

The Convolution Integral (6.6)

Sometimes in computing inverse Laplace transforms, we come across $H(s) = F(s)G(s)$. It is tempting to think that $\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{G(s)\}$, but this is false. To help us with this situation, we define the convolution.

Definition. Let $f(t)$ and $g(t)$ be functions defined on $0 \leq t < \infty$. Then the convolution of f and g

$$f * g = \int_0^t f(t-\tau) g(\tau) d\tau$$

A nice property is that $f * g = g * f$,

$$\text{so } \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-\tau) f(\tau) d\tau$$

It turns out that convolutions are the sort of "product" we need for splitting up the Laplace transform.

Thm 6.6.1 Suppose $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

Let $h(t) = (f * g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau$.

Then $H(s) = \mathcal{L}\{h(t)\}$ exists and $H(s) = F(s)G(s)$.

The proof is in the textbook.

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Ex 1. Compute $\mathcal{L}\{f(t)\}$ where

$$f(t) = \int_0^t \underbrace{e^{(t-\tau)}}_{g(t-\tau)} \underbrace{\sin 3\tau}_{h(\tau)} d\tau$$

so $g(t) = e^t$ and $h(t) = \sin(3t)$

so $f(t) = (g * h)(t)$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\} \\ &= \frac{1}{s-1} \cdot \frac{3}{s^2+9} \end{aligned}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{3}{(s-1)(s^2+9)}}$$

Ex 2. Find $\mathcal{L}^{-1}\{F(s)\}$ in terms of convolution integrals.

$$\begin{aligned} F(s) &= \frac{1}{(s-3)^3(s^2-16)} = \frac{1}{(s-3)^3} \cdot \frac{1}{(s^2-16)} \\ &= \underbrace{\left(\frac{1}{2} \cdot \frac{2!}{(s-3)^{2+1}}\right)}_{G(s)} \cdot \underbrace{\left(\frac{1}{4} \cdot \frac{4}{s^2-16}\right)}_{H(s)} \end{aligned}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{2} t^2 e^{3t} \quad (\text{line 11})$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4} \sinh(4t)$$

$$\begin{aligned} g(t-\tau) &= \frac{1}{2} (t-\tau)^2 e^{3(t-\tau)} \\ h(\tau) &= \frac{1}{4} \sinh(4\tau) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= g * h = \int_0^t \frac{1}{2} (t-\tau)^2 e^{3(t-\tau)} \cdot \frac{1}{4} \sinh(4\tau) d\tau \\ &= \boxed{\frac{1}{8} \int_0^t (t-\tau)^2 e^{3(t-\tau)} \sinh(4\tau) d\tau} \end{aligned}$$

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Ex 3. Solve the IVP in terms of convolution integrals.

$$y'' + 3y' + 2y = \cos \alpha t; y(0) = 1, y'(0) = 0$$

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\cos \alpha t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 6[sY(s) - y(0)] + 9Y(s) = \mathcal{L}\{\cos \alpha t\}$$

$$s^2 Y(s) - s + 6sY(s) - 6 + 9Y(s) = \mathcal{L}\{\cos \alpha t\}$$

$$(s^2 + 6s + 9)Y(s) = s + 6 + \mathcal{L}\{\cos \alpha t\}$$

$$Y(s) = \frac{s+6}{s^2+6s+9} + \frac{1}{s^2+6s+9} \mathcal{L}\{\cos \alpha t\}$$

$$s^2 + 6s + 9 = (s+3)^2$$

$$\frac{s+6}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$s+6 = A(s+3) + B = As + 3A + B$$

$$A = 1, 3A + B = 6 \Rightarrow B = 3$$

$$Y(s) = \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{1}{(s+3)^2} \mathcal{L}\{\cos \alpha t\}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-3t} + 3te^{-3t} + (f * \cos \alpha t)$$

$$\text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = te^{-3t}$$

$$y(t) = e^{-3t} + 3te^{-3t} + \int_0^t (t-\tau) e^{-3(t-\tau)} \cos \alpha \tau d\tau$$

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Fun extras: Convolutions can help us solve certain integrals.

Ex 4. Compute $\int_0^5 e^{-x} \sin x \, dx$

We would normally use integration by parts.
Instead, write as a convolution integral.

$$\int_0^5 e^{-x} \sin x \, dx = e^{-5} \int_0^5 e^{5-x} \sin x \, dx$$

where

$$h(t) = \int_0^t e^{t-\tau} \sin \tau \, d\tau$$

$$\text{Then } \int_0^5 e^{-x} \sin x \, dx = e^{-5} \int_0^5 e^{5-x} \sin x \, dx = e^{-5} h(5)$$

$$\begin{aligned} \mathcal{L}\{e^{-5} h(t)\} &= e^{-5} \mathcal{L}\{e^t\} \mathcal{L}\{\sin t\} \\ &= e^{-5} \left(\frac{1}{s-1} \right) \left(\frac{1}{s^2+1} \right) \end{aligned}$$

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s-1) = As^2 + A + Bs^2 + Cs - Bs - C$$

$$\begin{cases} A+B=0 \\ -B+C=0 \\ A-C=1 \end{cases} \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}, C = -\frac{1}{2}$$

$$\mathcal{L}\{e^{-5} h(t)\} = \left[\frac{1}{2} \left(\frac{1}{s-1} \right) - \frac{1}{2} \frac{s}{s^2+1} - \frac{1}{2} \frac{1}{s^2+1} \right] e^{-5}$$

$$e^{-5} h(t) = \left[\frac{1}{2} e^t - \frac{1}{2} \cos t - \frac{1}{2} \sin t \right] e^{-5}$$

$$\boxed{\int_0^5 e^{-x} \sin x \, dx = e^{-5} h(5) = e^{-5} \left[\frac{1}{2} e^5 - \frac{1}{2} \cos(5) - \frac{1}{2} \sin(5) \right]}$$