Lesson 4

Separable Equations (Sec 2.2)

If we have a first order differential equation, we can always write it in the form

$$f(x,y) + g(x,y)\frac{dy}{dx} = 0.$$

If f is a function of only x (no y dependence) and g is a function of only y (no x dependence), we get the form

$$f(x) + g(y)\frac{dy}{dx} = 0$$

Such a diff eq is called <u>separable</u>.

We all know the technique to solve this:

$$g(y)\frac{dy}{dx} = -f(x)$$

$$g(y)\,dy = -f(x)\,dx$$

$$\text{so } \int g(y)\,dy = -\int f(x)\,dx + C$$

But this isn't an incredibly valid technique, though it is often a good way to think about it.

We can show that this is valid rigorously by the chain rule. Let $F(x)$ and $G(y)$ be antiderivatives of $f(x)$ and $g(y)$, respectively.

If $F(x) + G(y) = C$ is differentiated with respect to x, we get

$$F'(x) + G'(y)\frac{dy}{dx} \text{ (chain rule) } = 0$$

$$f(x) + g(y)\frac{dy}{dx} = 0$$

This validates the technique.

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Let's try to use the technique in a few examples.

Ex 1. Solve the diff eq $y' + y^2 \sin x = 0$

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\frac{dy}{y^2} = -\sin x \, dx \qquad \underline{\text{if } y \neq 0}$$

$$\int \frac{dy}{y^2} = \int -\sin x \, dx$$

$$-\frac{1}{y} = \cos x + C$$

$$\frac{1}{y} = -\cos x + C$$

$$\boxed{y = \frac{1}{\cos x + C}}$$

$$\boxed{\text{Also } y = 0}$$

Often times, solutions will be too difficult to find explicitly, so we can leave our solutions implicit.

Ex 2. $\frac{dy}{dx} = \frac{x^2}{1+3y^2}$

$$(1+3y^2)\, dy = x^2 \, dx$$

$$\int (1+3y^2)\, dy = \int x^2 \, dx$$

$$y + y^3 = \tfrac{1}{3}x^3 + C$$

so leave solution as

$$\boxed{y + y^3 - \frac{x^3}{3} = C}$$

Note: When we have implicit solutions like this, we must be careful about the domain on which the solution is defined.

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Ex 3. Solve the IVP

$$\frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y}, \quad y(0)=1$$

$$(3y^2-6y)\,dy = (1+3x^2)\,dx$$

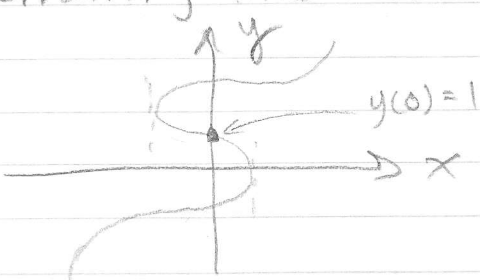
$$y^3-3y^2 = x+x^3+C$$

so $y^3-3y^2-x-x^3=C$

$$1-3-0-0=C, \text{ so } C=-2$$

$$y^3-3y^2-x-x^3=-2$$

Where is the solution valid?
Graph it (use Wolfram Alpha or MATLAB)
Looks something like this!



Derivatives do not exist where the function has vertical tangents. We can find these by finding where $\frac{dy}{dx}$ has a denominator equal to 0.

$3y^2-6y=0 \Rightarrow 3y(y-2)=0$ so at $y=0$ and $y=2$.

The initial condition $y(0)=1$ forces us to take the values in between them.

What are the x-values?

