Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X be a Hausdorff space and let A be a compact subset of X. **Prove** from the definitions that A is closed.

2. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

3. Show that if Y is compact, then the projection map $X \times Y \to X$ is a closed map.

4. Let X be a compact space and suppose we are given a nested sequence of subsets

   $$C_1 \supset C_2 \supset \cdots$$

   with all C_i closed. Let U be an open set containing $\cap C_i$.

   Prove that there is an i_0 with $C_{i_0} \subset U$.

5. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

6. Let X be a compact metric space and let $\mathfrak{U}$ be a covering of X by open sets.

   Prove that there is an $\epsilon > 0$ such that, for each set $S \subset X$ with diameter $< \epsilon$, there is a $U \in \mathfrak{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

7. Let S^1 denote the circle

   $$\{x^2 + y^2 = 1\}$$

   in $\mathbb{R}^2$. Define an equivalence relation on S^1 by

   $$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

   (you do *not* have to prove that this is an equivalence relation). **Prove** that the quotient space $S^1/\sim$ is homeomorphic to S^1.

   One way to do this is by using complex numbers.

8. Let X be a compact Hausdorff space and let $f : X \to X$ be a continuous function. Suppose f is 1-1. **Prove** that there is a nonempty closed set A with $f(A) = A$.

9. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. **Prove** that the quotient space $\mathbb{R}^2/\sim$ is compact but not Hausdorff.

10. Let X be a locally compact Hausdorff space. **Explain** how to construct the one-point compactification of X, and **prove** that the space you construct is really compact (you do not have to prove anything else for this problem).

11. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

12. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

    Prove that the map
    $$e : X \times \mathcal{C}(X, Y) \to Y$$
    defined by the equation
    $$e(x, f) = f(x)$$
    is continuous.

13. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

14. Let X be a topological space and $f : [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

15. Let X be a topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives a homomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this).

    Suppose that for every pair of paths α and β from x_0 to x_1 the homomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. **Prove** that $\pi_1(X, x_0)$ is abelian.

16. Let $p : E \to B$ be a covering map with B connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. Prove that, for every $b \in B$, $p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

17. Let B be a Hausdorff space.

    Let $p : E \to B$ be a covering map.

    Prove that E is Hausdorff.

18. Let $p : E \to B$ be a covering map. **Prove** that p takes open sets to open sets.

19. Let X be a topological space and let $f : X \to X$ be a homeomorphism for which $f \circ f$ is the identity map.

    Suppose also that each $x \in X$ has an open neighborhood V_x for which $V_x \cap f(V_x)$ is empty.

    Define an equivalence relation $\sim$ on X by: $x \sim y$ if and only if $x = y$ or $f(x) = y$. (You do **not** have to prove that this is an equivalence relation; this is the only place where the assumption that $f \circ f$ is the identity is used).

(a) **Prove** that the quotient map $q : X \to X/\sim$ takes open sets to open sets.

(b) **Prove** that q is a covering map. (You may use part (a) even if you didn't prove it.)

20. Let $p : E \to B$ be a covering map with E path-connected. Let $p(e_0) = b_0$.

    (a) Give the definition of the standard map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ constructed in Munkres (you do NOT have to prove that this is well-defined).

    (b) Suppose that α and β are two elements of $\pi_1(B, b_0)$ with $\phi(\alpha) = \phi(\beta)$. Prove that there is an element γ of $\pi_1(E, e_0)$ with $\beta = p_*(\gamma) \cdot \alpha$.

21. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

    (a) Give the definition of the function $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, including the proof that it is well-defined.

    (b) Prove that if f is a covering map then f_* is one-to-one.

22. Let X be a path-connected space.

    Let x_0 and x_1 be two different points in X.

    Suppose that every path from x_0 to x_1 is path-homotopic to every other path from x_0 to x_1.

    Prove that X is simply-connected.

23. Let X and Y be topological spaces, let $x_0 \in X$, $y_0 \in Y$, and let $f : X \to Y$ be a continuous function which takes x_0 to y_0.

    Is the following statement true? If f is 1-1 then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is 1-1. Prove or give a counterexample (and if you give a counterexample justify it). You may use anything in Munkres's book.

24. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

    Find an example in which f is onto but $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is not onto. **Prove** that your example really has this property. You may use any fact from Munkres.

25. Let D^2 be the unit disk $\{\, x^2 + y^2 \leq 1 \,\}$ and let S^1 be the unit circle $\{\, x^2 + y^2 = 1 \,\}$. Prove that S^1 is not a retract of D^2 (that is, prove that there is no continuous function $f : D^2 \to S^1$ whose restriction to S^1 is the identity function). You may use anything in Munkres for this.

26. Let X and Y be topological spaces and let $x \in X$, $y \in Y$.

    Prove that there is a 1-1 correspondence between

$$\pi_1(X \times Y, (x, y))$$

and

$$\pi_1(X, x) \times \pi_1(Y, y).$$

(You do **not** have to show that the 1-1 correspondence is compatible with the group structures.)

27. Let $p : Y \to X$ be a covering map, let $y \in Y$, and let $x = p(y)$.

    Let σ be a loop beginning and ending at x and let $[\sigma]$ be the corresponding element of $\pi_1(X, x)$.

    Let $\tilde{\sigma}$ be the unique lifting of σ to a path starting at y.

    Prove that if $[\sigma] \in p_*\pi_1(Y, y)$ then $\tilde{\sigma}$ ends at y.

28. Let $p : \mathbb{R} \to S^1$ be the usual covering map (specifically, $p(t) = (\cos 2\pi t, \sin 2\pi t)$). Let $b_0 \in S^1$ be the point $(1, 0)$. Recall that the standard map

    $$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

    is defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is a lifting of f with $\tilde{f}(0) = 0$.

    (a) **Prove** that ϕ is 1-1.

    (b) **Prove** that ϕ is a group homomorphism.

29. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

    $$\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}.$$

    Let x_0 be the point $(0, 0, 1)$ of S^2.

    Use the Seifert-van Kampen theorem to **prove** that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert-van Kampen theorem given in Munkres's book. You will **not** get credit for any other method.

30. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

    i) Calculate $H_1(X)$. (You may use any fact in Munkres, but be sure to be clear about what you're using.)

    ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?