Lesson 6

Differences Between Linear and Nonlinear Eqns (2.4)

When it comes to first order ODEs, there are many differences concerning existence, uniqueness, and domain of definition for solutions of linear vs. nonlinear diff eqs.

We start with a theorem concerning linear equations.

Theorem 2.4.1 Consider the IVP

$$y' + p(t)y = g(t), \; y(t_0) = y_0.$$

If $p(t)$ and $g(t)$ are continuous on the open interval $I = (\alpha, \beta)$ where $\alpha < t_0 < \beta$, then there exists a unique function $y = \phi(t)$ satisfying the IVP.

The proof of this theorem follows the method of integrating factors.

So, for a linear first order ODE, we can usually tell where solutions exist, if they are unique or not, and the domain of definition of said solutions just by looking at the diff eq itself.

Lesson 6

__Ex 1.__ Find an interval on which the solution to the IVP is guaranteed to exist and be unique

$$(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$$

Get in form: $y' + \underbrace{\frac{\ln t}{t-3}}_{p(t)} y = \underbrace{\frac{2t}{t-3}}_{g(t)}$

$p(t)$ is continuous when $t > 0$ and $t \neq 3$

$g(t)$ is continuous $t \neq 3$.

Since the initial condition requires $t = 1$, the largest interval for which the solution is defined is $\boxed{0 < t < 3}$.

Nonlinear equations are often not so nice.

All first order nonlinear diff eqs can be written in the form $y' = f(t, y)$ where f is a function of t and y.

__Theorem 2.4.2__ Suppose that you are given an IVP

$$y' = f(t,y), \quad y(t_0) = y_0.$$

If f and $\frac{\partial f}{\partial y} = f_y$ are continuous in the rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ where $\alpha < t_0 < \beta$ and $\gamma < y_0 < \delta$, then there exists a unique solution $y = \phi(t)$ to the IVP in some interval $t_0 - h < t < t_0 + h$, where

$$\alpha \leq t_0 - h < t_0 + h \leq \beta.$$

Note: If f is continuous, then existence is guaranteed, but uniqueness is only guaranteed if $\frac{\partial f}{\partial y}$ is continuous as well.

Lesson 6

Ex 2. State where in the ty-plane the hypotheses of Thm 2.4.2 are satisfied

$$y' = \underbrace{\frac{\cot(t)\,y}{1+y}}_{f(t,y)}$$

f is not continuous if either $\sin t = 0$, i.e., $t = n\pi$ for all integers n.

Also when $y=-1$.

use quotient rule for $\frac{\partial f}{\partial y}$.

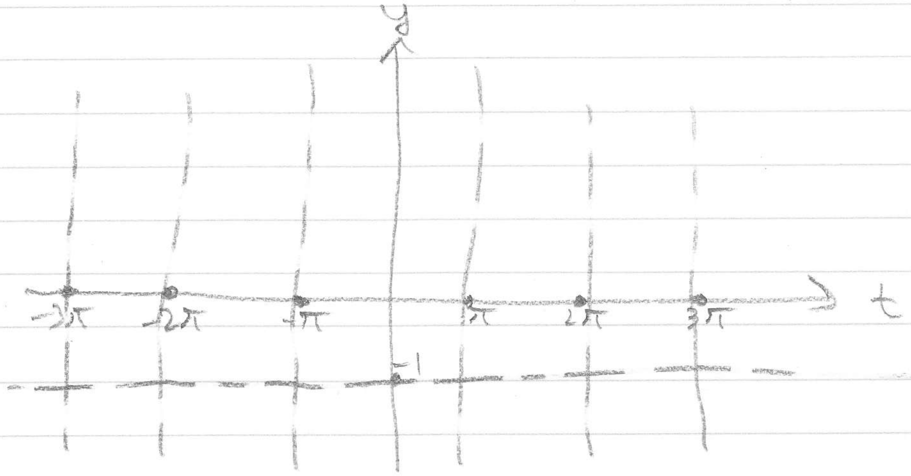
$$\frac{\partial f}{\partial y} = \frac{(1+y)\cot(t) - \cot(t)y}{(1+y)^2} = \frac{\cot(t)}{(1+y)^2}$$

which has the same discontinuities.

So the hypotheses of Thm 2.4.2 are satisfied when

$t \neq n\pi$ for any interger n

and $y \neq -1$



anywhere except the dotted lines

Lesson 6

Even though it may be hard to tell, Thm 2.4.2 is weaker than Thm 2.4.1.

