MATH 142 (SUMMER '21, SESH A2)

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LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

BREAK
~5–10 MIN

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2021/math142/index.html

SHORT URL: bit.ly/sahay142

COURSE INFORMATION

1. TEXTBOOK & REQUIREMENTS (WORKING WEBCAM!)

2. PREREQUISITES $\longrightarrow$ MTH 141

NEED $\begin{cases} \geq C- & \text{[IF GRADED]} \\ \text{PASS} & \text{[IF P/F]} \end{cases}$

3. (TENTATIVE) SCHEDULE & COURSE DESCRIPTION

* ALL LECTURES WILL BE RECORDED
* OFFICE HOURS WILL $\boxed{\text{NOT}}$ BE RECORDED

COURSE INFORMATION

4. EXAMS

MIDTERM 1 : MONDAY, JUNE 7th

MIDTERM 2 : MONDAY, JUNE 21st

FINAL EXAM : THURSDAY, JULY 1st

PART A OF FINAL WILL REPLACE A MIDTERM IF THIS IMPROVES YOUR GRADE.

5. HOMEWORK (VIA WEBWORK)

↳ USUALLY 2 PER WEEK

↳ DETAILS TBA

6. GRADING

7. POLICIES (a) ACADEMIC HONESTY

(b) DISABILITY SUPPORT

(c) PARTICIPATION

(d) ZOOM

OTHER NOTES

1. GOOGLE FORM
2. WILL UPLOAD LECTURE NOTES
3. MINIMAL USE OF BLACKBOARD
   (ONLY EMAILS & GRADES)
4. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!
5. DON'T FALL BACK (ONLY 6 WEEKS!)

$y = x^2 x + 1$

$y = \frac{x^2}{2x^2 - 1}$

§ 4.5 SUMMARY OF CURVE SKETCHING

GUIDING PRINCIPLES

A. DOMAIN

B. INTERCEPTS

C. SYMMETRY

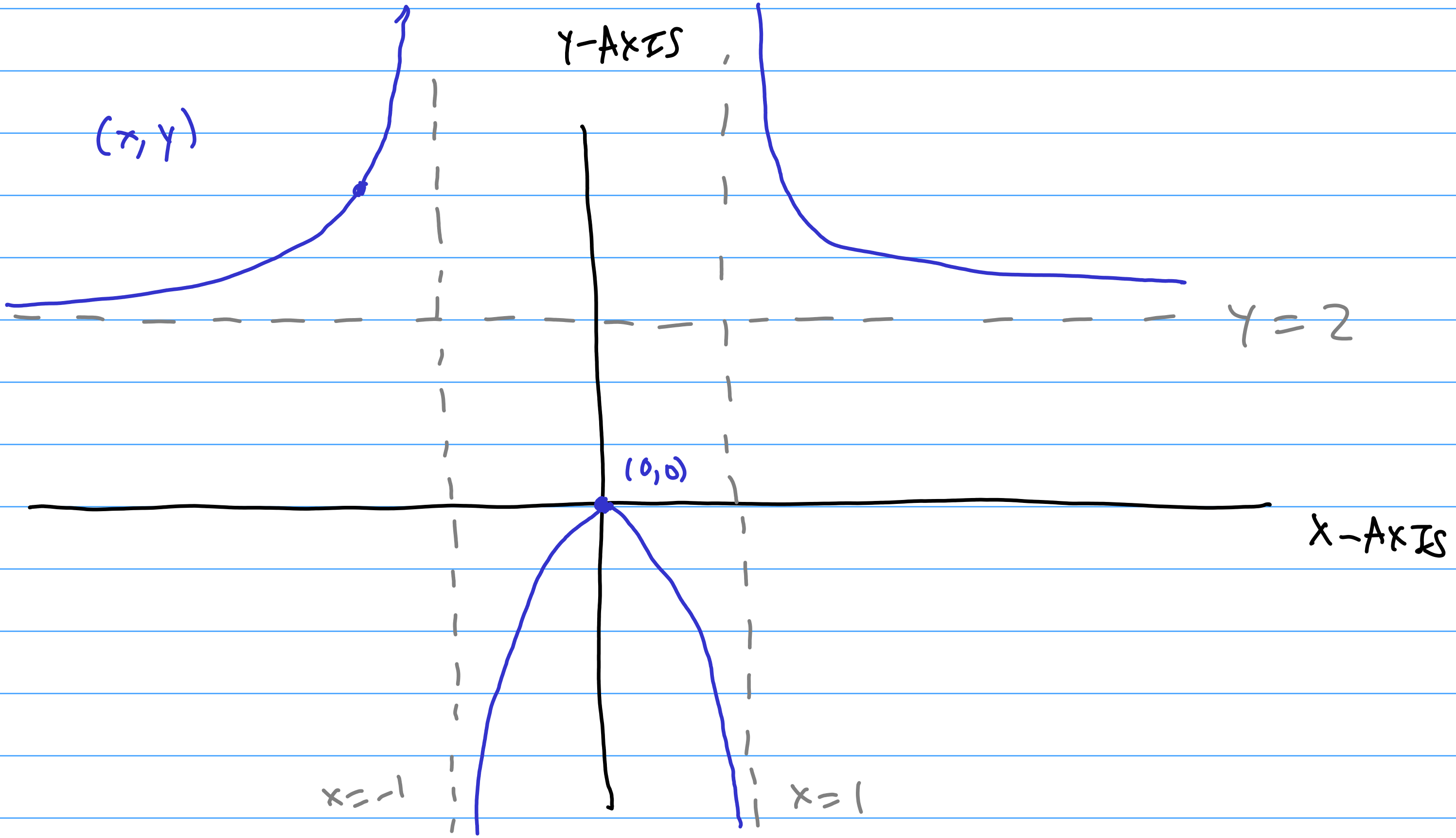
D. ASYMPTOTES

E. INTERVALS OF INCREASE / DECREASE

F. LOCAL MAXIMA & LOCAL MINIMA

G. CONCAVITY AND POINTS OF INFLECTION

EXAMPLE : $y = \dfrac{2x^2}{x^2 - 1}$



$y = f(x)$

A. **DOMAIN**: "THE SET OF x FOR WHICH $f(x)$ MAKES SENSE"

e.g. (a) NO DIVISION BY ZERO

(b) $\sqrt{(\cdot)} \Rightarrow (\cdot) \geq 0$

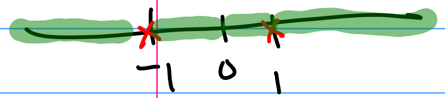
(c) $\log(\cdot) \Rightarrow (\cdot) > 0$

