Lesson 28

Homogeneous Linear Systems with Constant Coefficients (7.5)

For the next few lessons, we focus on such systems
$\vec{x}' = A\vec{x}$ where A is a constant matrix.

If A is 1×1, we have the system $x' = ax$,
which has the solution $x = ce^{at}$.

It makes sense, then, that a solution to $\vec{x}' = A\vec{x}$ could be of the form $\vec{\xi}e^{\lambda t}$ for some vector $\vec{\xi}$ and number λ.
Plugging in for $\vec{x}$, we get

$$\lambda \vec{\xi} e^{\lambda t} = A\vec{\xi}e^{\lambda t}$$

This statement is true, by the definition of eigenvalues and eigenvectors, when λ is an eigenvalue and $\vec{\xi}$ an associated eigenvector of A.

Fact. If λ is an eigenvalue of A and $\vec{\xi}$ is an associated eigenvector, then $c\vec{\xi}e^{\lambda t}$ is a solution to $\vec{x}' = A\vec{x}$.

If A is a 2×2 matrix and $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ are linearly independent eigenvectors (λ_1, λ_2) of A,
then $\left|\vec{\xi}^{(1)}\ \vec{\xi}^{(2)}\right| \neq 0$, so

$$\vec{x} = c_1\vec{\xi}^{(1)}e^{\lambda_1 t} + c_2\vec{\xi}^{(2)}e^{\lambda_2 t}$$

is the general solution.

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Ex 1. Solve the IVP.

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

First, find eigenvalues and assoc. eigenvectors of A.

$$\begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda) - (1)(-5) = -8 - 2\lambda + \lambda^2 + 5$$
$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\lambda_1 = -1, \quad \lambda_2 = 3$

$\lambda_1 = -1: \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -5 & 5 & 0 \end{array}\right): \quad \begin{matrix} -x_1 + x_2 = 0 \\ -5x_1 + 5x_2 = 0 \end{matrix} : \quad x_1 = x_2, \quad \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3: \left(\begin{array}{cc|c} -5 & 1 & 0 \\ -5 & 1 & 0 \end{array}\right): \quad -5x_1 + x_2 = 0: \quad x_1 = \frac{1}{5}x_2, \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

general solution: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{pmatrix}$$

obtain the system: $\begin{cases} c_1 + c_2 = 1 \\ c_1 + 5c_2 = 3 \end{cases} \Rightarrow c_1 = \frac{1}{2}, \; c_2 = \frac{1}{2}$

$$\boxed{\vec{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}}$$

Now, we consider the behavior of solutions for well-behaved systems of equations (A is a real 2×2 matrix and 0 is not an eigenvalue of A).

First, we develop the phase plane for a system of equations. This is very similar to phase lines and direction fields.

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A phase plane is an x_1x_2-plane. Given a system $\vec{x}' = A\vec{x}$, we can find a direction vector to a solution at (a,b) by letting $x_1 = a$, $x_2 = b$; i.e., $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$. We then plot the direction vector at the point (a,b) in the plane. Use pplane8 in MATLAB.

In all of these situations we consider, $(0,0) = \vec{0}$ is an equilibrium solution and the only one. How do we classify the equilibrium point, and how do solutions behave? These are important questions we consider over the next few lessons. In this lesson, we focus on the case where A has two distinct real eigenvalues (neither of which is 0).

Ex 2. Both eigenvalues positive:

$$\vec{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \vec{x}$$

We see that $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

If $c_1 = 0$, then $\vec{x}(t) = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$, so we get a solution along the eigenvector $\vec{\xi}^{(2)}$ and $\vec{x}$ is unbounded as $t \to \infty$.

If $c_2 = 0$, then $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$, so we get a solution along the eigenvector $\vec{\xi}^{(1)}$ and $\vec{x}$ is unbounded as $t \to \infty$.

Since the solutions along both eigenvectors have the same behavior, $\vec{0}$ is a node.

In general, as $t \to \infty$, $\vec{x}(t)$ moves away from $\vec{0}$, so $\vec{0}$ is an unstable node.

(Plot with pplane 8 to get a visual)

Ex 3. Both eigenvalues negative:

$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \vec{x}$

get $\lambda_1 = -1$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -2$, $\vec{\xi}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

so the general solution is

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$

Along $\vec{\xi}^{(2)}$ $(c_1 = 0)$, $\vec{x} = c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} \to \vec{0}$ as $t \to \infty$.

Along $\vec{\xi}^{(1)}$ $(c_2 = 0)$, $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \to \vec{0}$ as $t \to \infty$.

Since solutions behave the same way on both eigenvectors, $\vec{0}$ is a node.

In general, as $t \to \infty$, $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} \to \vec{0} + \vec{0} = \vec{0}$,

so $\vec{0}$ is an asymptotically stable node.

(Plot with pplane 8).

Ex 4. One eigenvalue positive, one negative:

$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$

get $\lambda_1 = 1$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1$, $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

Along $\vec{\xi}^{(2)}$ $(c_1 = 0)$, $\vec{x} = c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} \to \vec{0}$ as $t \to \infty$.

Along $\vec{\xi}^{(1)}$ $(c_2 = 0)$, $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$ diverges from $\vec{0}$ as $t \to \infty$.

Since the solutions along the eigenvectors behave differently, $\vec{0}$ is called a saddle point.

In general, as $t \to \infty$, $\vec{x}$ is unbounded, diverging from $\vec{0}$.

This is true for all saddle points.

So all saddle points are unstable.

(Plot with pplane 8).