MTH 142 **Final Exam** **December 17, 2016**

Last/Family Name: ______________________________

First/Given Name: ______________________________

Student ID Number: ______________________________

Instructor (circle): Hambrook (MW 3:25) Zeng (MW 9:00)

Honor Pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

You must write out and sign the honor pledge for your examination to be valid.

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Signature: ______________________________ Date: ______________________________

QUESTION	VALUE	SCORE
1	15	
2	10	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	15	
13	10	
14	10	
15	10	
TOTAL	165	

Instructions:

- Time: 3 hours.
- Write in pencil or pen.
- No notes, textbooks, phones, calculators, or other electronic devices.
- If you need extra space, use the back of the page, and indicate it.
- To receive full credit, you must show your work and justify your answers.
- The final page is a formula sheet. You may detach it.

1. (15 points) Consider the function f with

$$f(x) = \frac{4x^2}{x^2-4}, \quad f'(x) = \frac{-32x}{(x^2-4)^2}, \quad f''(x) = \frac{32(3x^2+4)}{(x^2-4)^3}.$$

(a) Find the domain of f.

$x^2 - 4 \neq 0 \quad \Rightarrow \quad x \neq \pm 2.$

Domain of f : $(-\infty,-2) \cup (-2,2) \cup (2,+\infty)$

(b) List all x- and y-intercepts of f.

y- intercept : 0.

x - intercept : 0.

(c) List all vertical asymptotes of f or explain why none exist.

$\lim_{x\to 2^+} \frac{4x^2}{x^2-4} = +\infty.$ $\quad \lim_{x\to 2^-} \frac{4x^2}{x^2-4} = -\infty.$

$\lim_{x\to -2^+} \frac{4x^2}{x^2-4} = -\infty$ $\quad \lim_{x\to -2^-} \frac{4x^2}{x^2-4} = +\infty.$

Vertical asymptote : $x = \pm 2$.

(d) List all horizontal asymptotes of f or explain why none exist.

$\lim_{x\to +\infty} \frac{4x^2}{x^2-4} = 4$ $\quad$ Horizontal asymptote : $y = 4$

$\lim_{x\to -\infty} \frac{4x^2}{x^2-4} = 4.$

(e) On what intervals is $f(x)$ increasing? decreasing?

$$f'(x) = \frac{-32x}{(x^2-4)^2}.$$

	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,+\infty)$
$f'(x)$	>0	>0	<0	<0.
	increasing	increasing	decreasing	decreasing

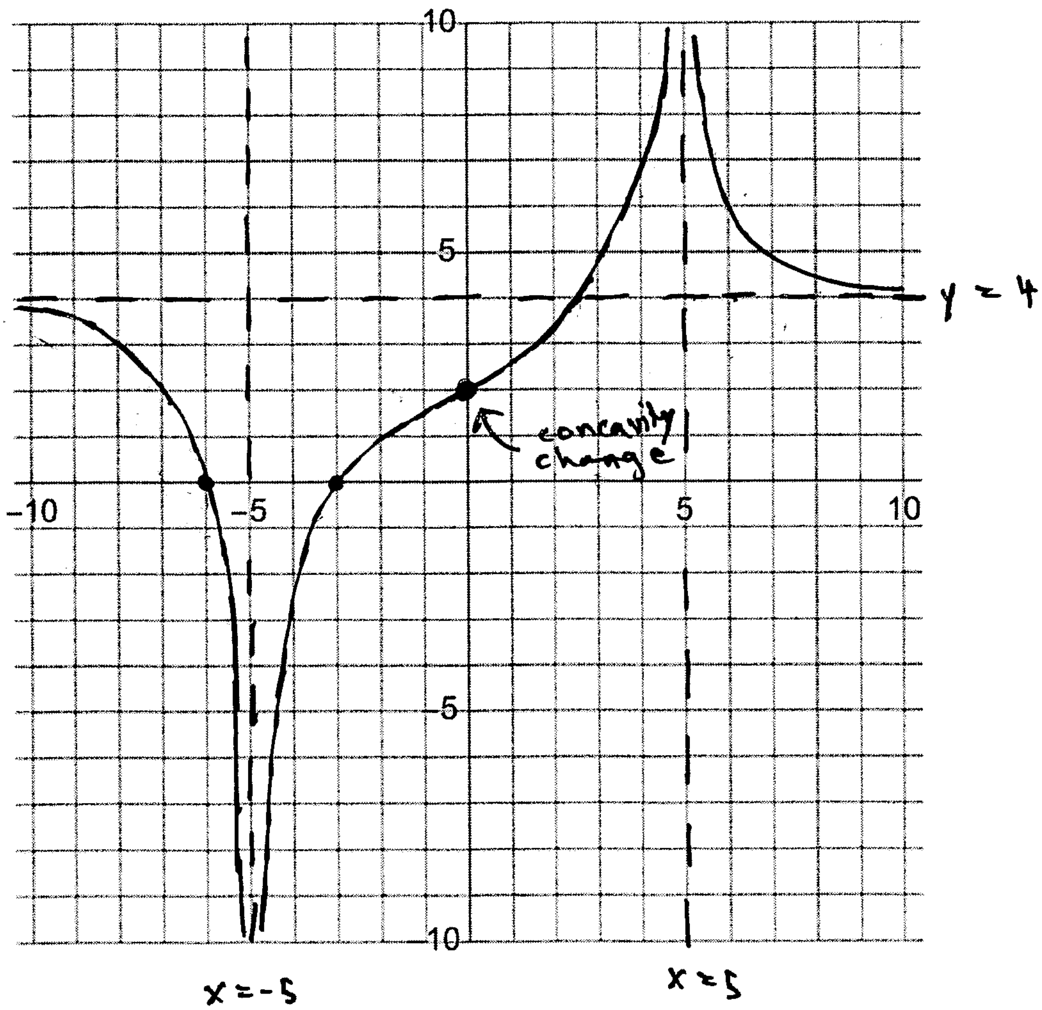
(f) On what intervals is $f(x)$ concave up? concave down?

