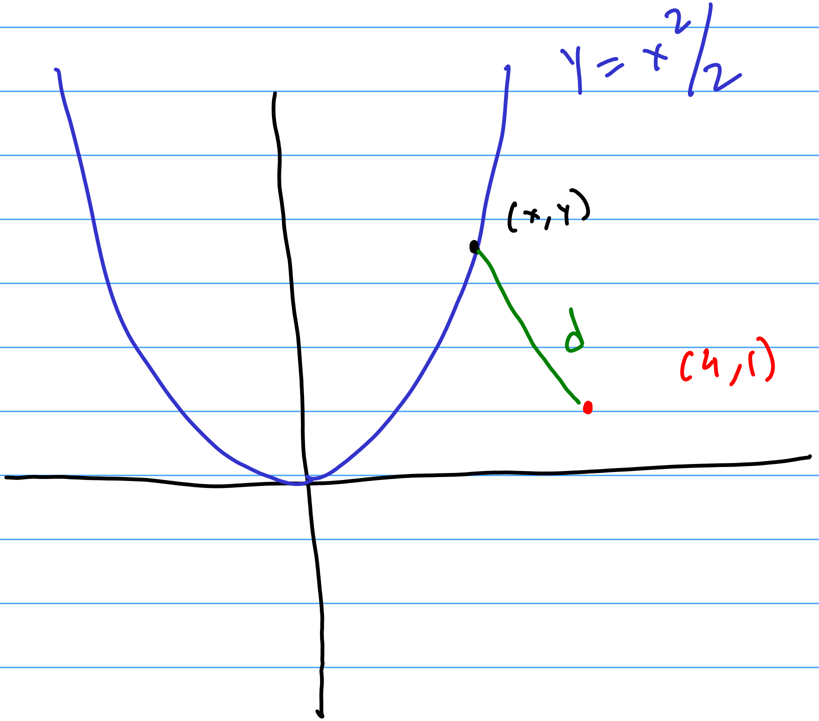
# MATH 142, SUMMER 2021 (A2)

FINAL EXAM

SOLUTIONS

**2. (15 points)**

Find the point on the parabola $y = x^2/2$ that is closest to the point $(4, 1)$.




$$d^2 = (x-4)^2 + (y-1)^2$$

$$y = \frac{x^2}{2}$$

$$\Rightarrow f(x) = d^2 = (x-4)^2 + \left(\frac{x^2}{2} - 1\right)^2$$

MINIMIZER OF $d$ = MINIMIZER OF $d^2$ !

$$f(x) = \frac{x^4}{4} - 8x + 17$$

$$\Rightarrow \quad f'(x) = x^3 - 8$$

$$\Rightarrow \quad f'(x) \begin{cases} > 0 & \text{IF } x > 2 \\ = 0 & \text{IF } x = 2 \\ < 0 & \text{IF } x < 2 \end{cases}$$

$\Rightarrow$ $x = 2$ IS A GLOBAL MINIMIZER

$$\Rightarrow y = \frac{x^2}{2} = \frac{2^2}{2} = 2$$

$\therefore (x, y) = (2, 2)$ HAS MIN. DISTANCE.