First, in Thm 2.4.1, if $p(t)$ and $g(t)$ are continuous on $\alpha < t < \beta$, then the solution exists on $\alpha < t < \beta$. For Thm 2.4.2, if f and $\frac{\partial f}{\partial y}$ are continuous on $\alpha < t < \beta$ and $\gamma < y < \delta$, we are not guaranteed existence on the whole interval $\alpha < t < \beta$. Rather, we are guaranteed existence on some (unspecified) subinterval.

So even though for ex 2 the hypotheses are satisfied on the rectangle $-\pi < t < \pi$, $-1 < y < \infty$, there is no guarantee that a solution exists to the IVP with $y(0) = 0$ on all of $-\pi < t < \pi$. Just that it exists on some interval $-h < t < h$ where $-\pi \leq -h < h \leq \pi$.

Ex 3. Consider the separable equation

$$y' = y^2 + 6y, \quad y(0) = 2$$

Find the largest interval on which the solution is defined.

$$\frac{dy}{y(y+6)} = dt$$

$$\frac{A}{y} + \frac{B}{y+6} = \frac{1}{y(y+6)} \Rightarrow A(y+6) + By = 1$$

$$y = 0 \Rightarrow A = \tfrac{1}{6},$$

$$y = -6 \Rightarrow B = -\tfrac{1}{6}$$

Lesson 6

$\frac{1}{6}\left(\frac{1}{y+6} - \frac{1}{y}\right) dy = dt$

$\frac{1}{6} \ln|y+6| - \ln|y| = t + C$

$\ln\left|\frac{y+6}{y}\right| = 6t + C$

$\frac{y+6}{y} = ce^{6t} \quad \left(\frac{8}{2} = c = 4\right)$

$y+6 = y\,4e^{6t}$

$y - y\,4e^{6t} = -6$

$y(1-4e^{6t}) = -6$

$y = \frac{-6}{1-4e^{6t}}$

Not defined when $1 - 4e^{6t} = 0$

$1 = 4e^{6t}$

$\frac{1}{4} = e^{6t}$

$\ln\left(\frac{1}{4}\right) = 6t$

$t = \frac{1}{6}\ln\left(\frac{1}{4}\right) \approx -0.23$

So the solution exists on the interval

$\frac{1}{6}\ln\left(\frac{1}{4}\right) < t < \infty$

(since $t=0$ is required in the interval, by the initial condition)

Ex 4. $y_1(t) = t$ and $y_2(t) = 1$ are distinct solutions to the IVP

$$y' = \frac{y^2-1}{t^2-1}, \quad y(1) = 1.$$

Why doesn't this contradict the uniqueness claim of Thm 2.4.2?

Check: $y_1' = 1$, $\frac{t^2-1}{t^2-1} = 1$, also $y_1(1) = 1$.
$y_2' = 0$, $\frac{1-1}{t^2-1} = 0$, also $y_2(1) = 1$.

$f(t,y) = \frac{y^2-1}{t^2-1}$ and $f(1,1) = \frac{1-1}{1-1} = \frac{0}{0}$, so f is not continuous at the initial condition $y(1)=1$.

Bernoulli Equations. These are diff eqs of the form $y' + p(t)y = q(t)\,y^n$.

To solve, make the substitution $v(t) = y^{1-n}$.

Ex 5. Solve $y' + \frac{1}{t}y = t^2 y^3$, $t > 0$

Let $v(t) = y^{1-3} = y^{-2}$ Then $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$

Diff eq: $y^{-3}\frac{dy}{dt} + \frac{1}{t}y^{-2} = t^2$

$$-\frac{1}{2}\frac{dv}{dt} + \frac{1}{t}v = t^2$$

$$\frac{dv}{dt} - \frac{2}{t}v = -2t^2$$

$$\mu(t) = \exp\left(\int -\frac{2}{t}\,dt\right) = e^{-2\ln|t|} = \frac{1}{t^2}$$

$$\frac{d}{dt}\left[\frac{1}{t^2}v\right] = -2$$

$$\frac{1}{t^2}v = -2t + c$$

$$v(t) = -2t^3 + ct^2$$

$$y^{-2} = -2t^3 + ct^2$$

$$\boxed{y(t) = \pm\frac{1}{\sqrt{-2t^3+ct^2}}}$$

Lesson 6

Bernoulli equations are nice since they are nonlinear but with an appropriate substitution can be solved like a linear equation.

In general, we cannot find an explicit solution for a nonlinear equation, even though we can for a linear equation.

This is another difference.