1. Consider a tank which has 400 gallons of a salt-water mixture. Initially, the tank has 15 lbs of salt in it. Water flows into the tank at a rate of 30 gallons per minute, and there is 1/2 lb of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the salt-water mixture. The salt-water mixture flows out of the tank at a rate of 30 gallons per minute. Find the concentration (lbs per gallon) of salt in the tank in the long run.

Let $q(t)$ be the amount of salt (in lbs) in the tank after $t$ min.
Let $c(t)$ be the concentration of salt (in lbs/gal) after $t$ min.
Let $v(t)$ be the amount of liquid (in gal) in the tank after $t$ min.

We can figure out $\frac{dq}{dt}$ = (rate of salt in) − (rate of salt out)

rate of salt in: $\frac{\frac{1}{2}\text{ lb}}{\text{gal}} \cdot \frac{30\text{ gal}}{\text{min}} = 15\text{ lb/min}$

rate of salt out: $\frac{q(t)\text{ lb}}{v(t)\text{ gal}} \cdot \frac{30\text{ gal}}{\text{min}} = \frac{30q}{400}\text{ lb/min}$ $(v(t) = 400)$

$\frac{dq}{dt} = 15 - \frac{3q}{40}$ (can use integrating factor or separable)

get $q(t) = Ce^{-3t/40} + 200$

Now, $q(0) = 15$, so $15 = C + 200$, making $C = -185$

$$q(t) = -185e^{-3t/40} + 200$$

Notice $c(t) = \frac{q(t)}{v(t)} = \frac{-185e^{-3t/40} + 200}{400}$

$$= \frac{-37}{80}e^{-3t/40} + \frac{1}{2}$$

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty}\Big(\underbrace{\frac{-37}{80}e^{-3t/40}}_{\to 0} + \frac{1}{2}\Big) = \frac{1}{2}$$

so $\boxed{\frac{1}{2}\text{ lb per gal}}$

2. Consider a tank which has 400 gallons of pure water, and has a capacity of 700 gallons. Salt water begins to flow into the tank at a rate of 5 gallons per minute and there are 10 grams of salt per gallon. There is a mixing device in the tank which keeps the salt evenly distributed throughout the tank. The mixture in the tank flows out at a rate of 3 gallons per minute. How much salt will be in the tank the instant it begins to overflow?

Use same symbols $q(t)$ and $V(t)$ as problem 1.

Notice: $V(t)$ is <u>not</u> constant.

$$\frac{dV}{dt} = (\text{rate in}) - (\text{rate out}) = 5\,\text{gal} - 3\,\text{gal} = 2\,\text{gal}$$

Thus, $V(t) = 2t + C$ and $V(0) = 400$ gal, so

$$V(t) = 2t + 400$$

$$\frac{dq}{dt} = (\text{rate in}) - (\text{rate out}) = \frac{10\,g}{\text{gal}} \cdot \frac{5\,\text{gal}}{\text{min}} - \frac{q(t)\,g}{V(t)\,\text{gal}} \cdot \frac{3\,\text{gal}}{\text{min}}$$

$$\frac{dq}{dt} = 50 - \frac{3q}{2t+400}$$

Use integrating factor method to obtain

$$q(t) = \frac{C}{(t+200)^{3/2}} + 20t + 4000$$

$q(0) = 0$, so $0 = \frac{C}{200^{3/2}} + 4000$, $C = -8{,}000{,}000\sqrt{2}$

$$q(t) = 20t - \frac{8{,}000{,}000\sqrt{2}}{(t+200)^{3/2}} + 4000$$

Tank overflows when $V(t) = 700$

$$700 = 2t + 400 \implies t = 150$$

$$q(150) \approx \boxed{5{,}272.16 \text{ grams}}$$

3. Pete stands at the top of a 40 meter building and throws a hammer upward with a speed of 5 m/s. Suppose there is a force due to air resistance acting on the hammer in the opposite direction of velocity with a magnitude of $\frac{|v|}{22}$ m/s. Set up a differential equation to model this scenario (use g = 9.8 m/s$^2$ as the magnitude of the acceleration due to gravity).

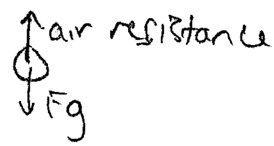
Let $V(t)$ be velocity at $t$ seconds.

Let $m$ be the mass of the hammer.