**3. (30 points)** Consider the region bounded in the first quadrant by $y = x^3$ and $y = 9x$.

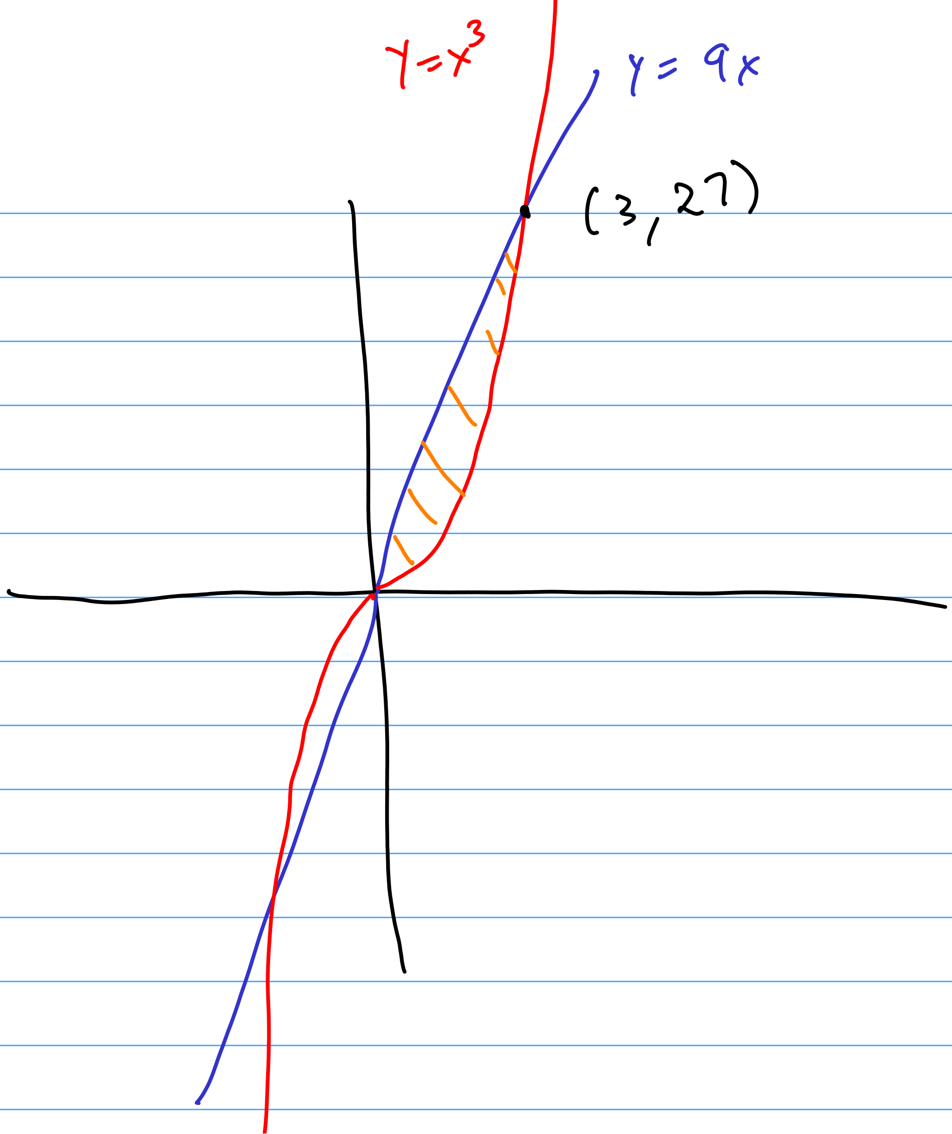
(a) Sketch the curves and shade the region described above.

(b) Write (but do NOT evaluate) an integral that is equal to the area of the region.

(c) Write (but do NOT evaluate) an integral using the shell method for the volume of the solid obtained by revolving the region about the $y$-axis.

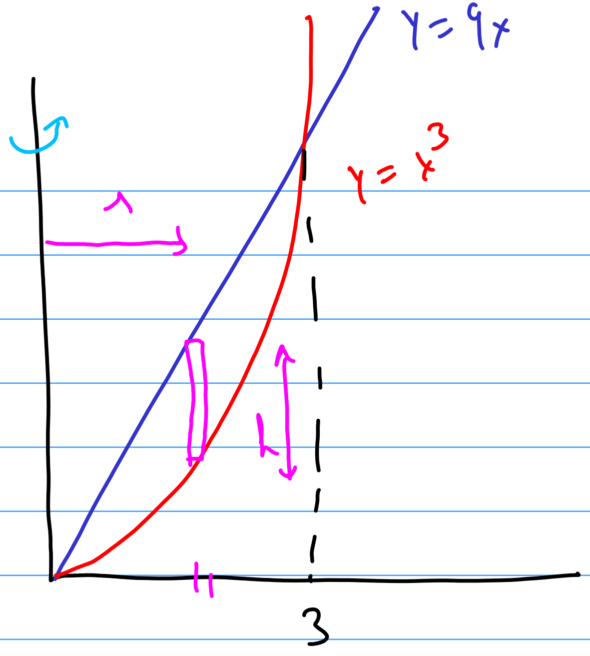
(d) Write (but do NOT evaluate) an integral using the washer method for the volume of the solid obtained by revolving the region about the line $x = 3$.