$f(x) = \dfrac{x}{x-1}$

$x - 1 = 0$ ✗

$x \neq 1$

$(-\infty, 1) \cup (1, \infty)$



$y = \dfrac{2x^2}{x^2 - 1}$

DOMAIN: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$x^2 - 1 = 0 \Leftrightarrow x = \pm 1$ ✗

$$y = \sqrt{x+3} \iff x+3 \geq 0$$

$$\Updownarrow$$

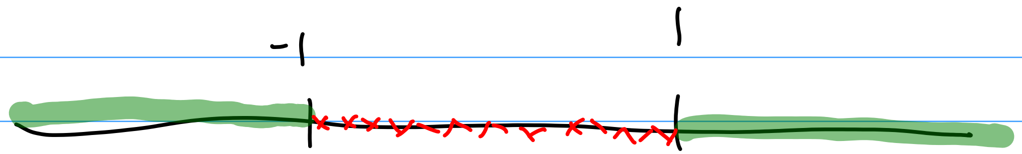
$$x \geq -3$$

$$x \in [-3, \infty)$$

$$y = \log(x^2 - 1) \iff x^2 - 1 > 0 \iff x \in (-\infty, -1) \cup (1, \infty)$$

$$(x-1)(x+1) > 0$$

$(x-1)(x+1) > 0$

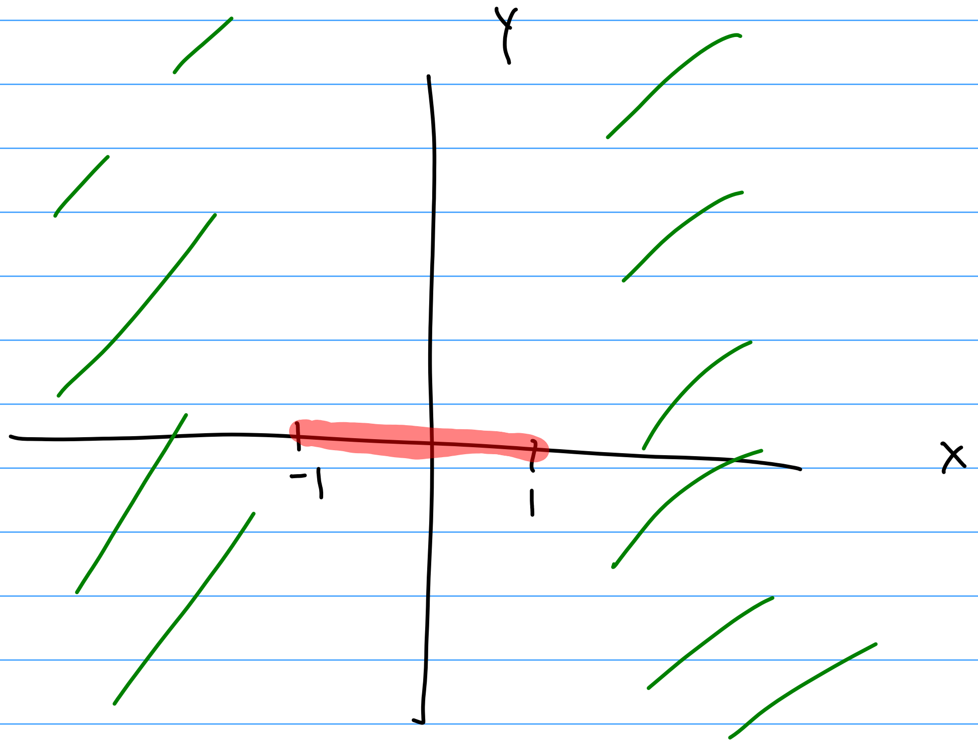


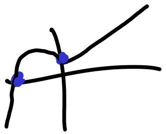
$(x-1)$	$-$	$-$	$-$	0	$+$
$(x+1)$	$-$	0	$+$	$+$	$+$
	$+$	0	$-$	0	$+$

* IN THE PICTURE

$\log(x^2-1)$

$x \in (-\infty, -1) \cup (1, \infty)$





B. **INTERCEPTS** "THE POINTS WHERE THE GRAPH CUTS THE AXES"

(a) Y-INTERCEPT $\longrightarrow$ $= f(0)$

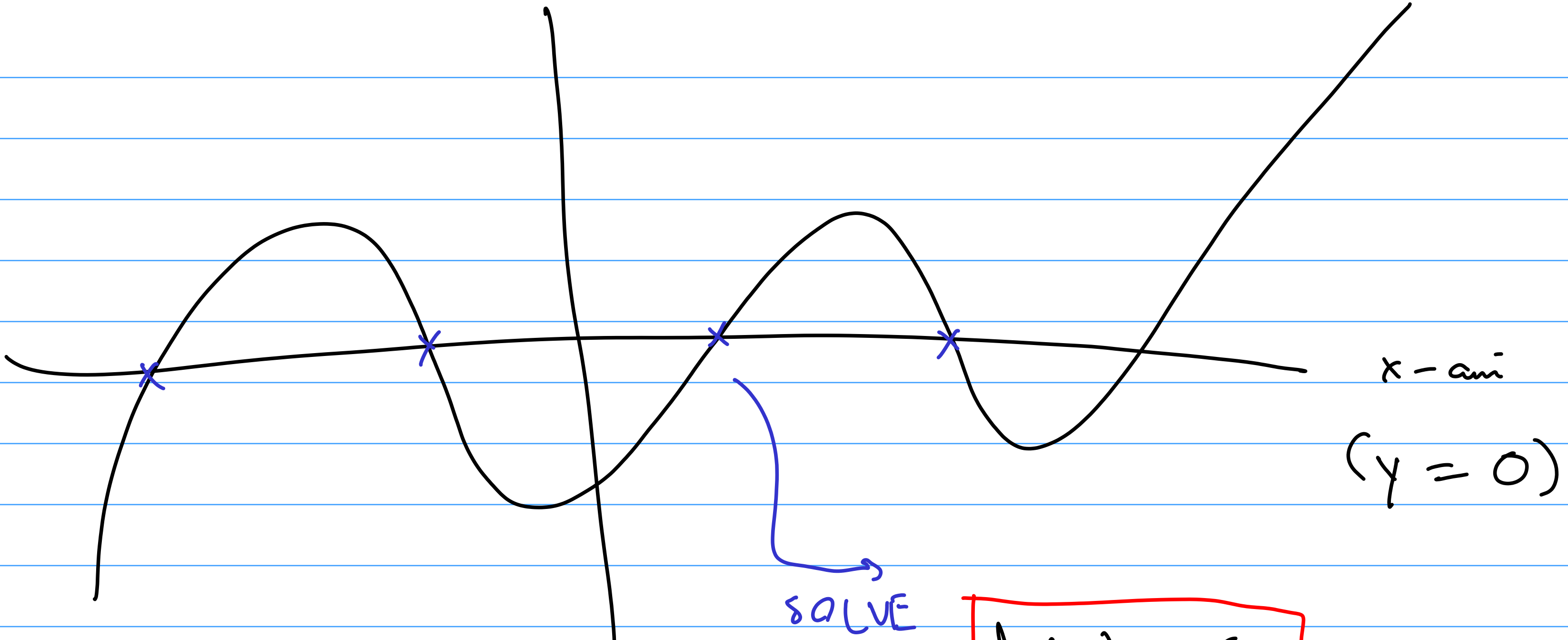
Y-axis $\leftrightarrow$ $x=0$ $\qquad (0, f(0))$