Governing equation: $F = ma$

Going up:


going down:




$a = \frac{dv}{dt}$, so $$-mg - \frac{v}{22} = m\frac{dv}{dt}$$

Or, if you prefer,

$$\frac{dv}{dt} = -9.8 - \frac{v}{22m}$$

Notice, if $v > 0$, then $\frac{-v}{22} < 0$

if $v < 0$, then $\frac{-v}{22} > 0$

So air resistance is always opposite to velocity.

Thus, this equation works for both going up and down.

Initial conditions: $v(0) = 5$, and if $h(t)$ is height, $h(0) = 40$

$$\boxed{\frac{dv}{dt} = -9.8 - \frac{v}{22m}, \quad v(0) = 5, \ h(0) = 40}$$

4. Suppose that the rate of change of a function $f$ is proportional to a function $g$. Write a differential equation which expresses this situation.

$f' = kg$, $k$ a constant.

5. Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of tea obeys Newton's Law of Cooling. Assume the tea has a temperature of 190° F when freshly poured, and 2 minutes later has cooled to 175° F in a room at 72° F. Find a function for the temperature $T$ of the tea at time $t$.

Let $T$ = temperate (in °F) of tea after $t$ minutes

$T_s$ = surrounding temperature

$$\frac{dT}{dt} = k(T - T_s) = k(T - 72)$$

Using lesson 2 or separable, get

$$T(t) = Ce^{kt} + 72$$

$T(0) = 190$, so $190 = C + 72$, $C = 118$

$$T(t) = 118e^{kt} + 72$$

$T(2) = 175$, so $175 = 118e^{2k} + 72$

$e^{2k} = \frac{103}{118}$, so $k = \frac{1}{2}\ln\left(\frac{103}{118}\right) = \ln\sqrt{\frac{103}{118}}$

$$T(t) = 118\left(e^{\ln\sqrt{\frac{103}{118}}}\right)^t + 72$$

$$\boxed{T(t) = 118\left(\sqrt{\frac{103}{118}}\right)^t + 72}$$

6. Suppose that a rocket is launched straight up from the surface of the Earth with an initial velocity of $v_0 = \sqrt{2gR}$, where $R$ is the radius of the Earth. Neglect air resistance. Find an expression for the velocity $v$ in terms of the distance $x$ from the surface of the Earth. Find the time required for the rocket to go 140,000,000 miles (the approximate distance from Earth to Mars). Assume that $R = 4000$ miles. Assume that the acceleration due to gravity $g = 32$ ft/s$^2$ (There are 5280 feet in a mile.)

Such a situation is governed by

$$\frac{dv}{dt} = -\frac{gR^2}{(R+x)^2} \quad \left(\text{Think: } F=ma, \text{ so } m\frac{dv}{dt} = w(x),\right.$$

$$\left. w(x) \text{ is gravitational force, } w(x) = \frac{-k}{(x+R)^2},\ w(0) = -mg\right)$$

By the Chain Rule, $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$ (since $v = \frac{dx}{dt}$)

$$v\frac{dv}{dx} = \frac{gR^2}{(R+x)^2}, \quad v(0) = \sqrt{2gR}$$

Solving, we get $v(x) = \frac{R\sqrt{2g}}{\sqrt{R+x}}$

But $\frac{dx}{dt} = v$, so $\frac{dx}{dt} = \frac{R\sqrt{2g}}{\sqrt{R+x}} \Rightarrow (R+x)^{1/2}dx = R\sqrt{2g}\,dt$

Get: $\frac{2}{3}(R+x)^{3/2} = Rt\sqrt{2g} + C$

Notice $x(0) = 0$, so $\frac{2}{3}(R+x)^{3/2} = Rt\sqrt{2g} + \frac{2}{3}R^{3/2}$

To solve for time, use $x = 140{,}000{,}000$, $R = 4000$,

$$g = \frac{32\text{ ft}}{s^2}\cdot\frac{1\text{ mi}}{5280\text{ ft}} \quad (\text{miles per } s^2)$$

This will give time in seconds.

Convert to hours.

$\approx 696{,}600.57$ hrs

($\approx 79.5$ years)

7. If Jack weighs 200 lbs, what is his mass?

Weight is the force due to gravity,

so $200 \text{ lbs} = mg$

$$200 = 32m$$

$$\Rightarrow \quad m = \frac{200}{32} = \boxed{6.25 \text{ slugs}}$$