$$f''(x) = \frac{32(3x^2+4)}{(x^2-4)^3}$$

	$(-\infty,-2)$	$(-2,2)$	$(2,+\infty)$
$f''(x)$	>0	<0	>0.
	Concave up	Concave down	Concave up

2. (10 points) Sketch the graph of a function $g(x)$ that satisfies the following properties:

- g is continuous at all points of its domain
- x-intercepts: $-6, -3$
- y-intercepts: 2
- $\lim_{x \to -5^-} g(x) = -\infty$ and $\lim_{x \to -5^+} g(x) = -\infty$
- $\lim_{x \to 5^-} g(x) = +\infty$ and $\lim_{x \to 5^+} g(x) = +\infty$
- $\lim_{x \to -\infty} g(x) = 4$ and $\lim_{x \to +\infty} g(x) = 4$
- increasing on $(-5, 5)$
- decreasing on $(-\infty, -5) \cup (5, \infty)$
- concave up on $(0, 5) \cup (5, \infty)$
- concave down ~~on~~ $(-\infty, -5) \cup (-5, 0)$



3. (15 points) A cylindrical can is being designed to have volume 20π m^3. The material for the top and bottom costs \$10/m^2 and material for the side costs \$8/m^2. Find the radius r and height h of the most economical (lowest cost) can.

Suppose T represents the total cost.

$$T = 2\pi r^2 \cdot 10 + 2\pi r \cdot h \cdot 8$$

We have. $\pi r^2 h = 20\pi \Rightarrow h = \frac{20}{r^2}$

$$\Rightarrow T(r) = 2\pi r^2 \cdot 10 + 2\pi r \cdot \frac{20}{r^2} \cdot 8$$

$$= 20\pi r^2 + 320\pi \cdot \frac{1}{r}.$$

$$\Rightarrow T'(r) = 40\pi r - \frac{320\pi}{r^2}.$$

Set $T'(r) = 0$ to find the critical pt.

$$40\pi r = \frac{320\pi}{r^2} \Rightarrow r = 2.$$

	$(0,2)$	$(2, +\infty)$
$T'(r)$	< 0	> 0

Thus when $r = 2$, $h = 5$.

it ~~cost~~ is the most economical.

4. (10 points)

The table gives the values of a function f obtained from an experiment.

x	2	5	8	11	14
$f(x)$	-1.5	-0.5	1.0	2.1	2.4

(a) Use the table to approximate $\int_2^{14} f(x)dx$ by a Riemann sum with right endpoints and four equal subintervals.

$3\times(-0.5) + 3\times 1.0 + 3\times 2.1 + 3\times 2.4$

$= -1.5 + 3 + 6.3 + 7.2$

$= 15$

(b) If f is known to be an increasing function, is your estimate larger or smaller than the exact value of the integral.

larger.