(b) X-INTERCEPT $\longrightarrow$ SET OF x s.t. $f(x) = 0$

MIGHT BE HARD / MIGHT NOT HAVE SOLUTIONS

$Y = \dfrac{2x^2}{x^2-1}$

Y-INTERCEPT ($x=0$)	X-INTERCEPT
$Y = \dfrac{2\cdot 0^2}{0^2-1} = 0$	$\dfrac{2x^2}{x^2-1} = 0$
$(0,0)$	$\Rightarrow x = 0$ $\quad (0,0)$



$$\boxed{f(x) = 0}$$

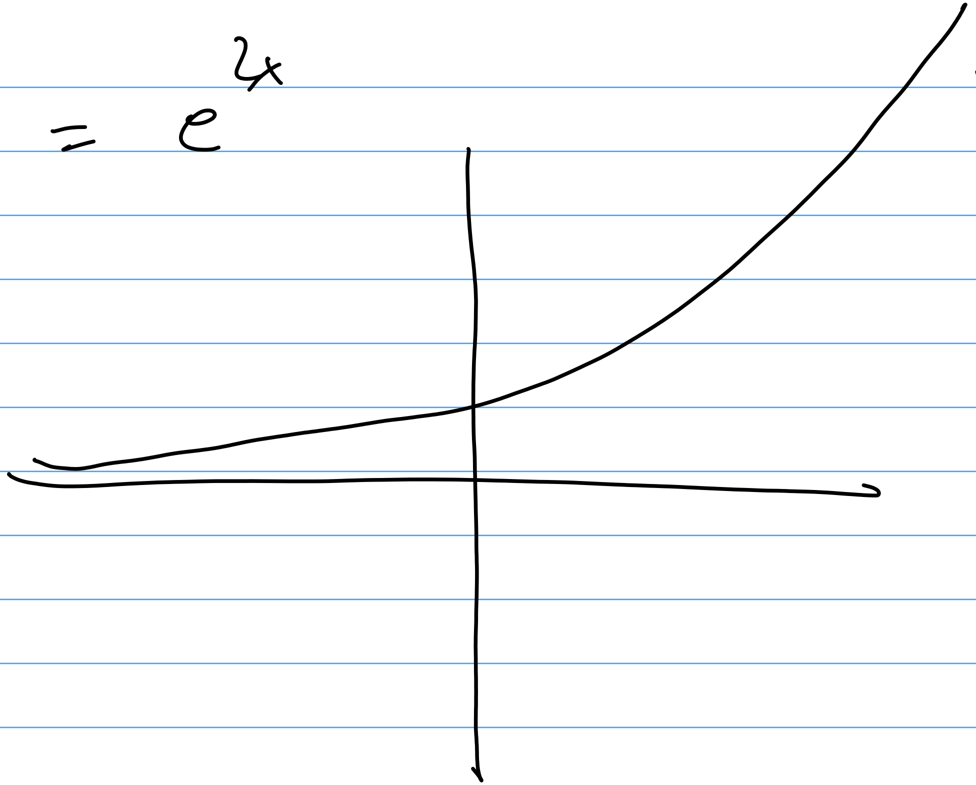
$$y = (x-1)(x+2)e^x$$

$$(x, y) = (x, f(x))$$

$$(x-1)(x+2)e^x = 0$$

$$x = 1, -2$$

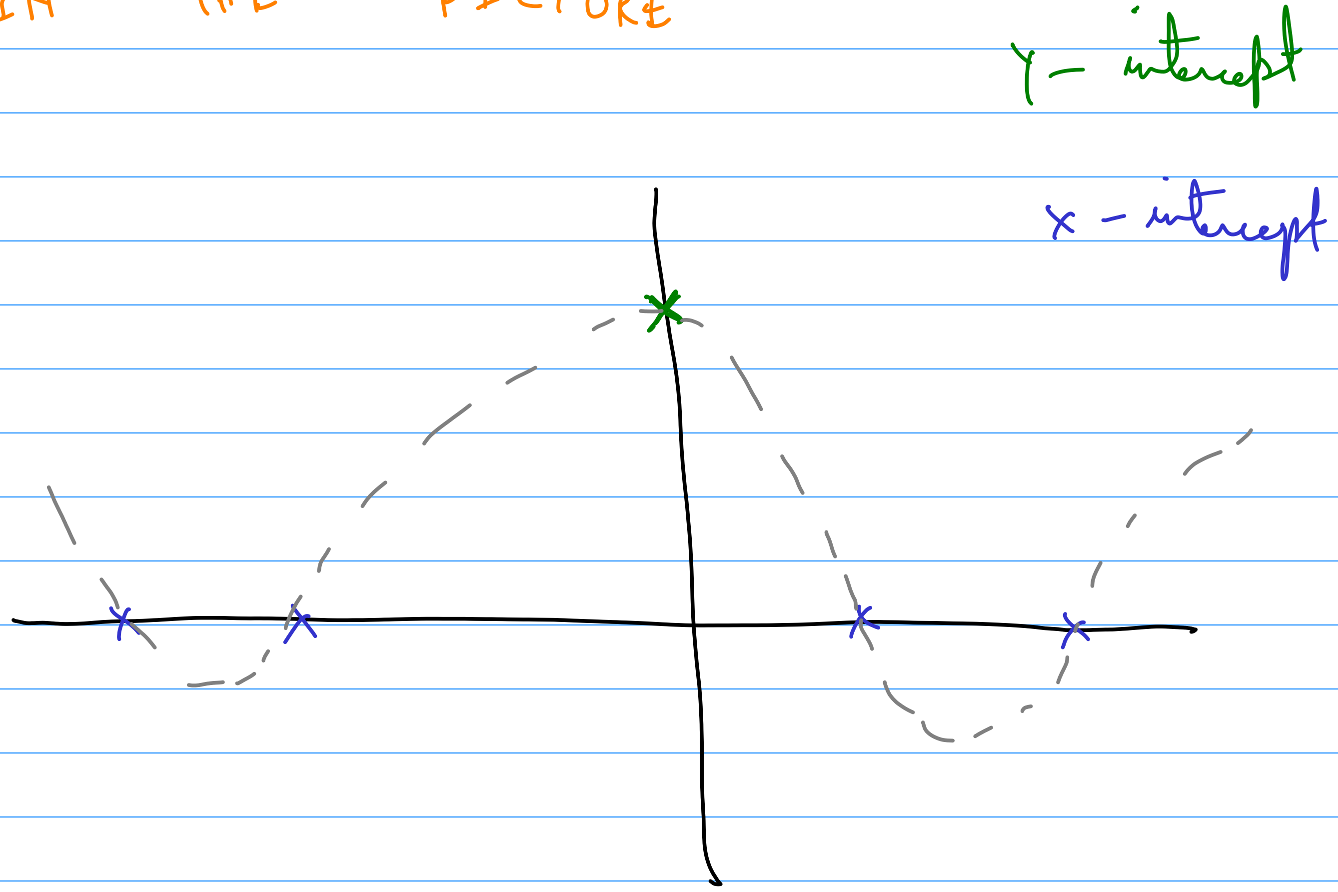
$$Y = e^{2x}$$



$Y = e^{2x}$

$e^{2x} \neq 0$ FOR ALL $x \in \mathbb{R}$

* IN THE PICTURE



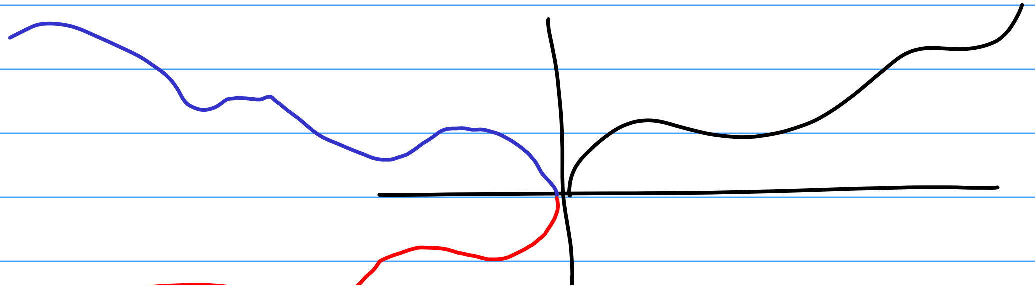
C. <u>SYMMETRY</u> "EXPLOIT SYMMETRY, IF IT EXISTS"

