$

So roots are also roots of $r^2 + 1$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

so $r_1 = 2,\ r_2 = -3,\ r_3 = i,\ r_4 = -i$

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-3t} + c_3 \cos(t) + c_4 \sin(t)}$$

Ex 4. Solve the IVP. Describe the behavior of the solution as $t \to \infty$.

$6y''' + 5y'' + y' = 0;\ y(0) = -2,\ y'(0) = 2,\ y''(0) = 0$

$6r^3 + 5r^2 + r = 0$
$r(6r^2 + 5r + 1) = 0$
$r(3r+1)(2r+1) = 0$
$r_1 = 0,\ r_2 = -\frac{1}{3},\ r_3 = -\frac{1}{2}$

$y(t) = c_1 + c_2 e^{-t/3} + c_3 e^{-t/2}$

$y'(t) = -\frac{c_2}{3} e^{-t/3} - \frac{c_3}{2} e^{-t/2}$
$y''(t) = \frac{c_2}{9} e^{-t/3} + \frac{c_3}{4} e^{-t/2}$

$$-2 = y(0) = c_1 + c_2 + c_3$$
$$2 = y'(0) = -\tfrac{1}{3}c_2 - \tfrac{1}{2}c_3$$
$$0 = y''(0) = \tfrac{1}{9}c_2 + \tfrac{1}{4}c_3$$

$$\begin{cases} c_1 + c_2 + c_3 = -2 \\ -\tfrac{1}{3}c_2 - \tfrac{1}{2}c_3 = 2 \\ \tfrac{1}{9}c_2 + \tfrac{1}{4}c_3 = 0 \end{cases}$$

I'm fine with you solving this with matrices or computers: put $\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -\frac{1}{3} & -\frac{1}{2} & 2 \\ 0 & \frac{1}{9} & \frac{1}{4} & 0 \end{bmatrix}$ in RREF

By hand:

$$\begin{aligned} -\tfrac{1}{3}c_2 - \tfrac{1}{2}c_3 &= 2 \\ \tfrac{1}{3}c_2 + \tfrac{3}{4}c_3 &= 0 \\ \hline \tfrac{1}{4}c_3 &= 2 \\ c_3 &= 8 \end{aligned}$$

$$-\tfrac{1}{3}c_2 - \tfrac{1}{2}(8) = 2$$
$$-\tfrac{1}{3}c_2 - 4 = 2$$
$$c_2 = -18$$

$$c_1 + (-18) + (8) = -2$$
$$c_1 = 8$$

$$\boxed{y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}}$$

Graph this!

y initially decreases exponentially, then increases slowly, tending toward 8 as $t \to \infty$.