

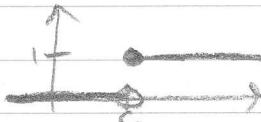
# Lesson 22

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## Step (Heaviside) Functions (6.3)

The real power of the Laplace Transform comes from being able to handle discontinuous forcing functions. Before, the techniques we worked with did not allow the nonhomogeneous part of an ODE to have discontinuities. To solve diff eqs with discontinuous forcing functions, we look at step functions.

The unit step function or Heaviside function with a jump at  $c$  is defined as

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$


Ex 1. Sketch the function and describe it as a piecewise defined function

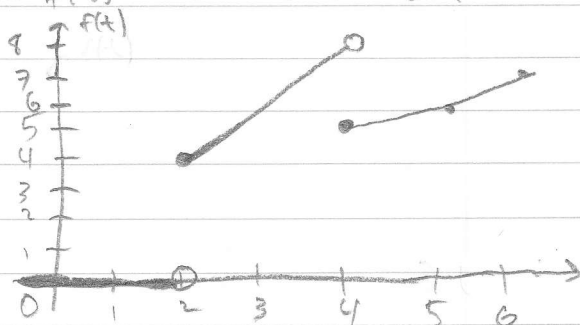
$$f(t) = 2t u_2(t) - (t-1) u_4(t)$$

When  $t < 2$ ,  $u_2(t) = u_4(t) = 0$ , so  $f(t) = 0$

$2 \leq t < 4$ ,  $u_2(t) = 1$ , so  $f(t) = 2t$

$t \geq 4$ ,  $u_4(t) = 1$ , so  $f(t) = 2t - (t-1) = t+1$

$$f(t) = \begin{cases} 0, & t < 2 \\ 2t, & 2 \leq t < 4 \\ t+1, & t \geq 4 \end{cases}$$



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Ex 2. Write  $f(t)$  in terms of step functions.

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ t-1, & 1 \leq t < 2 \\ t-2, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

First part  $f(t) = u_0(t) \cdot t + \text{stuff}$

For  $1 \leq t < 2$ , want  $t-1 = t + g(t)u_1(t)$

Clearly,  $g(t) = -1$

So  $f(t) = u_0(t)t - u_1(t) + \text{stuff}$

For  $2 \leq t < 3$ , want  $t-2 = t-1 + g(t)u_2(t)$

Need  $g(t) = -1$  again

so  $f(t) = t u_0(t) - u_1(t) - u_2(t) + \text{stuff}$

For  $t \geq 3$ , want  $0 = t-2 + g(t)u_3(t)$

Need  $g(t) = -t+2$

$$\text{so } \boxed{f(t) = t u_0(t) - u_1(t) - u_2(t) - (t-2)u_3(t)}$$

Now, how does the Laplace Transform interact with step functions?

Thm 6.3.1 If  $F(s) = \mathcal{L}\{f(t)\}$  exists for  $s > a \geq 0$ ,

and if  $c$  is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s).$$

Conversely, if  $f(t) = \mathcal{L}^{-1}\{F(s)\}$ , then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}.$$

$$\begin{aligned} \text{Proof. } \mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^{\infty} e^{-st} u_c(t) f(t-c) dt \\ &= \int_c^{\infty} e^{-st} f(t-c) dt \quad (\text{since } u_c(t) = 0 \text{ for } t < c) \end{aligned}$$

Let  $\xi = t-c$ . Then  $d\xi = dt$ , and  $t = \xi + c$

$$= \int_0^{\infty} e^{-(\xi+c)s} f(\xi) d\xi$$

$$= \int_0^{\infty} e^{-cs} e^{-\xi s} f(\xi) d\xi$$

$$= e^{-cs} \int_0^{\infty} e^{-\xi s} f(\xi) d\xi = e^{-cs} \mathcal{L}\{f(\xi)\}$$

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Notice that we have  $f(t-c)$ . This is important!