<u>TYPES</u>

ⓐ EVEN $f(-x) = f(x)$ FOR ALL x

ⓑ ODD $f(-x) = -f(x)$ FOR ALL x

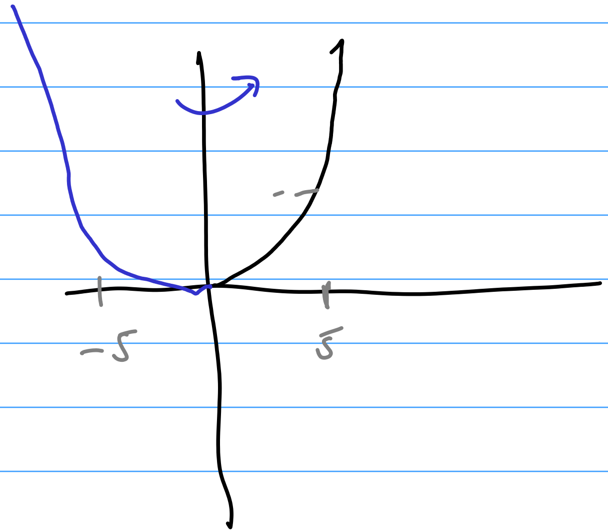
ⓒ PERIODIC FOR SOME <u>FIXED</u> P
$f(x+P) = f(x)$ FOR ALL x



EVEN

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

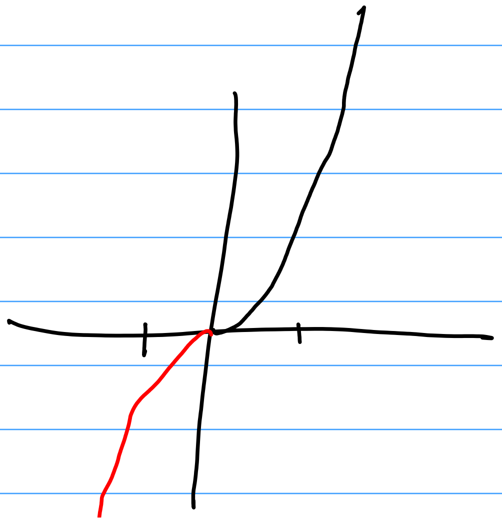


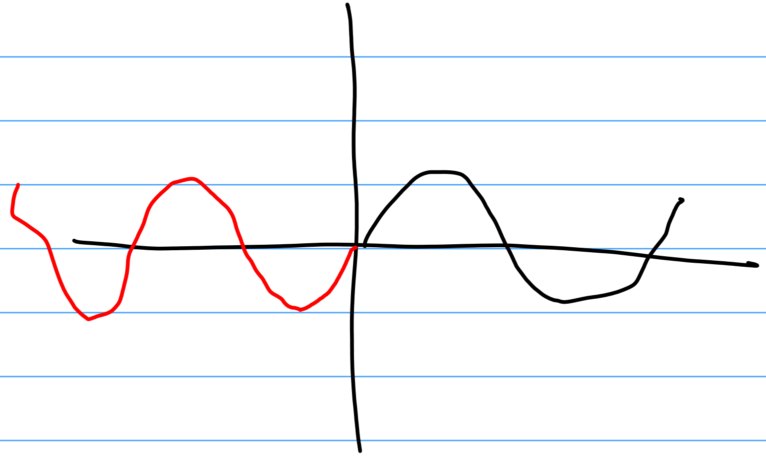
ODD

$$f(x) = x^3$$

$$f(x) = \sin x \qquad \sin(-x) = -\sin x$$

$$f(-x) = (-x)^3 = (-x)(-x)^2 = (-x)\,x^2 = -x^3$$
$$= -f(x)$$





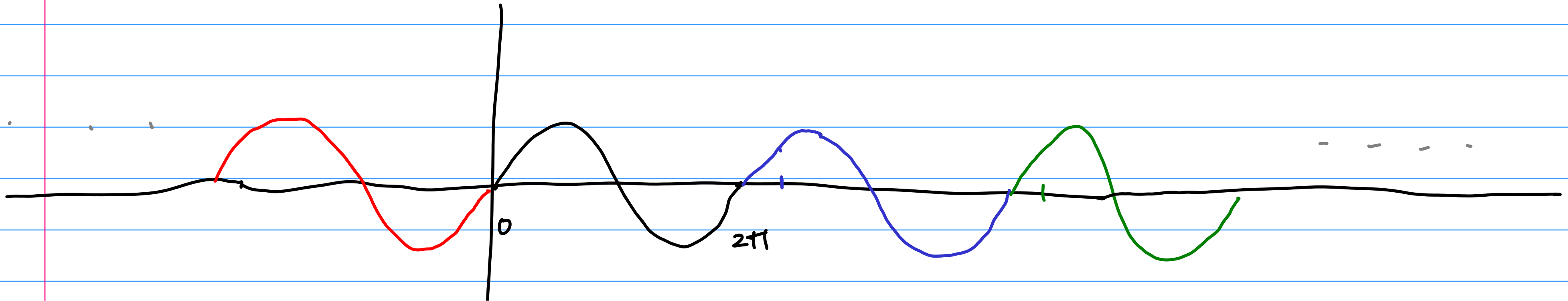
$f(x) = \sin x$

$\sin(x + 2\pi) = \sin x$ FOR ALL x

$\uparrow$

$\sin$ IS 2π - PERIODIC

PERIODIC W/ PERIOD $P = 2\pi$



* IN THE PICTURE

EVEN → MIRROR SYMMETRY AROUND Y-AXIS

ODD → POINT SYMMETRY AROUND ORIGIN

PERIODIC → ENOUGH TO LOOK AT A SMALL INTERVAL OF LENGTH = PERIOD

$$y = \frac{2x^2}{x^2-1}$$

$$\boxed{\frac{2(x+P)^2}{(x+P)^2-1} = \frac{2x^2}{x^2-1}} \Rightarrow \text{CONTRADICTION}$$

$$\frac{(-x)^2}{(-x)^2-1} = \frac{x^2}{x^2-1}$$

$$\uparrow \qquad\qquad \uparrow$$

$$f(-x) = f(x)$$

EVEN ✓

ODD ✗

PERIODIC ✗

(THERE EXISTS)

$\exists P$

$f(x+P) = f(x)$

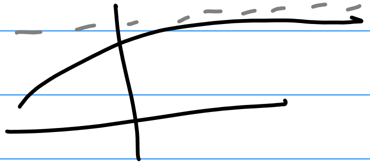
FOR ALL x

$f(-x) = -f(x)$ [ODD]

$f(-x) = f(x)$ [EVEN]

(N.B. : DON'T HAVE TO EXIST)

D. <u>ASYMPTOTES</u> : "LINES THAT THE GRAPH APPROACHES"

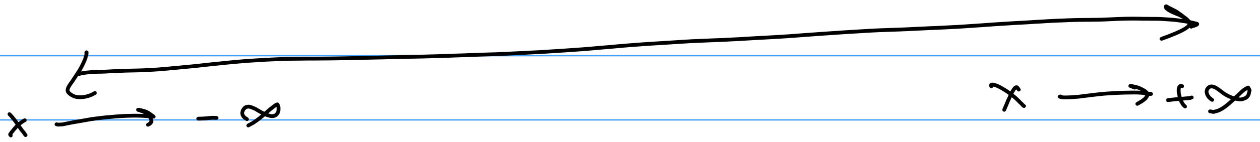


(a) <u>HORIZONTAL</u> :