5. (10 points) Evaluate:

(a) $\dfrac{d}{dt}\displaystyle\int_0^{t^2} \frac{1}{x+\cos x}\,dx$

$$= \frac{1}{t^2+\cos t^2} \cdot 2t$$

(b) $\displaystyle\int_0^{t^2} \frac{d}{dx}\left(\frac{1}{x+\cos x}\right)\,dx$

$$= \left.\frac{1}{x+\cos x}\right|_0^{t^2} = \frac{1}{t^2+\cos t^2} - 1$$

6. (10 points) The velocity function (in meters per second) for a particle moving in one dimension is

$$v(t) = t^2 - t - 2$$

Find the position $s(t)$ of the particle at time t, given that $s(2) = 2$.

$$s(t) - s(2) = \int_2^t v(s)\,ds$$

$$\Rightarrow \quad s(t) = s(2) + \int_2^t v(s)\,ds$$

$$= 2 + \int_2^t (s^2 - s - 2)\,ds$$

$$= 2 + \left(\frac{s^3}{3} - \frac{s^2}{2} - 2s\right)\Big|_2^t$$

$$= 2 + \left(\frac{t^3}{3} - \frac{t^2}{2} - 2t\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)$$

$$= \frac{t^3}{3} - \frac{t^2}{2} - 2t + \frac{16}{3}.$$

7. (10 points) Evaluate the integral.

$$\int_1^3 x^{-1} \ln x \, dx$$

$u = \text{Ln} x$

$du = \frac{1}{x} dx$

$dx = x \cdot du$

$$= \int_{x=1}^{x=3} \cancel{x}^{-1} \cdot u \cdot \cancel{x} \, du$$

$$= \int_{\text{Ln}1=u}^{\text{Ln}3=u} u \, du$$

$$= \frac{1}{2} u^2 \Big|_0^{\text{Ln}3} = \frac{1}{2}(\text{Ln}3)^2$$

8. (10 points) Evaluate the integral.

$$\int_1^3 x \ln x \, dx$$

$$= \left.\frac{x^2 \operatorname{Ln} x}{2}\right|_1^3 - \int_1^3 \frac{1}{2}x^2 \cdot \frac{1}{x} dx \qquad u = \operatorname{Ln} x \quad dv = x\,dx$$

$$du = \frac{1}{x} dx \quad v = \frac{1}{2}x^2.$$

$$= \frac{9 \operatorname{Ln} 3}{2} - 0 - \int_1^3 \frac{1}{2} x \, dx$$

$$= \frac{9 \operatorname{Ln} 3}{2} - \left.\frac{1}{4}x^2\right|_1^3$$

$$= \frac{9 \operatorname{Ln} 3}{2} - \left(\frac{9}{4} - \frac{1}{4}\right)$$

$$= \frac{9 \operatorname{Ln} 3}{2} - 2.$$

9. (10 points) Evaluate the integral.

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

$$= \int \frac{\sqrt{(2\sec\theta)^2-4}}{2\sec\theta}\, 2\sec\theta\tan\theta\, d\theta$$

$x = 2\sec\theta.$

$dx = 2\sec\theta\cdot\tan\theta\, d\theta.$

$$= \int \frac{\sqrt{4(\sec^2\theta-1)}}{\cancel{2}\cancel{\sec\theta}}\, \cancel{2}\cancel{\sec\theta}\cdot\tan\theta\, d\theta$$

$$= \int 2\tan^2\theta\, d\theta. = 2\int (\sec^2\theta - 1)\, d\theta$$

$$= 2\int \sec^2\theta\, d\theta - 2\theta$$

$$= 2\tan\theta - 2\theta + C$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{2}\right)^2 - 1}$$

$$\theta = \arctan\sqrt{\left(\frac{x}{2}\right)^2 - 1}$$

Thus the final answer is $2\cdot\sqrt{\left(\frac{x}{2}\right)^2-1} - 2\arctan\sqrt{\left(\frac{x}{2}\right)^2-1} + C.$

10. (10 points) Evaluate the integral.

$$\int \frac{5x}{(x+1)(x^2+4)}dx$$

Suppose

$$\frac{5x}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow \quad 5x = A(x^2+4) + (Bx+C)(x+1)$$

$$= Ax^2 + 4A + Bx^2 + (B+C)x + C$$

$$= (A+B)x^2 + (B+C)x + 4A + C$$

$$\Rightarrow \begin{cases} A+B=0 \\ B+C=5 \\ 4A+C=0 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=+1 \\ C=\cancel{6}4 \end{cases}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+4)} dx = \int -\frac{1}{1+x} dx + \int \frac{x+4}{x^2+4} dx$$

$$= -\ln|x+1| + \frac{1}{2}\ln|x^2+4| + 4 \cdot \frac{1}{2}\arctan\frac{x}{2}$$

$$+ C$$

11. (10 points) Evaluate the integral.

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}dx$$

$$= \lim_{t\to+\infty} \int_0^t \frac{e^x}{(1+e^x)^2}dx$$

$u = 1+e^x$

$du = e^x\,dx$.

