

Impulse Functions (6.5)

Sometimes we want to deal with a forcing function $g(t)$ which is very large in magnitude for a very small amount of time, say $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and $g(t) = 0$ elsewhere.

We can describe the impulse of this function,

$$I(\tau) = \int_{t_0 - \tau}^{t_0 + \tau} g(t) dt = \int_{-\infty}^{\infty} g(t) dt$$

Let's assume $t_0 = 0$, and that

$$g(t) = d\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{elsewhere} \end{cases}$$

where τ is a small positive constant.

$$I(\tau) = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left[\frac{t}{2\tau} \right]_{t=-\tau}^{t=\tau} = \frac{\tau}{2\tau} - \frac{-\tau}{2\tau} = 1$$

so the impulse of this function is 1, regardless of the value of τ .

We now define a function $\delta(t) = \lim_{\tau \rightarrow 0^+} d\tau(t)$.

Notice: $\lim_{\tau \rightarrow 0^+} I(\tau) = \lim_{\tau \rightarrow 0^+} \int_{-\tau}^{\tau} d\tau(t) dt = \lim_{\tau \rightarrow 0^+} 1 = 1$,
so we define $\int_{-\infty}^{\infty} \delta(t) dt = 1$.

So we obtain the unit impulse function or Dirac delta function $\delta(t)$ which is defined by the properties:

$$\delta(t) = 0 \text{ for all } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

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So if we have a situation which calls for an impulse of value A at time t_0 , we can represent that by

$A\delta(t-t_0)$, which has the properties

$$A\delta(t-t_0) = 0 \text{ for all } t \neq t_0$$

$$\int_{-\infty}^{\infty} A\delta(t-t_0) dt = A \int_{-\infty}^{\infty} \delta(t-t_0) dt = A$$

To work with such functions in the context of differential equations, we need to define $\mathcal{L}\{\delta(t-t_0)\}$. Like with the impulse of δ , we define:

$$\mathcal{L}\{\delta(t-t_0)\} := \lim_{\tau \rightarrow 0^+} \mathcal{L}\{d\tau(t-t_0)\}$$

$$\mathcal{L}\{d\tau(t-t_0)\} = \int_0^{\infty} e^{-st} d\tau(t-t_0) dt$$

$$= \int_{t_0-\tau}^{t_0+\tau} e^{-st} \frac{1}{2\tau} dt$$

$$= \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st} dt$$

$$= \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau})$$

$$= \frac{\sinh s\tau}{s\tau} e^{-st_0}$$

$$\text{Thus, } \mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau \rightarrow 0^+} \left(\frac{\sinh s\tau}{s\tau} \right) e^{-st_0}$$

$$\stackrel{\text{L'H}}{=} \lim_{\tau \rightarrow 0^+} \frac{s \cosh s\tau}{s} e^{-st_0}$$

$$= \frac{s}{s} e^{-st_0}$$

$$= e^{-st_0}$$

$$\text{So } \boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}}$$

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Ex 1. Solve the IVP.

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi)\} - \mathcal{L}\{\delta(t - 2\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$(s^2 + 4)Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 4} - \frac{e^{-2\pi s}}{s^2 + 4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2 + 4}\right\}$$

$$= u_{\pi}(t) f(t - \pi) - u_{2\pi}(t) f(t - 2\pi)$$

$$\text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} = \frac{1}{2} \sin(2t)$$

$$f(t - \pi) = \frac{1}{2} \sin[2(t - \pi)]$$

$$f(t - 2\pi) = \frac{1}{2} \sin[2(t - 2\pi)]$$

$$y(t) = u_{\pi}(t) \left[\frac{1}{2} \sin[2(t - \pi)] \right] - u_{2\pi}(t) \left[\frac{1}{2} \sin[2(t - 2\pi)] \right]$$

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Ex 2. Solve the IVP.

$$y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); y(0) = 0, y'(0) = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\delta(t - 3\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 3Y(s) = \frac{1}{s^2 + 1} + e^{-3\pi s}$$

$$(s^2 + 2s + 3)Y(s) = \frac{1}{s^2 + 1} + e^{-3\pi s}$$

$$Y(s) = \frac{1}{(s^2 + 1)(s^2 + 2s + 3)} + \frac{e^{-3\pi s}}{(s^2 + 2s + 3)}$$

$$\frac{1}{(s^2 + 1)(s^2 + 2s + 3)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 3}$$

$$1 = (As + B)(s^2 + 2s + 3) + (Cs + D)(s^2 + 1)$$

$$1 = As^3 + Bs^2 + 2As^2 + 2Bs + 3As + 3B + Cs^3 + Ds^2 + Cs + D$$

$$\begin{cases} A + C = 0 \\ 2A + B + D = 0 \\ 3A + 2B + C = 0 \\ 3B + D = 1 \end{cases} \Rightarrow A = -\frac{1}{4}, B = C = D = \frac{1}{4}$$

$$2A + B + D = 0$$

$$3A + 2B + C = 0$$

$$3B + D = 1$$

$$\text{Also } s^2 + 2s + 3 = s^2 + 2s + \underline{1} + 3 - \underline{1} = (s+1)^2 + 2$$

$$Y(s) = \frac{1}{4} \left(\frac{-\frac{s+1}{s^2+1} + \frac{s+1}{(s+1)^2+2}}{\frac{-s}{s^2+1} + \frac{1}{s^2+1}} \right) + e^{-3\pi s} \left(\frac{1}{\frac{1}{\sqrt{2}} \frac{\sqrt{2}}{(s+1)^2+2}} \right)$$

$$y(t) = \frac{1}{4} (-\cos t + \sin t + e^{-t} \cos(\sqrt{2}t)) + \frac{1}{\sqrt{2}} u_{3\pi}(t) f(t - 3\pi)$$

$$\text{where } f(t) = \mathcal{L}^{-1} \left\{ \frac{\sqrt{2}}{(s+1)^2+2} \right\} = e^{-t} \sin(\sqrt{2}t)$$

$$y(t) = \frac{1}{4} (\sin t - \cos t + e^{-t} \cos(\sqrt{2}t)) + \frac{1}{\sqrt{2}} u_{3\pi} [e^{-(t-3\pi)} \sin[\sqrt{2}(t-3\pi)]]$$