Ex 3. Compute  $\mathcal{L}\{f(t)\}$  where

$$f(t) = u_1(t) + t u_2(t) + (t^2+1) u_4(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{t u_2(t)\} + \mathcal{L}\{(t^2+1) u_4(t)\}$$

$$t = g(t-2) \text{ for some function } g(t).$$

$$\text{Plug in } t+2 \text{ for } t. \text{ Then } t+2 = g(t)$$

$$t^2+1 = h(t-4) \text{ for some function } h(t).$$

$$\text{Plug in } t+4 \text{ for } t. \text{ Then } t^2+8t+17 = (t+4)^2+1 = h(t)$$

$$\text{So } \mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} + e^{-2s} G(s) + e^{-4s} H(s)$$

$$\text{where } G(s) = \mathcal{L}\{g(t)\}, H(s) = \mathcal{L}\{h(t)\}$$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t+2\} = \mathcal{L}\{t\} + \mathcal{L}\{2\} = \frac{1}{s^2} + \frac{2}{s} = \frac{1+2s}{s^2}$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{t^2\} + 8\mathcal{L}\{t\} + \mathcal{L}\{17\} = \frac{2}{s^3} + \frac{8}{s^2} + \frac{17}{s} = \frac{2+8s+17s^2}{s^3}$$

$$\text{So } \mathcal{L}\{f(t)\} = \frac{e^{-s}s^2}{s^3} + \frac{e^{-2s}(1+2s)s}{s^2} + \frac{e^{-4s}(2+8s+17s^2)}{s^3}$$

$$= \left[ e^{-s}s^2 + e^{-2s}(1+2s) + e^{-4s}(2+8s+17s^2) \right] / s^3$$

Ex 4. Compute  $\mathcal{L}^{-1}\{F(s)\}$  where

$$F(s) = \frac{e^{-2s}}{s^2+s-2}$$

$$F(s) = e^{-2s} \left( \frac{1}{s^2+s-2} \right) = e^{-2s} \left( \frac{1}{(s+2)(s-1)} \right)$$

$$\frac{1}{(s+2)(s-1)} = \frac{a}{s+2} + \frac{b}{s-1} \Rightarrow a(s-1) + b(s+2) = 1$$

$$s=1 \Rightarrow 3b=1 \Rightarrow b = \frac{1}{3}, \quad s=-2 \Rightarrow -3a=1 \Rightarrow a = -\frac{1}{3}$$

$$F(s) = -\frac{1}{3} e^{-2s} \left( \frac{1}{s+2} \right) + \frac{1}{3} e^{-2s} \left( \frac{1}{s-1} \right)$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{3} \mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s+2}\} + \frac{1}{3} \mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s-1}\}$$

$$= -\frac{1}{3} u_2(t) g(t-2) + \frac{1}{3} u_2(t) h(t-2)$$

$$\text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}, \quad h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

$$\text{so } g(t-2) = e^{-2(t-2)} = e^{-2t+4}, \quad h(t-2) = e^{(t-2)} = e^{t-2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \left( -\frac{1}{3} e^{-2t+4} + \frac{1}{3} e^{t-2} \right) u_2(t)$$

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Thm 6.3.2. If  $F(s) = \mathcal{L}\{f(t)\}$  for  $s > a \geq 0$

and if  $c$  is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad s > a+c.$$

Conversely, if  $f(t) = \mathcal{L}^{-1}\{F(s)\}$ , then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof.  $\mathcal{L}\{e^{ct} f(t)\} = \int_0^{\infty} e^{-st} e^{ct} f(t) dt$   
 $= \int_0^{\infty} e^{-(s-c)t} f(t) dt$   
 $= F(s-c)$

Ex 5. Compute  $\mathcal{L}\{f(t)\}$  where

$$f(t) = e^{3t} \sin(t)$$

$$\mathcal{L}\{f(t)\} = G(s-3) \quad \text{where } G(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\text{Then } G(s-3) = \frac{1}{(s-3)^2+1} = \frac{1}{s^2-6s+10}$$

$$\text{So } \boxed{\mathcal{L}\{f(t)\} = \frac{1}{s^2-6s+10}}$$

Ex 6. Compute  $\mathcal{L}^{-1}\{F(s)\}$  where

$$F(s) = \frac{3s+6}{s^2+4s+8}$$

$$\text{Notice, } F(s) = \frac{3(s+2)}{s^2+4s+4+4-\frac{4}{4}} = \frac{3(s+2)}{(s+2)^2+4} = 3G(s+2)$$

$$\text{where } G(s) = \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 3 \mathcal{L}^{-1}\{G(s+2)\} = 3e^{2t} g(t)$$

$$\text{where } g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$\text{So } \boxed{\mathcal{L}^{-1}\{F(s)\} = 3e^{2t} \cos 2t}$$