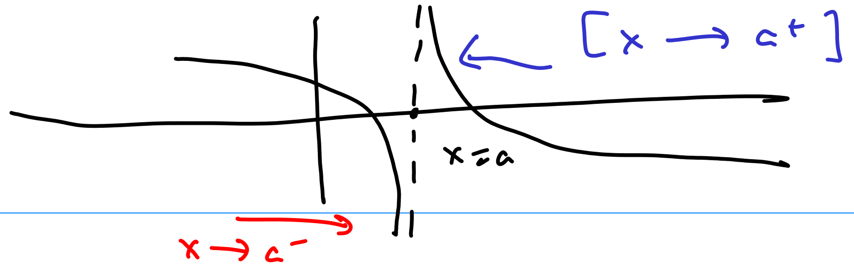
COMPUTE $\boxed{\lim_{x \to \infty} f(x)}$ & $\boxed{\lim_{x \to -\infty} f(x)}$

IF EITHER EXIST, AND ARE EQUAL TO L THEN $\boxed{y = L}$ IS A HORIZONTAL ASYMPTOTE

$L = \lim_{x \to \infty} f(x)$



(b) <u>VERTICAL:</u>



FOR a ON THE EDGE OF DOMAIN, FIND

$$\lim_{x \to a^+} f(x) \quad \& \quad \lim_{x \to a^-} f(x)$$

IF $\lim_{x \to a^{+/-}} f(x) = \pm \infty$ THEN $x = a$ IS A VERTICAL ASYMPTOTE

$+/$ $+/-$

(c) SLANT → SKIP.

$$x^2 = \frac{1}{x^{-2}}$$

$$y = \frac{2x^2}{x^2 - 1}$$

$$x = 16^{1000}$$

$$x^2 - 1 \approx x^2$$

HORIZONTAL

$$\lim_{x \to \infty} \frac{2x^2}{x^2 - 1} \approx \frac{2x^2}{x^2} = 2$$

$$\lim_{x \to \infty} \frac{2x^2}{x^2 - 1} = \frac{x^2 \cdot [2]}{x^2 \left[1 - \frac{1}{x^2}\right]} = \lim_{x \to \infty} \frac{2}{1 - \frac{1}{x^2}} = \frac{2}{1 - 0} = 2$$

$\frac{1}{x^2} \to 0$

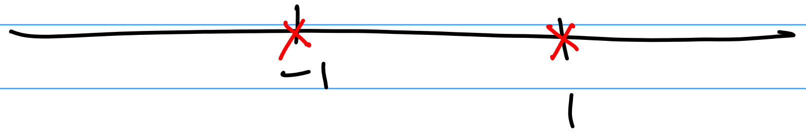
$\boxed{y = 2}$ IS A HORIZONTAL ASYMPTOTE.

$$\lim_{x \to -\infty} \frac{2x^2}{x^2 - 1} = 2$$

[SIMILAR TO PREVIOUS]

$y = 2$ (HORIZONTAL ASYM)

$$y = \frac{2x^2}{x^2-1} \qquad x \neq 1, -1$$

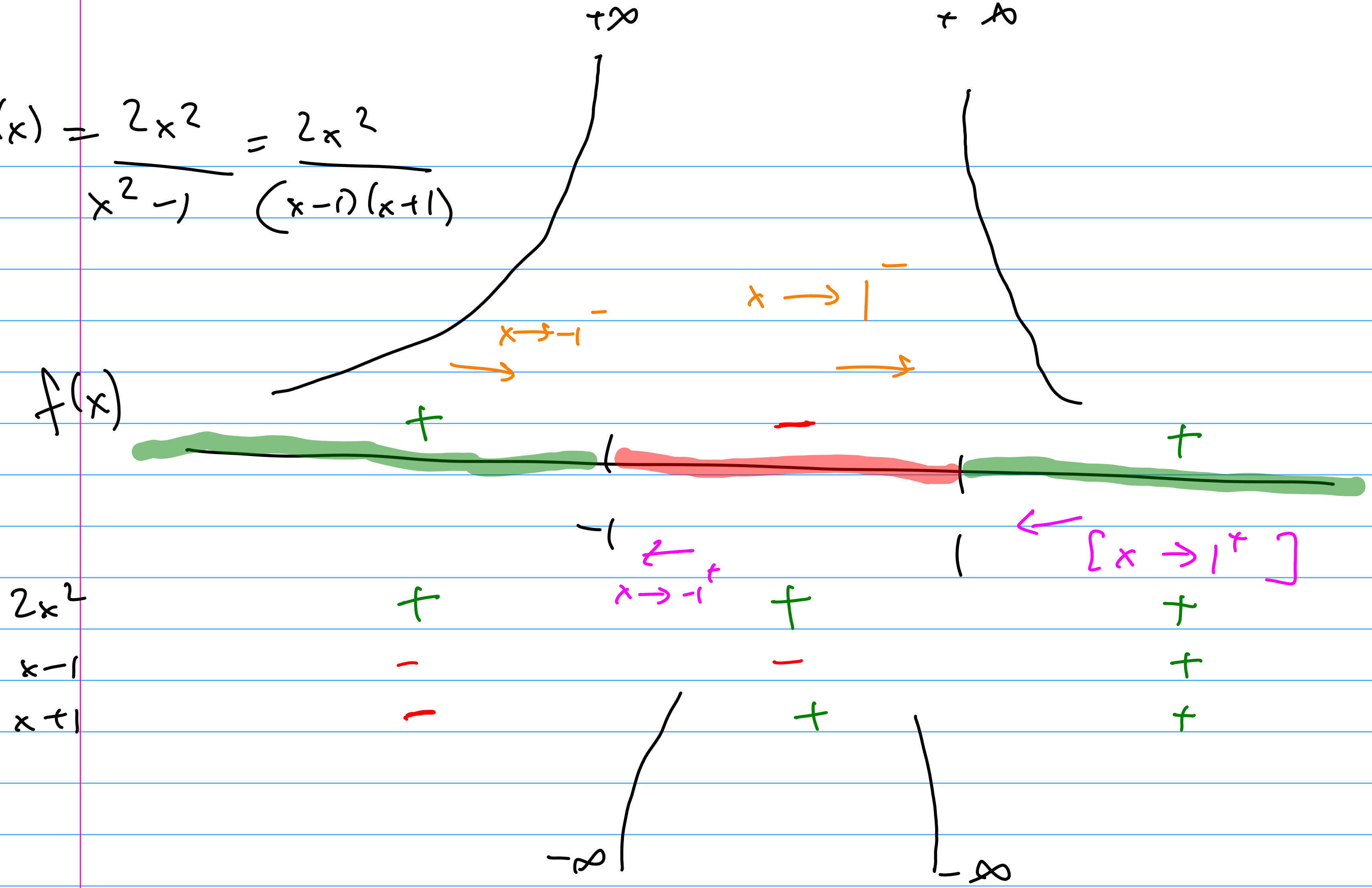


$\lim\limits_{x \to -1^{(-)}} = +\infty$	$\lim\limits_{x \to -1^{(+)}} \frac{2x^2}{x^2-1} = -\infty$
$\lim\limits_{x \to 1^{(-)}} = -\infty$	$\lim\limits_{x \to 1^{(+)}} = +\infty$