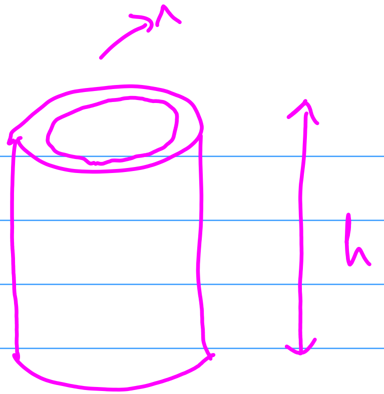
(a)




(b) $\displaystyle\int_0^3 (9x - x^3)\,dx$

(c)



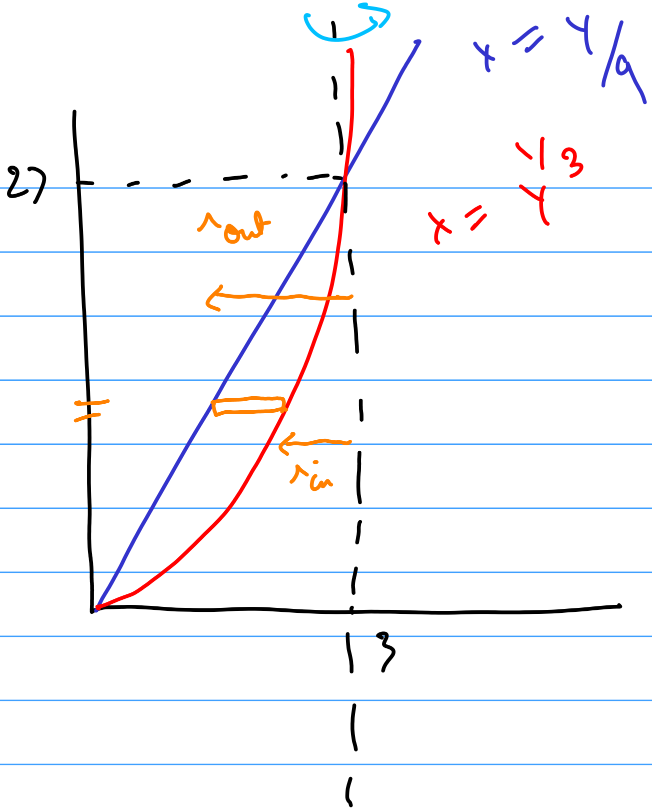



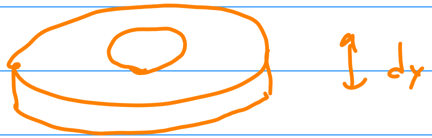

$r = x$

$h = (9x - x^3)$

$$V = \int_0^3 2\pi x \left( 9x - x^3 \right) dx$$

(d)




$$r_{out} = 3 - \frac{y}{9}$$

$$r_{in} = 3 - y^{1/3}$$

$$V = \int_0^{27} \pi \left[ \left(3 - y/9\right)^2 - \left(3 - y^{1/3}\right)^2 \right] dy$$

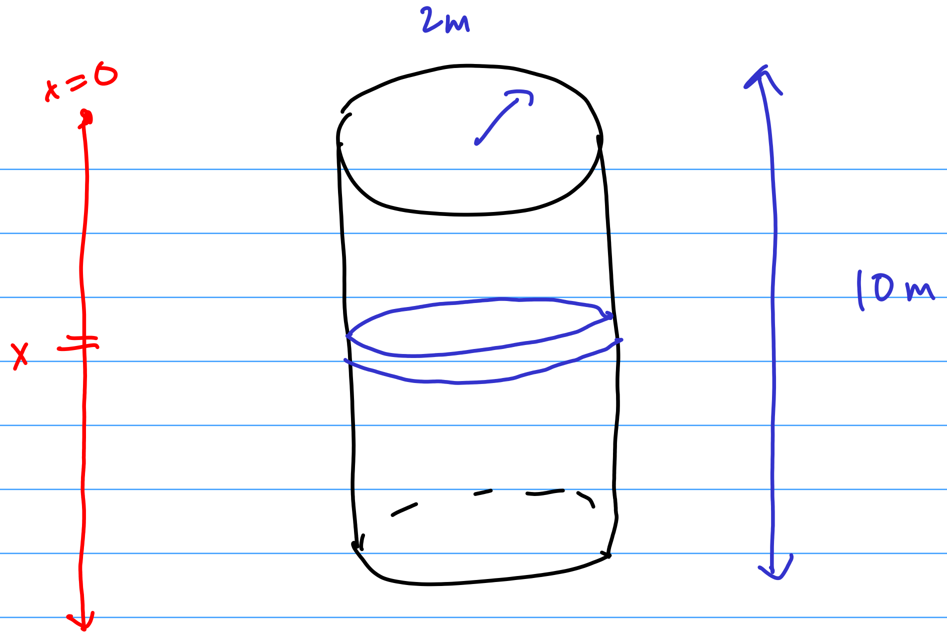
**4. (20 points)**

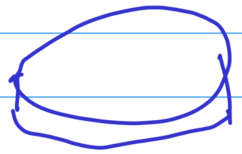
A cylindrical tank of height $h = 10\,\mathrm{m}$, and radius $r = 2\,\mathrm{m}$. The water is pumped out of a hole at the top of the tank over time.