When $y=0$, $-x-x^3=-2$. We solve to get $x=1$.
When $y=2$, $-x-x^3=2$. We solve to get $x=-1$.

$$\boxed{\text{So the solution is } y^3-3y^2-x-x^3+2=0 \text{ for } -1<x<1}$$

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<u>Ex 4</u>. Find the solution to the IVP

$$\frac{dy}{dx} = \frac{1-2x}{y}, \quad y(1) = -2$$

$$y\,dy = (1-2x)\,dx$$

$$\frac{y^2}{2} = x - x^2 + C$$

$$\frac{(-2)^2}{2} = 1 - 1 + C, \text{ so } C = 2$$

$$\frac{y^2}{2} = x - x^2 + 2$$

<u>Can</u> solve explicitly:

$$y^2 = 2x - 2x^2 + 4$$

$$y = \pm\sqrt{2x - 2x^2 + 4}$$

We require that when $x = 1$, $y = -2$, so we have to choose the negative root.

y is defined when

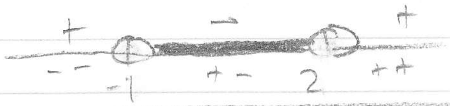
$$2x - 2x^2 + 4 > 0$$

$$\Rightarrow \quad 2(x^2 - x - 2) \quad 0$$

$$\Rightarrow \quad x^2 - x + 2 < 0$$

$$\Rightarrow \quad (x-2)(x+1) < 0$$

(y cannot equal 0 - see original diff eq)



So $\boxed{y = -\sqrt{2x - 2x^2 + 4}, \quad -1 < x < 2}$

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<u>Homogeneous Equations</u> Given a differential equation $\frac{dy}{dx} = f(x,y)$, we say it is <u>homogeneous</u> if $f(x,y) = g(\frac{y}{x})$ (i.e., f can be expressed as a function of the "variable" $\frac{y}{x}$). We can often solve such equations by making a substitution $u(x) = \frac{y}{x}$.

<u>Ex 5.</u> Consider $\frac{dy}{dx} = \frac{y-4x}{x-y}$.
(a) Show the equation is homogeneous.
(b) Solve using a substitution.

(a) Multiply numerator and denominator by $\frac{1}{x}$:

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

(b) Let $u(x) = \frac{y}{x}$. Then $x\,u(x) = y$.
By the product rule

$$x\frac{du}{dx} + u = \frac{dy}{dx}$$

so we substitute to get:

$$x\frac{du}{dx} + u = \frac{u-4}{1-u}$$

$$x\frac{du}{dx} = \frac{u-4}{1-u} - u = \frac{u-4}{1-u} - \frac{u-u^2}{1-u}$$

$$x\frac{du}{dx} = \frac{u^2-4}{1-u}$$

$$\frac{(1-u)\,du}{(u+2)(u-2)} = \frac{1}{x}\,dx$$

Partial fractions: $\frac{1-u}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$

$1 - u = (u-2)A + (u+2)B$

$u = 2 \Rightarrow -1 = 4B$, so $B = -\frac{1}{4}$

$u = -2 \Rightarrow 3 = -4A$, so $A = -\frac{3}{4}$

$$\left(-\frac{3}{4}\cdot\frac{1}{u+2} + -\frac{1}{4}\cdot\frac{1}{u-2}\right)du = \frac{1}{x}\,dx$$

$$-\frac{3}{4}\ln|u+2| - \frac{1}{4}\ln|u-2| = \ln|x| + C$$

$$3\ln|u+2| + \ln|u-2| = \ln x^4 + C$$

$$\ln|(u+2)^3(u-2)| = \ln x^4 + C$$

$$(u+2)^3(u-2) = Cx^4$$

$$\boxed{\left(\frac{y}{x}+2\right)^3\left(\frac{y}{x}-2\right) = Cx^4}$$

Sometimes just a substitution is needed.

Ex 6. Use $u(x) = y^4$ to solve

$$4y^3\frac{dy}{dx} + \frac{y^4}{x} = \frac{1}{x}, \quad x > 0$$

If $u = y^4$, then $\frac{du}{dx} = 4y^3\frac{dy}{dx}$

$$\frac{du}{dx} + \frac{u}{x} = \frac{1}{x}$$

$$\mu(x) = \exp\left(\int\frac{1}{x}\,dx\right) = e^{\ln|x|} = |x| = x \quad (x>0)$$

$$\frac{d}{dx}[xu] = 1$$

$$xu = x + C$$

$$u = 1 + \frac{C}{x}$$

$$\boxed{y^4 = 1 + \frac{C}{x}}$$