$dx = \frac{1}{e^x}\,du$

$$= \lim_{t\to+\infty} \int_0^{x=t} \frac{\cancel{e^x}}{u^2} \cdot \frac{1}{\cancel{e^x}} \cdot du.$$

$$= \lim_{t\to+\infty} \int_1^{1+e^t} \frac{1}{u^2}\,du.$$

$$= \lim_{t\to+\infty} -\frac{1}{u}\Big|_1^{1+e^t} = \lim_{t\to+\infty} -\frac{1}{1+e^t}+1$$

$$= 1.$$

The improper integral is convergent and it equals to 1.

12. (15 points) Find the length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 2.$$

$$y' = x^2 - \frac{1}{4x^2}$$

$$L = \int_1^2 \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + x^4 - 2 \cdot x^2 \cdot \frac{1}{4x^2} + \left(\frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{x^4 + \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{x^4 + 2 \cdot x^2 \cdot \frac{1}{4x^2} + \left(\frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

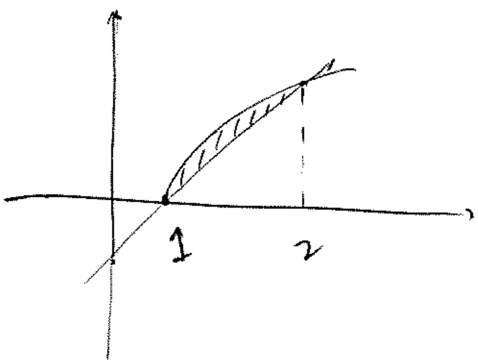
$$= \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx = \left.\frac{x^3}{3} - \frac{1}{4x}\right|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{7}{3} + \frac{1}{8} = \frac{59}{24}$$

13. (10 points) Sketch the region enclosed by the given curves. Then set up an integral equal to its area. Do **not** evaluate the integral.

$$y = \sqrt{x-1}, \quad x - y = 1.$$



Find their intersection:

$$\begin{cases} y = \sqrt{x-1} \\ x - y = 1 \end{cases}$$

$$\Rightarrow \quad \sqrt{x-1} = y = -1 + x$$

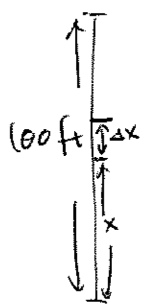
$$\Rightarrow \quad x - 1 = x^2 - 2x + 1$$

$$\Rightarrow \quad x^2 - 3x + 2 = 0$$

$$\Rightarrow \quad \begin{cases} x = 1, \\ y = 0 \end{cases} \text{or} \begin{cases} x = 2 \\ y = 1 \end{cases}$$

$$\text{Area} = \int_1^2 \left[\sqrt{x-1} - (x-1)\right] dx$$

14. (10 points) A rope measuring 120 ft long and weighing 0.5 lb/ft sits coiled on the ground at the base of a 100 ft cliff. A climber attaches one end of the rope to herself and leaves the other end sitting on the ground as she climbs to the top of the cliff. Set up an integral equal to the work done on the rope when the climber reaches the top. Do **not** evaluate the integral.



Consider the ~~climber~~ work to climbs from heigh x ft to $x+\Delta x$ ft.

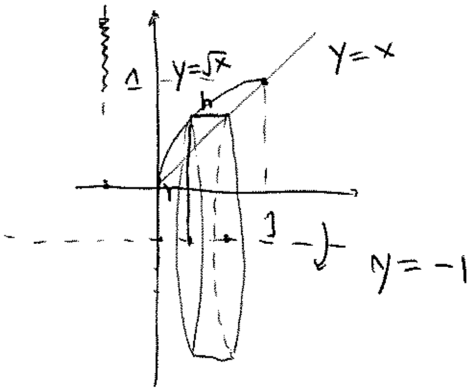
$$\Delta W = (0.5 \cdot x) \cdot \Delta x.$$

$$\Rightarrow \quad W = \int_0^{100} 0.5x \, dx.$$

15. (10 points) Consider the solid obtained when the region bounded by

$$y = x, \quad y = \sqrt{x}$$

is rotated about the line $y = -1$. Use the method of **cylindrical shells** to set up an integral equal to the volume of the described solid. Do **not** evaluate the integral.



$$\int_0^1 2\pi\cdot(y+1)(y-y^2)\,dy$$

Formula Sheet

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}$$

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin^2 x = \frac{1}{2}(1-\cos(2x)) \qquad \sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$$
$$\cos^2 x = \frac{1}{2}(1+\cos(2x)) \qquad \cos A\cos B = \frac{1}{2}(\cos(A-B)+\cos(A+B))$$
$$\sin x\cos x = \frac{1}{2}\sin(2x) \qquad \sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \cot x\,dx = -\ln|\csc x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$