(a) Write (but do NOT evaluate) an integral that represents the work done to empty the tank.

(b) Write (but do NOT evaluate) an integral that represents the work done to empty half the tank.

Recall that the density of water is $1000\,\mathrm{kg\,m^{-3}}$ and that the acceleration due to gravity is $9.8\,\mathrm{m\,s^{-2}}$.



$$V = \pi(2^2)\,dx$$
$$= 4\pi\,dx$$

$$m = \rho V = (1000)(4\pi)\,dx$$

$$F = mg = (4000\pi\,dx)(9.8)$$
$$= 39200\,\pi\,dx$$

$$\therefore\ W = (F)(d) = 39200\pi\,x\,dx$$

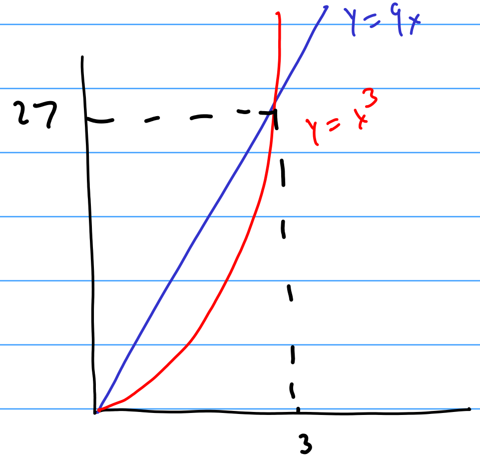
(a) $$\int_0^{10} 39200\pi \, x \, dx$$

(b) $$\int_0^{5} 39200\pi \, x \, dx$$

**5. (15 points)**

Recall from a previous question the region bounded in the first quadrant by $y = x^3$ and $y = 9x$.

Express the perimeter of the region as the sum of two integrals, one of which is against $dx$ and the other of which is against $dy$. Do NOT evaluate the integrals.




RED (IN x) $= \displaystyle\int_0^3 \sqrt{1 + (3x^2)^2}\, dx$

BLUE (IN y) $= \displaystyle\int_0^{27} \sqrt{1 + \left(\tfrac{1}{9}\right)^2}\, dy$

**6. (30 points)**

Compute the following indefinite integrals:

(a)

$$\int \frac{\sin 2\theta}{\sqrt{3 - 2\sin\theta - \sin^2\theta}}\, d\theta$$

(b)

$$\int \cos(\ln x)\, dx$$

(a)

$$\int \frac{\sin 2\theta}{\sqrt{3 - 2\sin\theta - \sin^2\theta}} \, d\theta$$

$$= 2 \int \frac{\sin\theta \quad (\cos\theta \, d\theta)}{\sqrt{3 - 2\sin\theta - \sin^2\theta}}$$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

$$\Rightarrow \quad 2\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

$$3-2x-x^2 = 4-(1+x)^2$$

[$\because$ COMPLETING THE SQUARE]

$$\therefore \quad 2\int \frac{x}{\sqrt{4-(1+x)^2}}\,dx$$

$u = x+1$

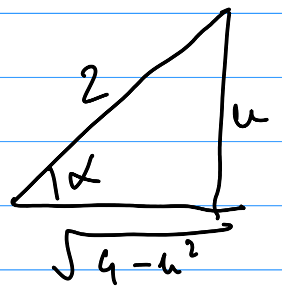
$$\Rightarrow \quad 2\int \frac{(u-1)\,du}{\sqrt{4-u^2}}$$

$$u = 2\sin\alpha \quad \Rightarrow \quad du = 2\cos\alpha\, d\alpha \;\&\; \sqrt{4-u^2} = 2\cos\alpha$$

$$\therefore \quad 2\int \frac{(2\sin\alpha - 1)(2\cos\alpha)}{(2\cos\alpha)}\,d\alpha = \int 4\sin\alpha\, d\alpha - \int 2\,d\alpha$$

