

Lesson 20

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The Laplace Transform (6.1)

Review of Improper Integrals

In this section, we deal with integrals of the form $\int_a^\infty f(t) dt$. Recall that these integrals are defined as

$$\int_a^\infty f(t) dt = \lim_{A \rightarrow \infty} \int_a^A f(t) dt$$

If the limit converges, the integral converges, and if the limit diverges, the integral diverges.

Ex 1. Find the values of s so that $\int_0^\infty e^{st} dt$ converges and find the value of the integral when it does.

$$\int_0^\infty e^{st} dt = \lim_{A \rightarrow \infty} \int_0^A e^{st} dt$$

$$= \lim_{A \rightarrow \infty} \left. \frac{e^{st}}{s} \right|_0^A$$

$$= \lim_{A \rightarrow \infty} \left[\frac{e^{sA}}{s} - \frac{1}{s} \right]$$

$$= \frac{(\lim_{A \rightarrow \infty} e^{sA}) - 1}{s}$$

This converges only when $s < 0$
and converges to

$$\boxed{-\frac{1}{s}}$$

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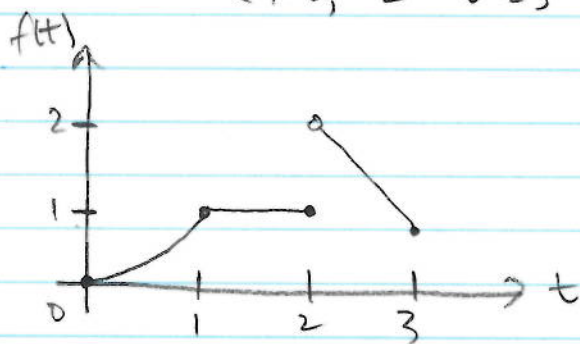
In order to take the integral of a function, we need the function to be mainly continuous. It doesn't have to be continuous everywhere, just almost everywhere. This leads to the notion of piecewise continuity:

Defn. A function $f(t)$ is piecewise continuous on an interval $\alpha < t < \beta$ (including possibly $\alpha = -\infty$ and/or $\beta = \infty$) if there exist finitely many points t_1, \dots, t_n in $\alpha < t < \beta$ such that $f(t)$ is continuous on $t_1 < t < t_2$, $t_2 < t < t_3$, \dots , $t_{n-1} < t < t_n$ and that for all k , $\lim_{t \rightarrow t_k^+} f(t)$ exists and $\lim_{t \rightarrow t_{k+1}^-} f(t)$ exists.

In other words, $f(t)$ is piecewise continuous on $\alpha < t < \beta$ if $f(t)$ has at most finitely many discontinuities and each discontinuity is either a jump or removable discontinuity (no infinite discontinuities).

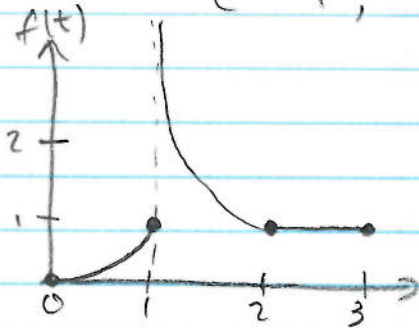
Ex 2. Are the following functions piecewise continuous?

$$(a) f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 2 \\ 4-t, & 2 < t \leq 3 \end{cases}$$



yes, piecewise continuous on $0 < t < 3$.

$$(b) f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ (t-1)^{-1}, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \end{cases}$$



not piecewise continuous
since it has an infinite
discontinuity

If a function $f(t)$ is piecewise continuous on $\alpha < t < \beta$ and its discontinuities occur at t_1, \dots, t_n , then

$$\int_{\alpha}^{\beta} f(t) dt = \int_{\alpha}^{t_1} f(t) dt + \int_{t_1}^{t_2} f(t) dt + \dots + \int_{t_n}^{\beta} f(t) dt$$

Definition. If $f(t)$ is piecewise continuous or continuous, the Laplace Transform of $f(t)$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

provided that the integral converges for some values of s .

The Laplace transform $F(s)$ is a function of s , but s is treated like a constant in the actual integral, since we are integrating with respect to t .

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Ex 3. Let $f(t) = e^{at}$ for a constant a .
Compute $\mathcal{L}\{f(t)\}$ and give the domain.

$$F(s) = \mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \lim_{A \rightarrow \infty} \int_0^A e^{(-s+a)t} dt$$

$$= \lim_{A \rightarrow \infty} \left[\frac{e^{(-s+a)t}}{(-s+a)} \right]_{t=0}^{t=A}$$

$$= \lim_{A \rightarrow \infty} \left[\frac{e^{(-s+a)A}}{(-s+a)} - \frac{1}{(-s+a)} \right]$$

by example 1, converges
if and only if $-s+a < 0$, i.e., $s > a$

When $s > a$,

$$0 - \frac{1}{-s+a} = \frac{1}{s-a}$$

$$\text{So } \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a}$$

$$\text{e.g., if } a=5, \mathcal{L}\{e^{5t}\} = \frac{1}{s-5}, s > 5$$

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Ex. 5

Ex 5. Let $f(t) = \sin(bt)$. Compute $\mathcal{L}\{\sin(bt)\}$.

By Euler's Formula, $\sin bt = \frac{e^{ibt} - e^{-ibt}}{2i}$

$$\mathcal{L}\{\sin bt\} = \int_0^{\infty} e^{-st} \left(\frac{e^{ibt} - e^{-ibt}}{2i} \right) dt$$

$$= \frac{1}{2i} \lim_{A \rightarrow \infty} \int_0^A (e^{(-s+ib)t} - e^{(-s-ib)t}) dt$$

$$= \frac{1}{2i} \lim_{A \rightarrow \infty} \left[\frac{e^{(-s+ib)t}}{-s+ib} - \frac{e^{(-s-ib)t}}{-s-ib} \right]_{t=0}^{t=A}$$

$$= \frac{1}{2i} \lim_{A \rightarrow \infty} \left[\frac{e^{(-s+ib)A}}{-s+ib} - \frac{e^{(-s-ib)A}}{-s-ib} - \frac{1}{-s+ib} + \frac{1}{-s-ib} \right]$$

By Euler's Formula, $e^{(-s+ib)A} = e^{-sA}(\cos(bA) + i\sin(bA))$

which converges if and only if e^{-sA} converges,

so when $-s < 0$, i.e., $s > 0$.

$$= \frac{1}{2i} \left[-\frac{1}{-s+ib} + \frac{1}{-s-ib} \right]$$

$$= \frac{1}{2i} \left[\frac{s+ib}{(s^2+b^2)} + \frac{-s+ib}{(s^2+b^2)} \right]$$

$$= \frac{1}{2i} \left[\frac{2ib}{s^2+b^2} \right] = \frac{b}{s^2+b^2}$$

$$\text{so } \boxed{\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad s > 0}$$