$$\frac{2(-1)^2}{(-1)^2-1}$$

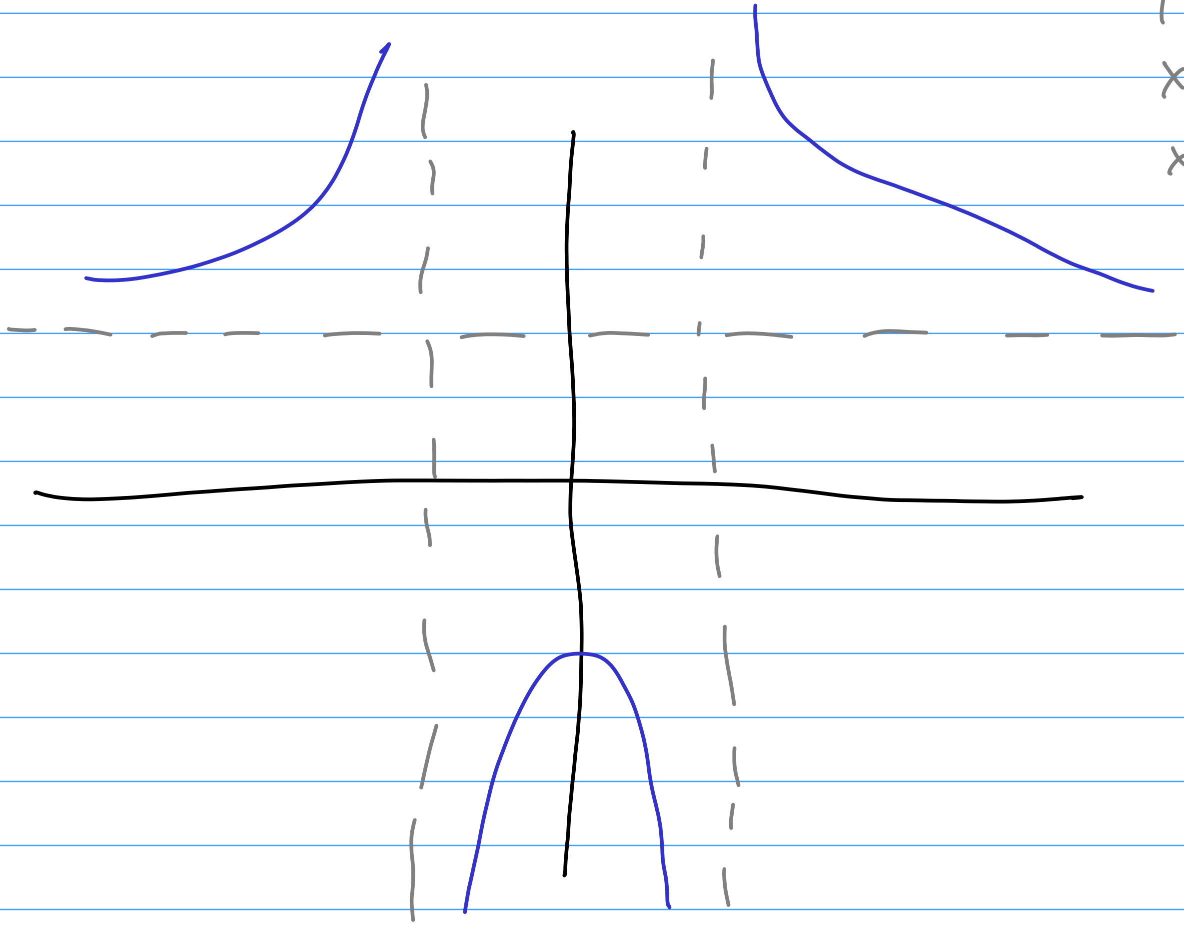
$$\frac{2(1)^2}{1^2-1} = \frac{2}{0}$$

$$f(x) = \frac{2x^2}{x^2-1} = \frac{2x^2}{(x-1)(x+1)}$$



$x = -1$ & $x = 1$ ARE VERTICAL ASYMPTOTES.

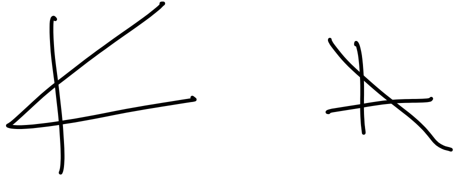
* IN THE PICTURE



$y = 2$

$x = 1$

$x = -1$



I. INTERVALS OF INCREASE / DECREASE

"WHERE IS $y = f(x)$ GOING UP / DOWN?"

DEFN: f IS INCREASING ON I IF

$x_1, x_2 \in I$, $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

DEFN: f IS DECREASING ON I IF

$x_1, x_2 \in I$, $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$

HOW?

ANS: USE CALCULUS

COMPUTE $f'(x)$

$f'(x) > 0 \Rightarrow f \uparrow$

$f'(x) < 0 \Rightarrow f \downarrow$

$f'(x) = 0 \Rightarrow f -$

$$y = \frac{2x^2}{x^2 - 1}$$

$$f(x) = \frac{2x^2}{x^2 - 1}$$

$$f'(x) = \frac{\left[\frac{d}{dx} 2x^2\right](x^2-1) - (2x^2)\left(\frac{d}{dx}(x^2-1)\right)}{(x^2-1)^2} \quad [\text{QUOTIENT}]$$

$$= \frac{(4x)(x^2-1) - (2x^2)(2x)}{(x^2-1)^2}$$

$$= \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

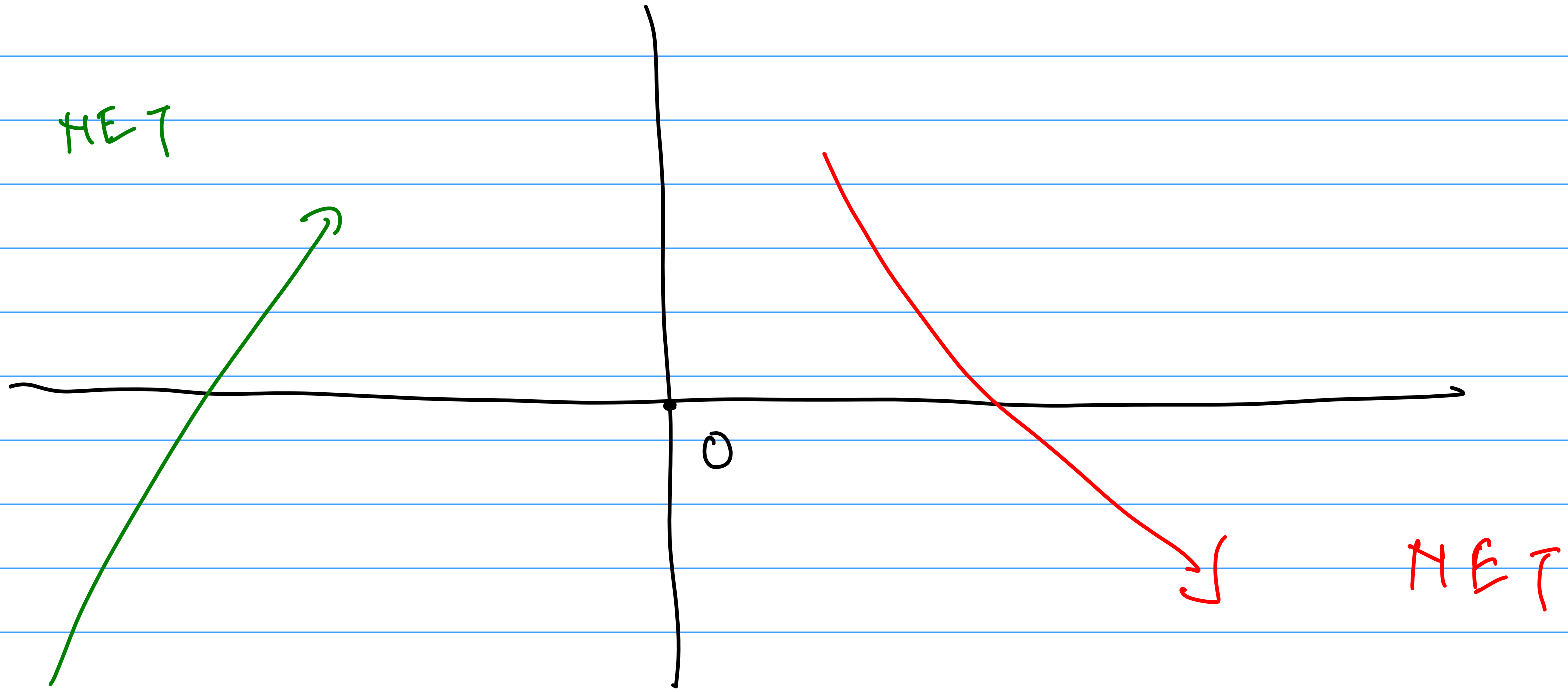
$$f'(x) = \frac{\boxed{-4x}}{\boxed{(x^2-1)^2}} \quad \begin{matrix} > 0 \\ < 0 \end{matrix}$$

