MA 35100 (Sec: 130): Elementary Linear Algebra

Final Exam

April 30th, 2024

- The exam will be 120 minutes long.
- There are 15 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- For the Long Answer Questions, **show all your work and provide justifications.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. You may use any theorem proved in class provided you accurately state it before using it.
- For True or False (T/F) questions or Multiple Choice Questions (MCQs), you do not need to provide any justification.
- **Be sure to write your final answers in the designated spots.**
- Please use a writing instrument which is dark enough to be picked up by the scanner.
- When the time is over, all students must put down their writing instruments immediately.
- Violating these instructions will be considered an act of academic dishonesty. Any act of dishonesty will be reported to the Office of the Dean of Students. Penalties for such behaviour can be severe and may include an automatic F in the course.

NAME: ____________________

PUID: ____________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

QUESTION	VALUE	SCORE
1	20	
2	5	
3	5	
4	5	
5	5	
6	6	
7	7	
8	10	
9	7	
TOTAL	70	

True or False

1. (20 points)

Determine if the following statements are true or false. Fill in your answers in the following table. You do not need to justify your answers.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

(a) All three eigenvalues of a 3×3 matrix with real entries can be non-real (i.e., have non-zero imaginary part).

(b) If A is a non-square matrix (i.e., $A \in M_{m \times n}$ with $m \neq n$), then either A or A^T will have a nontrivial nullspace.

(c) The set $\{t, 1+t^2, (1+t)^2\}$ is a basis of $P_2(\mathbb{R})$.

(d) For any 2×2 matrices A and B, we have that $\det(A+B) = \det(A)\det(B)$.

(e) The Cauchy-Schwarz inequality states that if $\vec{a}, \vec{b} \in \mathbb{R}^n$, then $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$.

(f) If a square matrix A satisfies $\det A \neq 0$, then A is diagonalizable.

(g) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then T is injective if and only if T is surjective.

(h) If a 3×3 upper triangular matrix has only one eigenvalue (i.e., a single eigenvalue with algebraic multiplicity 3), then it is diagonalizable only if it was diagonal to begin with.

(i) If $A^{-1} = A^T$, then $\det A = 1$.

Multiple Choice Questions

2. (5 points)

Suppose that A is a 4×3 matrix, and $\vec{b}$ is a vector in $\mathbb{R}^4$. Further suppose that the system of equations $A\vec{x} = \vec{b}$ has a unique solution for $\vec{x} \in \mathbb{R}^3$. In this case, which of the following is true?

(a) The equation $A\vec{x} = \vec{b}_1$ always has a unique solution for every $\vec{b}_1 \in \mathbb{R}^4$.

(b) There is a vector $\vec{x}_1 \neq \vec{0}$ such that $A\vec{x}_1 = \vec{0}$.

(c) There is a vector $\vec{b}_2 \in \mathbb{R}^4$ such that $A\vec{x} = \vec{b}_2$ has infinitely many solutions.

(d) The vector $\vec{b}$ is not in the column space of A.

(e) The rank of the matrix A is 3.

Final Answer:

3. (5 points)

Which of the following matrices is *not* diagonalizable over $\mathbb{C}$?

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}.$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 8 \end{bmatrix}$

Final Answer:

4. (5 points)

Let M_{ij} be the (i, j)th minor and C_{ij} be the (i, j)th cofactor of the following matrix:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

Which of the following is correct?

(a) $M_{23} = 4$ and $C_{14} = 4$.

(b) $M_{23} = -4$ and $C_{14} = 4$.

(c) $M_{23} = 4$ and $C_{14} = -4$.

(d) $M_{23} = -4$ and $C_{14} = -4$.

(e) None of the above.

Final Answer:

5. (5 points)

Let $\vec{x} = (0, 3, -1)$ and $\vec{y} = (0, -2, 4)$. Then one can write

$$\vec{x} = \vec{x}_{\|} + \vec{x}_{\perp}$$

where $\vec{x}_{\|}$ is the projection of $\vec{x}$ onto $\vec{y}$ and $\vec{x}_{\perp}$ is the component of $\vec{x}$ orthogonal to $\vec{y}$. Then, $\vec{x}_{\|}$ and $\vec{x}_{\perp}$ are given by:

(a) $\vec{x}_{\|} = (0, 2, 1)$ and $\vec{x}_{\perp} = (0, 1, -2)$.

(b) $\vec{x}_{\|} = (0, 1, -2)$ and $\vec{x}_{\perp} = (0, 2, 1)$.

(c) $\vec{x}_{\|} = (0, -3, 1)$ and $\vec{x}_{\perp} = (0, 1, 3)$.

(d) $\vec{x}_{\|} = (0, 1, 3)$ and $\vec{x}_{\perp} = (0, -3, 1)$.

(e) $\vec{x}_{\|} = (0, 1, -2)$ and $\vec{x}_{\perp} = (0, -3, 1)$.

Final Answer:

Long Answer Questions

6. (6 points)

Compute the LU decomposition of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 7 & 11 \\ -1 & -4 & -9 & -16 & -25 \\ 0 & 0 & 4 & 8 & 12 \end{bmatrix}$$

Final Answer:

(Continued)

7. (7 points)

Find all (complex) solutions of the following system of equations:

$$\begin{array}{cccccc} iz_1 & +z_2 & -(1+i)z_3 & = & 4+4i \\ 3z_1 & +iz_2 & +(1-i)z_3 & = & 4 \\ 2z_1 & +2iz_2 & +(2-2i)z_3 & = & 4i \end{array}$$

Final Answer:

8. (10 points)

Consider the sequence given by $a_0 = 1$, $a_1 = 3$, $a_2 = 4$ and

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \qquad \text{for } n \geq 3.$$

Find a formula for a_n.

Final Answer:

(Continued)

9. (7 points)

Find the general solution to the following system of ODEs:

$$\begin{aligned} x_1' &= x_1 + 2x_2, \\ x_2' &= -2x_1 + x_2. \end{aligned}$$

Your answer should be purely real (i.e., no complex-valued functions).

Final Answer:

(Continued)

Scratch Work