$$= -4\cos\alpha - 4\alpha$$

$\sin\alpha = \frac{u}{2}$




$$= -4\left[\frac{\sqrt{4-u^2}}{2}\right] - 4\arcsin\left(\frac{u}{2}\right) + C$$

$$= -2\sqrt{4-u^2} - 4\arcsin(u/2) + C$$

$\cos\alpha = \frac{\sqrt{4-u^2}}{2}$

$$= -2\sqrt{3-2x-x^2} - 4\arcsin\left(\frac{x+1}{2}\right) + C \quad [\because u = x+1]$$

$$= -2\sqrt{3-2\sin\theta-\sin^2\theta} - 4\arcsin\left(\frac{\sin\theta+1}{2}\right) + C \quad [\because x = \sin\theta]$$

(b) $\int \cos(\ln x)\,dx$

SOLN #1 : $u = \ln x \iff x = e^u \implies dx = e^u$

$$\therefore \int \cos(u)\,(e^u\,du) = \int e^u \cos u\,du$$

$$\int e^u \cos u\,du = \frac{1}{2} e^u (\cos u + \sin u) + C$$

[SEE BOOK / NOTES / MIDTERM II]

$$\therefore \int \cos(\ln x)\,dx = \frac{1}{2} x \left[\cos(\ln x) + \sin(\ln x)\right] + C$$

SOLN # 2 :

$$\int \cos(\ln x)\, dx$$

$$u = \cos(\ln x) \Rightarrow du = -\frac{\sin(\ln x)}{x}\, dx$$

$$dv = dx \Rightarrow x = v$$

$$\therefore \quad x \cos(\ln x) - \int (x)\left(-\frac{\sin(\ln x)}{x}\right) dx = x\cos(\ln x) + \int \sin(\ln x)\, dx$$

$$u = \sin(\ln x) \Rightarrow du = \frac{\cos(\ln x)}{x}\,dx$$

$$dv = dx \Rightarrow v = x$$

$$\therefore \quad x\cos(\ln x) + \left[x\sin(\ln x) - \int (x)\left(\frac{\cos(\ln x)}{x}\right)dx\right]$$

$\therefore$ IF $\displaystyle I = \int \cos(\ln x)\,dx$

$$I = x\cos(\ln x) + x\sin(\ln x) - I$$

$$\Rightarrow I = \frac{1}{2} x \left( \cos(\ln x) + \sin(\ln x) \right) + C$$

**7. (30 points)**

Compute the following definite integrals:

(a)

$$\int_2^3 \frac{dv}{v^2(v^2-1)}$$

(b)

$$\int_0^{2\pi} (\sin x)^2 (\cos x)^4 \, dx$$

(c)

$$\int_0^1 \arctan x \, dx$$

(a) $$\int_2^3 \frac{dv}{v^2(v^2-1)}$$

$$v^2(v^2-1) = v^2(v-1)(v+1)$$

$$\therefore \frac{1}{v^2(v^2-1)} = \frac{a}{v} + \frac{b}{v^2} + \frac{c}{v-1} + \frac{d}{v+1}$$

$$\Rightarrow 1 = a\,v(v-1)(v+1) + b(v-1)(v+1) + c\,v^2(v+1) + d\,v^2(v-1)$$

SET

(i) $v = 0 \Rightarrow 1 = -b \Rightarrow b = -1$

(ii) $v = 1 \Rightarrow 1 = 2c \Rightarrow c = 1/2$

(iii) $v = -1 \Rightarrow 1 = -2d \Rightarrow d = -1/2$

(iv) $v = 2 \Rightarrow 1 = 6a + 3b + 12c + 4d$

$$\Rightarrow a = 1 - (3)(-1) - (12)(1/2) - (4)(-1/2)$$

$$= 0$$

$$\therefore \int_2^3 \frac{1}{2(v-1)} - \frac{1}{2(v+1)} - \frac{1}{v^2} \, dv$$

$$= \frac{1}{2}\ln(v-1)\Big]_2^3 - \frac{1}{2}\ln(v+1)\Big]_2^3 + \frac{1}{v}\Big]_2^3$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left(\ln 4 - \ln 3\right) + \frac{1}{3} - \frac{1}{2}$$

