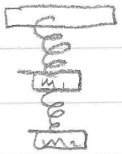
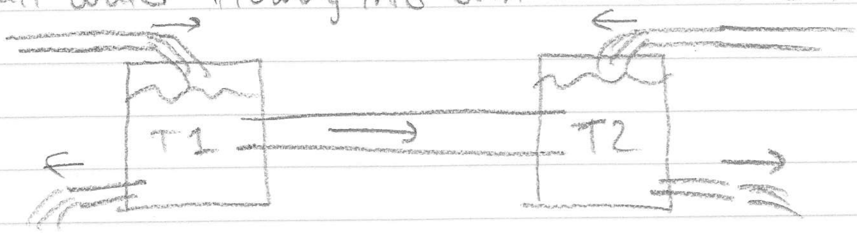
Lesson 26

Systems and Matrices (7.1, 7.2)

Suppose we have a mass attached to a spring which is attached to another mass attached to a spring



This situation is more complicated than with one mass. Suppose you have two interconnected tanks with salt water flowing into both and out of both:



In both of these examples, the behavior of object 1 is affected by the behavior of object 2 and vice versa.

To handle these, we use systems of differential equations.

Ex 1. Salt water with concentration 3 g/L of salt flows into Tank 2 at a rate of 4 L/min. The well-stirred mixture from Tank 2 flows into Tank 1 at a rate of 4 L/min and the well-stirred mixture of Tank 1 flows out at a rate of 4 L/min. If Tank 1 has 20 L, Tank 2 has 30 L, Tank 1 initially contains fresh water, and Tank 2 initially has 6 g of salt, write a system of linear diff eqs representing this situation.

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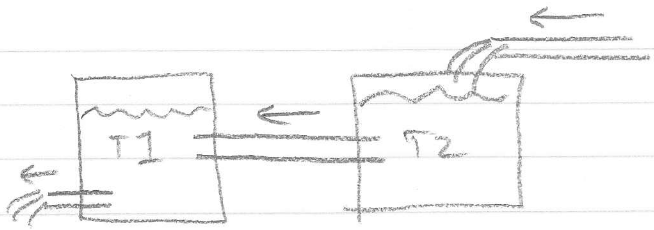

Let $x_1(t)$ and $x_2(t)$ be the amount of salt in Tanks 1 and 2, respectively, at time t minutes.

$$x_2' = \text{amount in} - \text{amount out}$$
$$= \frac{3\,g}{L} \cdot \frac{4L}{min} - \frac{x_2(t)\,g}{30\,L} \cdot \frac{4L}{min}$$
$$= 12 - \frac{2}{15}x_2 \qquad , \quad x_2(0) = 6$$

$$x_1' = \text{amount in} - \text{amount out}$$
$$= \frac{x_2(t)\,g}{30\,L} \cdot \frac{4L}{min} - \frac{x_1(t)\,g}{20\,L} \cdot \frac{4L}{min}$$
$$= \frac{2}{15}x_2 - \frac{1}{5}x_1 \quad , \qquad x_1(0) = 0$$

$$\boxed{\begin{cases} x_1' = -\frac{1}{5}x_1 + \frac{2}{15}x_2, & x_1(0) = 0 \\ x_2' = -\frac{2}{15}x_2 + 12, & x_2(0) = 6 \end{cases}}$$

We can convert 2nd order equations into 1st order systems and vice versa.

<u>Ex 2</u>. Convert $3u'' + 2u' - u = \sin t$, $u(0) = 3$, $u'(0) = 2$ into a system of first order equations.

Let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$.

$$3x_2' + 2x_2 - x_1 = \sin t \qquad x_1(0) = u(0) = 3,$$
$$3x_2' = x_1 - 2x_2 + \sin t \qquad x_2(0) = u'(0) = 2$$

$$\boxed{\begin{cases} x_1' = x_2, & x_1(0) = 3 \\ x_2' = \frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}\sin t, & x_2(0) = 2 \end{cases}}$$

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Ex 3. Convert the system into a 2nd order equation.

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3 \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \frac{1}{2}. \end{cases}$$

First, take the derivative of the second equation:

$$x_2'' = 2x_1' - 2x_2'$$

Substitute the first equation in.

$$x_2'' = 2(3x_1 - 2x_2) - 2x_2' = 6x_1 - 4x_2 - 2x_2'$$

Solve the second equation for x_1

$$2x_1 = x_2' + 2x_2$$

Plug in

$$x_2'' = 3(x_2' + 2x_2) - 4x_2 - 2x_2'$$

$$x_2'' = 3x_2' - 2x_2' + 6x_2 - 4x_2$$

$$x_2'' - x_2' - 2x_2 = 0$$

Already have $x_2(0) = \frac{1}{2}$. Need $x_2'(0)$.

By the second equation,

$$x_2'(0) = 2x_1(0) - 2x_2(0) = 2(3) - 2(\tfrac{1}{2}) = 5$$

$$\boxed{x_2'' - x_2' - 2x_2 = 0, \quad x_2(0) = \tfrac{1}{2}, \quad x_2'(0) = 5}$$

We will often work with systems of equations in matrix form.

An $m \times n$ matrix $A = (a_{ij})$ is an array with m rows and n columns

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

a_{ij} is the (i,j)-th entry located in row i, column j.

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In this class, we restrict our attention to 2×2 matrices, and 2-vectors.
An n-dimensional row vector is a $1\times n$ matrix and an m-dimensional column vector is an $m\times 1$ matrix

If A and B are both $m\times n$ matrices, $A+B$ is defined by component-wise addition. i.e., if $(c_{ij}) = A+B$, then $c_{ij} = a_{ij}+b_{ij}$ for all i and j.

If A is an $m\times n$ matrix and c is a constant, then $cA = (ca_{ij})$, i.e., each component of A is multiplied by c.

Multiplying two matrices is tricky. AB only exists is A has as many columns as B has rows.
If $(c_{ij}) = AB$, the c_{ij} is the dot product of the i^{th} row of A with the j^{th} column of B.

In general, $AB \neq BA$.

<u>Ex 4</u>. Let $A = \begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}$

Compute
(a) $2A - B$

$$2\begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2i & -2i \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} -i & -1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+i & -1-2i \\ 5 & -2 \end{pmatrix}$$

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(b) AB

$$\begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix}\begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2i-1-i & 2+i-2i \\ 3i+0 & 3+0 \end{pmatrix}$$

$$= \begin{pmatrix} -1+i & 2-i \\ 3i & 3 \end{pmatrix}$$

(c) BA

$$\begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2+i & -i \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2i-1+3 & 1+0 \\ 2+i+6 & -i+0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2i & 1 \\ 8+i & -i \end{pmatrix}$$

We can write systems of differential equations in terms of matrices

$$\begin{cases} x_1' = 2x_1 + 3x_2 \\ x_2' = 4x_1 - 2x_2 \end{cases} \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or simply $\vec{x}' = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

<u>Ex 5</u>. Verify that $\vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-4t}$ is a solution to the above system.

$$\vec{x}' = \begin{pmatrix} 1 \\ -2 \end{pmatrix} -4e^{-4t} = \begin{pmatrix} -4 \\ 8 \end{pmatrix} e^{-4t}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-4t} = \begin{pmatrix} 2-6 \\ 4+4 \end{pmatrix} e^{-4t} = \begin{pmatrix} -4 \\ 8 \end{pmatrix} e^{-4t} \quad \checkmark$$

If you want, you can rewrite as

$x_1(t) = e^{-4t}, \quad x_2(t) = -2e^{-4t}$