≥ 0

$f'(x) > 0 \iff -4x > 0 \iff x < 0 \iff (-\infty, 0)$ INCREASING

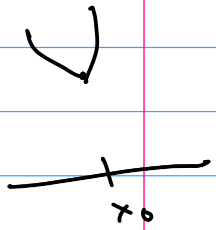
$f'(x) < 0 \iff -4x < 0 \iff x > 0 \iff (0, \infty)$ DECREASING

* IN THE PICTURE



F. LOCAL MAXIMA & LOCAL MINIMA

"USE CRITICAL POINTS + 1st/2nd DERIVATIVE TEST"

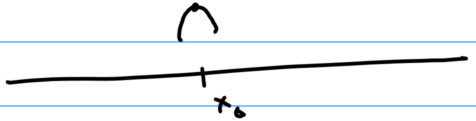


DEFN: x_0 IS A LOCAL MINIMUM IF :-

FOR ALL x CLOSE TO x_0 $\quad f(x) \geq f(x_0)$

DEFN: x_0 IS A LOCAL MAXIMUM IF :-

FOR ALL x CLOSE TO x_0 $\quad f(x) \leq f(x_0)$



HOW?

ANS: USE CALCULUS [DERIVATIVE TESTS]

1. FIND CRITICAL POINTS (i.e. $f'(c) = 0$ OR $f'(c)$ DNE)

2. USE 1st DERIVATIVE TEST TO FIND MAXIMA & MINIMA

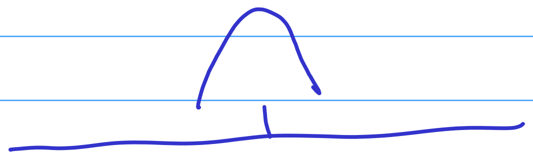
3. IF $f'(c) = 0$ BUT $f''(c) \neq 0$, CAN INSTEAD USE 2nd DERIVATIVE TEST

1st derivative test

x_0

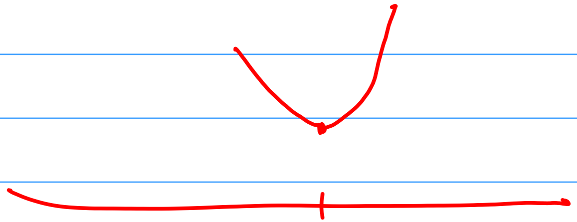
$x < x_0 \Rightarrow f'(x) > 0$

$x > x_0 \Rightarrow f'(x) < 0$



x_0

$\Rightarrow$ LOCAL MAXIMUM



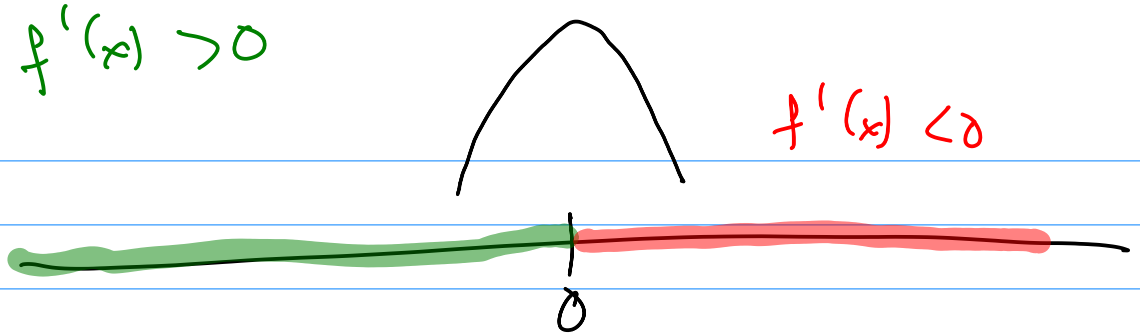
x_0

$x < x_0 \Rightarrow f'(x) < 0$

$x > x_0 \Rightarrow f'(x) > 0$

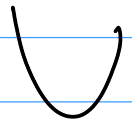
$\Rightarrow$ LOCAL MINIMUM

$$y = \frac{2x^2}{x^2-1}$$



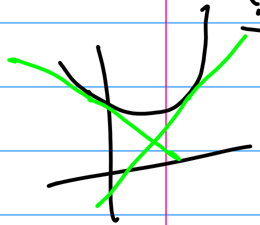
$\Rightarrow x = 0$ IS A LOCAL MAXIMUM

* IN THE PICTURE



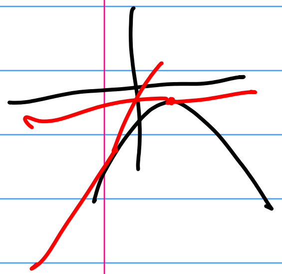
G. CONCAVITY AND POINTS OF INFLECTION

"USE $f''(x)$ FOR CONCAVITY"



DEFN: IF $y = f(x)$ IS ABOVE (resp BELOW) ALL ITS TANGENTS THEN IT IS CONCAVE UPWARDS (resp. DOWNWARDS)

DEFN: A POINT WHERE CONCAVITY CHANGES IS CALLED A POINT OF INFLECTION



HOW?

ANS : USE CALCULUS

FIND $f''(x)$

$f''(x) > 0 \Rightarrow f \cup$ [CONCAVE UP]

$f''(x) < 0 \Rightarrow f \cap$ [CONCAVE DOWN]

f'' FLIPS SIGN $\Rightarrow$ POINT OF INFLECTION

$y = \frac{2x^2}{x^2-1}$ $\qquad f'(x) = \frac{-4x}{(x^2-1)^2}$

$$f''(x) = \frac{\left[\frac{d}{dx}(-4x)\right](x^2-1)^2 - (-4x)\frac{d}{dx}(x^2-1)^2}{\left[(x^2-1)^2\right]^2}$$

$$\frac{\left[\,-4\,\right](x^2-1)^2 - (-4x)\;4x\,(x^2-1)}{\left[(x^2-1)^2\right]^2}$$

$$\frac{d}{dx}\left[x^2-1\right]^2 = \overbrace{\left\{\frac{d}{dx}(x^2-1)\right\}}^{h'(x)} \cdot \overbrace{2\left\{x^2-1\right\}}^{g'(h(x))}$$