(b) $$\int_0^{2\pi} (\sin^2 x)(\cos^4 x)\, dx$$

$$(\sin^2 x)(\cos^4 x) = (\sin x \cos x)^2 \cdot (\cos^2 x)$$

$$= \left(\frac{\sin 2x}{2}\right)^2 \left(\frac{1 + \cos 2x}{2}\right)$$

$( \because$ DOUBLE - ANGLE $)$

$$\int_0^{2\pi} \left(\frac{\sin^2 2x}{4}\right)\left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int_0^{2\pi} \sin^2 2x \; dx + \frac{1}{8}\int_0^{2\pi} (\sin^2 2x)(\cos 2x)\, dx$$

$$= \frac{1}{8}\int_0^{2\pi} \left(\frac{1 - \cos 4x}{2}\right) dx + \left.\frac{1}{8}\int_0^{0} (u^2)\left(\frac{du}{2}\right)\right\} \rightarrow = 0$$

(DOUBLE - ANGLE) $\qquad$ $(u = \sin^2 2x)$

$$\therefore \quad \frac{1}{16}\int_0^{2\pi} dx \quad - \quad \frac{1}{16}\int_0^{2\pi} \cos 4x \, dx$$

$$= \frac{1}{16}\left(2\pi\right) \quad - \quad \frac{1}{16}\left[\sin 8\pi - \sin 0\right]$$

$$= \frac{\pi}{8}$$

(c) SEE MIDTERM II SOLNS.

Q 6. (c)

**8. (30 points)**

(a) Is the following integral improper? In either case, compute the value of the integral.

$$\int_0^2 \frac{dx}{(x-1)^{1/3}}$$

(b) Determine if the following integral converges or diverges.

$$\int_1^\infty \frac{(\cos x)^4}{x^2}\,dx.$$

(c) Determine if the following integral converges or diverges.

$$\int_e^\infty \frac{\ln x}{x}\,dx.$$

(a) IT IS IMPROPER $\rightarrow$ DISCONTINUITY AT

$x = 1 \in [0,2]$

$$\therefore \int_0^2 = \int_0^1 + \int_1^2$$

$$\int_0^1 = \lim_{t \to 1^-} \int_0^t \frac{dx}{(x-1)^{1/3}} = \lim_{t \to 1^-} \frac{3}{2}(x-1)^{2/3}\Big]_0^t$$

$$= \lim_{t \to 1^-} \frac{3}{2}\left[(t-1)^{2/3} - 1\right] = -\frac{3}{2}$$

$$\int_1^2 = \lim_{t\to 1^+} \int_t^2 \frac{dx}{(x-1)^{1/3}} = \lim_{t\to 1^+} \frac{3}{2}(x-1)^{2/3}\Big]_t^2$$

$$= \lim_{t\to 1^+} \frac{3}{2}\left[1 - (t-1)^{2/3}\right] = \frac{3}{2}$$

$$\therefore \int_0^2 = \int_0^1 + \int_1^2 = -\frac{3}{2} + \frac{3}{2} = 0$$

(b)

$$0 \leq \cos^4 x \leq 1 \quad \forall x$$

$$\Rightarrow \quad 0 \leq \frac{\cos^4 x}{x^2} \leq \frac{1}{x^2} \quad \forall x \geq 1$$

$$\therefore \quad \int_1^\infty \frac{\cos^4 x}{x^2} \leq \int_1^\infty \frac{dx}{x^2}$$

[ $\because$ COMPARISON TEST ]

$$= \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = \lim_{t \to \infty} \left. \frac{-1}{x} \right]_1^t$$

$$= \lim_{t \to \infty} \left( 1 - \frac{1}{t} \right) = 1 < \infty$$

$$\Rightarrow \int_1^\infty \frac{\ln^4 x}{x^2} \, dx \quad \text{IS} \quad \text{CONVERGENT.}$$

(c) FOR $x \geq e$, $\ln x \geq \ln e = 1 \geq 0$

$$\therefore \quad 0 \leq \frac{1}{x} \leq \frac{\ln x}{x}$$

$$\Rightarrow \int_e^{\infty} \frac{dx}{x} \leq \int_e^{\infty} \frac{\ln x}{x}\, dx$$ [$\because$ COMPARISON TEST]

$$\int_e^{\infty} \frac{dx}{x} = \lim_{t \to \infty} \int_e^{t} \frac{dx}{x} = \lim_{t \to \infty} \ln x \Big]_e^t = \lim_{t \to \infty} \left[ \ln t - 1 \right]$$

$$= +\infty$$

$$\Rightarrow \int_{e}^{\infty} \frac{\ln x}{x}\, dx \quad \text{IS} \quad \text{DIVERGENT.}$$