Homework 12

MA 35100 (Spring 2025, §§130–131)

April 11th, 2025

Instructions

- Due: Saturday, April 19th at 11 PM Eastern Time.
- Total Score: 25 points.
- The three lowest homework scores will be dropped from the final grade.
- One late submission is permitted (over the course of the semester) with no questions asked.
- Submissions can be hand-written or typed in LaTeX and must be submitted on Gradescope.
- You are allowed to discuss and collaborate on problems. However, each student must work on the final submission on their own. **In particular, copying someone else's final submission will be considered cheating and will be reported to the Office of the Dean of Students.**

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> I have not made use of any unauthorized resources (including online resources) while working on this submission. Any collaboration with other students conforms with the policies of this course.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

The following is relevant for Problem 1:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Problem 1. [12 points] For each of the matrices $X \in \{A, B, C\}$ find a formula for X^k where $k \geqslant 1$ is a natural number. Using this formula, find A^3, B^{10} and C^{100}. You do not need to simplify any powers of numbers in your final answer.
Hint: Try to diagonalize X by writing $X = PDP^{-1}$ where D is a diagonal. Find a formula for X^k in terms of P and D^k and use the fact that D is diagonal to compute D^k easily.

Problem 2. [3 points] Two $n \times n$ matrices A and B are said to commute if $AB = BA$. On the other hand, two $n \times n$ matrices are said to be simultaneously diagonalizable if they share an eigenbasis $\{\vec{b}_1, \cdots, \vec{b}_n\}$, and hence can be written as

$$A = PDP^{-1}, \qquad B = P\Delta P^{-1},$$

where P is an $n \times n$ matrix whose jth column is the jth element of the shared eigenbasis b_j and D and Δ are $n \times n$ diagonal matrices whose (jj)th entries are respectively the eigenvalues of A and B for the eigenvector b_j. In other words, two matrices are simultaneously diagonalizable if there is a coordinate system in which both of them are represented by diagonal matrices.

Show that two simultaneously diagonalizable matrices must commute.

Problem 3. [10 points] A complex number $z \in \mathbb{C}$ is called a root of unity if $z^n = 1$ for some n.

(a) Show that if z is a root of unity, then it must have modulus equal to 1.

(b) Using de Moivre's formula or Euler's formula, find a formula for all nth roots of unity. How many nth roots of unity are there? [**Hint:** if $\cos\theta = 1$ and $\sin\theta = 0$, then $\theta = 2\pi k$ for some integer k.]

(c) Explicitly compute all the nth roots for $n = 1, 2, 3, 4, 8$.

(d) A root of unity is called a *primitive* nth root of unity if it is an nth root of unity but it is not an mth root of unity for $m < n$. It is called imprimitive otherwise. If ω is an imprimitive nth root of unity, then it must be a primitive mth root of unity for some m dividing n (You may assume this fact). Compute the primitive 8th roots of unity.

(e) If s is a positive real number, find all complex roots to the equation $z^n = s$. Your answer should be in terms of s, roots of unity, and can use radicals of real numbers (e.g. $\sqrt[5]{3}$ or $\sqrt[n]{s+1}$).