$$= (2x)\left\{2(x^2-1)\right\} = 4x\,(x^2-1)$$

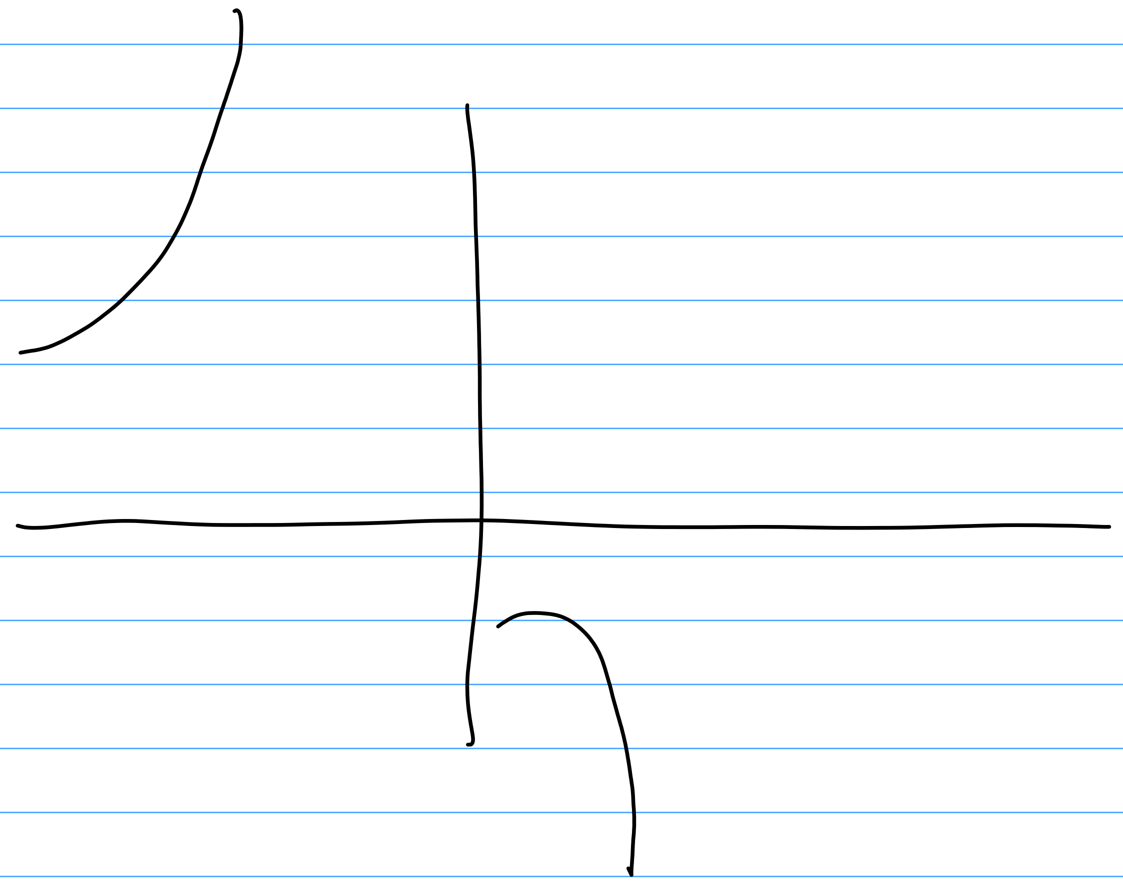
$$(x^2-1)^2 = g(h(x))$$

$$h(x) = x^2-1 \qquad g(x) = x^2$$

$$f''(x) = \frac{12x^2 + 4}{(x^2-1)^3} = \frac{12x^2+4}{(x-1)^3\,(x+1)^3}$$

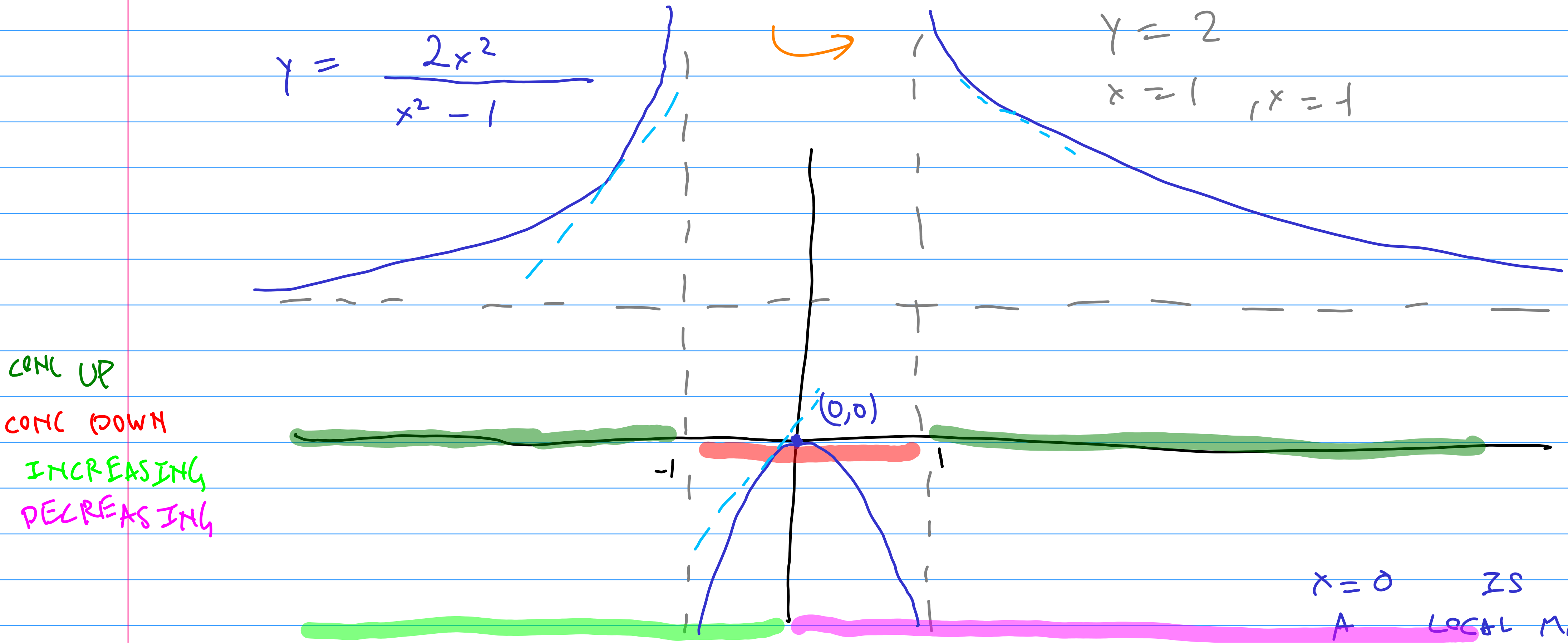
	$x < -1$	-1	$-1 < x < 1$	1	$x > 1$
$f''(x)$	$+$ UP		$-$ DOWN		$+$ UP
$x-1$	$-$		$-$		$+$
$x+1$	$-$		$+$		$+$

* IN THE PICTURE



FINISHING THE EXAMPLE

$$y = \frac{2x^2}{x^2 - 1}$$



x = 0 IS A LOCAL MAX

GENERAL HOW - TO

1. FIND DOMAIN, COMPUTE $f(0)$ & SOLVE $f(x) = 0$
2. LOOK FOR SYMMETRY
3. COMPUTE LIMITS (AT $\pm\infty$, ENDPOINTS)
4. COMPUTE $f'(x)$ [+/- ve?]
5. COMPUTE $f''(x)$ [+/- ve?]

Eg 2

$$y = xe^{x}$$

A. DOMAIN

B. INTERCEPTS

C. SYMMETRY

D. ASYMPTOTES

E. INTERVALS OF INCREASE / DECREASE

F. LOCAL MAXIMA & LOCAL MINIMA

G. CONCAVITY AND POINTS OF INFLECTION

Eg. 3

$$y = \ln(4 - x^2)$$

A. DOMAIN

B. INTERCEPTS

C. SYMMETRY

D. ASYMPTOTES

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