

Lesson 23

Diff Eqs with Discontinuous Forcing Functions (6.4)

In this lesson, we use our knowledge of step functions, the Laplace Transform, and partial fractions to solve IVPs which have discontinuous forcing functions (nonhomogeneous part).

Directions for Examples: Solve the IVP

Ex 1. $y'' + 3y' + 2y = \begin{cases} 1, & 0 \leq t < 10 \\ 0, & t \geq 10 \end{cases}$; $y(0) = 0, y'(0) = 0$
 $A(t)$

$$A(t) = 1 - u_{10}(t)$$

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{10}(t)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$s^2 Y(s) + 3s Y(s) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$(s^2 + 3s + 2) Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} - \frac{e^{-10s}}{s(s^2 + 3s + 2)} \quad s^2 + 3s + 2 = (s+2)(s+1)$$

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} \Rightarrow 1 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$$

$$s=0 \Rightarrow 2A=1 \Rightarrow A=\frac{1}{2}, s=-2 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}, s=-1 \Rightarrow -C=1 \Rightarrow C=-1$$

$$Y(s) = \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{s+1} + e^{-10s} \left(\frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{s+1} \right)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}(1) + \frac{1}{2}e^{-2t} - e^{-t} + u_{10}(t) \left[\frac{1}{2}g(t-10) + \frac{1}{2}h(t-10) - j(t-10) \right]$$

$$\text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t},$$

$$j(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$g(t-10)=1, h(t-10)=e^{-2t+20}, j(t-10)=e^{-t+10}$$

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t) \left[\frac{1}{2} + \frac{1}{2}e^{-2t+20} - e^{-t+10} \right]$$

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Ex 2. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \frac{\pi}{2})$; $y(0) = 0$, $y'(0) = 0$
 $t - \frac{\pi}{2} = f(t - \frac{\pi}{2})$, so $f(t) = t$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{t\} - \mathcal{L}\{u_{\pi/2}(t)f(t - \frac{\pi}{2})\}$$

$$s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) + \frac{5}{4}Y(s) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$$

$$(s^2 + s + \frac{5}{4})Y(s) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2 + s + \frac{5}{4})} - \frac{e^{-\pi/2 s}}{s^2(s^2 + s + \frac{5}{4})}$$

$$\frac{1}{s^2(s^2 + s + \frac{5}{4})} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + s + \frac{5}{4}}$$

$$1 = As(s^2 + s + \frac{5}{4}) + B(s^2 + s + \frac{5}{4}) + (Cs + D)s^2$$

$$1 = \underbrace{As^3 + As^2 + \frac{5}{4}As}_{A+C} + \underbrace{Bs^2 + Bs + \frac{5}{4}B}_{A+B+D} + \underbrace{Cs^3 + Ds^2}_{\frac{5}{4}A+B}$$

$$\begin{cases} A+C=0 \\ A+B+D=0 \\ \frac{5}{4}A+B=1 \end{cases} \Rightarrow B = \frac{4}{5}, A = -\frac{16}{25}, C = \frac{16}{25}, D = \frac{4}{25}$$

Also, $\frac{\frac{16}{25}s + \frac{4}{25}}{s^2 + s + \frac{5}{4}} = \frac{\frac{16}{25}(s+1) - \frac{8}{25}}{(s+1)^2 + \frac{1}{4}}$

$$= \frac{16}{25} \frac{(s+1)}{(s+1)^2 + \frac{1}{4}} - \frac{16}{25} \frac{\frac{1}{4}}{(s+1)^2 + \frac{1}{4}}$$

$$Y(s) = \left[-\frac{16}{25} \cdot \frac{1}{s} + \frac{4}{5} \cdot \frac{1}{s^2} + \frac{16}{25} \frac{(s+1)}{(s+1)^2 + \frac{1}{4}} - \frac{16}{25} \cdot \frac{\frac{1}{4}}{(s+1)^2 + \frac{1}{4}} \right] (1 - e^{-\pi/2 s})$$

$$y(t) = -\frac{16}{25} + \frac{4}{5}t + \frac{16}{25}e^{-t/2}\cos(t) - \frac{16}{25}e^{-t/2}\sin(t) - u_{\pi/2}(t) \left[-\frac{16}{25} + \frac{4}{5}(t - \frac{\pi}{2}) + \frac{16}{25}e^{-t/2 + \pi/4}\cos(t - \frac{\pi}{2}) - \frac{16}{25}e^{-t/2 + \pi/4}\sin(t - \frac{\pi}{2}) \right]$$

Ex 3. $y'' + y = u_{3\pi}(t)$; $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{3\pi}(t)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}$$

$$s^2 Y(s) - s + Y(s) = \frac{e^{-3\pi s}}{s}$$

$$(s^2 + 1) Y(s) = s + \frac{e^{-3\pi s}}{s}$$

$$Y(s) = \frac{s}{s^2 + 1} + \frac{e^{-3\pi s}}{s(s^2 + 1)}$$

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \Rightarrow 1 = As^2 + A + Bs^2 + Cs$$

$$\begin{cases} A + B = 0 \\ C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1, B = -1, C = 0$$

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$Y(s) = \frac{s}{s^2 + 1} + e^{-3\pi s} \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

$$y(t) = \cos t + u_{3\pi}(t) [f(t - 3\pi) - g(t - 3\pi)]$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad g(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} = \cos t$$

$$f(t - 3\pi) = 1, \quad g(t - 3\pi) = \cos(t - 3\pi) = \cos(t - \pi) = -\cos(t)$$

$$y(t) = \cos t + u_{3\pi}(t) [1 + \cos t]$$

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Ex 4. $y'' - y = u_3(t)$ $y(0) = 0, y'(0) = 0$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{u_3(t)\}$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{e^{-3s}}{s}$$

$$(s^2 - 1) Y(s) = \frac{e^{-3s}}{s}$$

$$Y(s) = \frac{e^{-3s}}{s(s^2 - 1)}$$

$$\frac{1}{s(s^2 - 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 1}$$

$$1 = A(s^2 - 1) + (Bs + C)s = As^2 - A + Bs^2 + Cs$$

$$\begin{cases} A + B = 0 \\ C = 0 \\ -A = 1 \end{cases} \Rightarrow A = -1, B = 1, C = 0$$

$$Y(s) = e^{-3s} \left(-\frac{1}{s} + \frac{s}{s^2 - 1} \right)$$

$$y(t) = u_3(t) [-f(t-3) + g(t-3)]$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad g(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 1}\right\} = \cosh t$$

$$y(t) = u_3(t) [\cosh(t-3) - 1]$$

$$\text{check: } y(t) = \begin{cases} 0, & 0 \leq t < 3 \\ \cosh(t-3) - 1, & t \geq 3 \end{cases} = u_3(t) [\cosh(t-3) - 1]$$

$$y'(t) = \begin{cases} 0, & 0 \leq t < 3 \\ \sinh(t-3), & t \geq 3 \end{cases}$$

$$y''(t) = \begin{cases} 0, & 0 \leq t < 3 \\ \cosh(t-3), & t \geq 3 \end{cases} = u_3(t) \cosh(t-3)$$

$$y''(t) - y(t) = u_3 \cosh(t-3) - u_3(t) [\cosh(t-3) - 1] = u_3(t) \quad \checkmark$$