MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, F 4-5PM ;
BY APPOINTMENT

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

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979-4693-0650

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COURSE PAGE: bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK DEADLINES :
   (a) WW 5 → TODAY, 11 PM
   (b) WW 6 → FRIDAY, 11 PM
2. MIDTERM 1 ANSWER KEY (EXAMS / SCHEDULE ON WEBPAGE)
3. ONE-ON-ONE MEETINGS (SET ONE UP !)

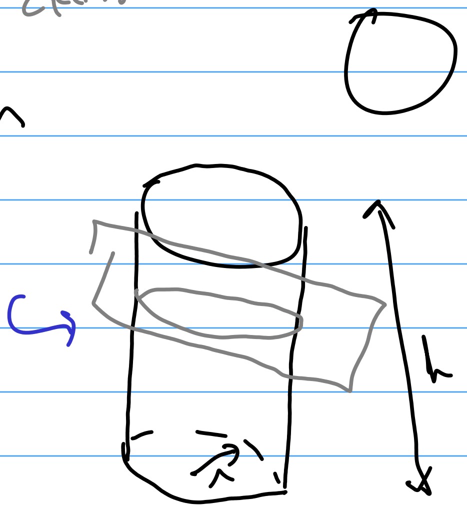
VOLUMES

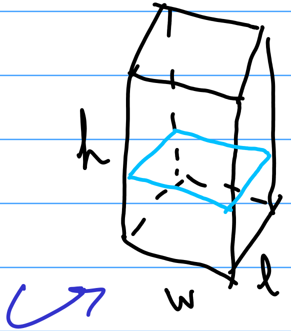
VOLUME OF CYLINDRICAL OBJECTS.

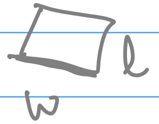
(IDENTICAL CROSS-SECTION

AREA OF THE CROSS-SECTION

$\boxed{\pi r^2} h$

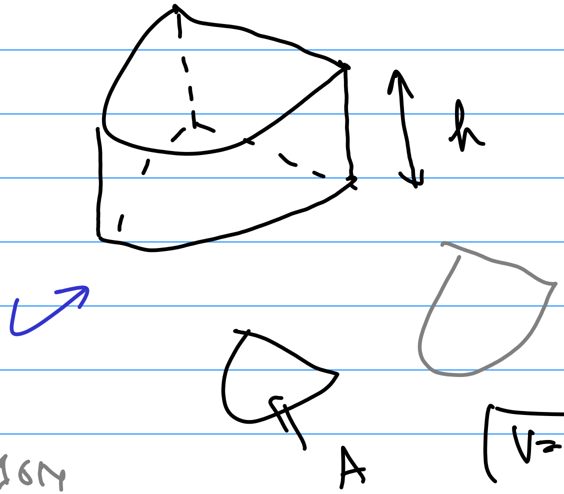






$V = \boxed{(wl)}\, h$

AREA OF THE CROSS-SECTION



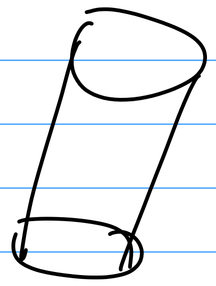
$\boxed{V = Ah}$

(CYLINDRICAL OBJECT)

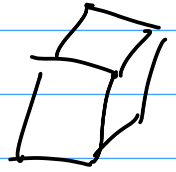
CONSTANT CROSS-SECTIONAL AREA $= A$

HEIGHT $= h$

$\Rightarrow \quad V = Ah$



WHAT IF AREA IS NOT CONSTANT?

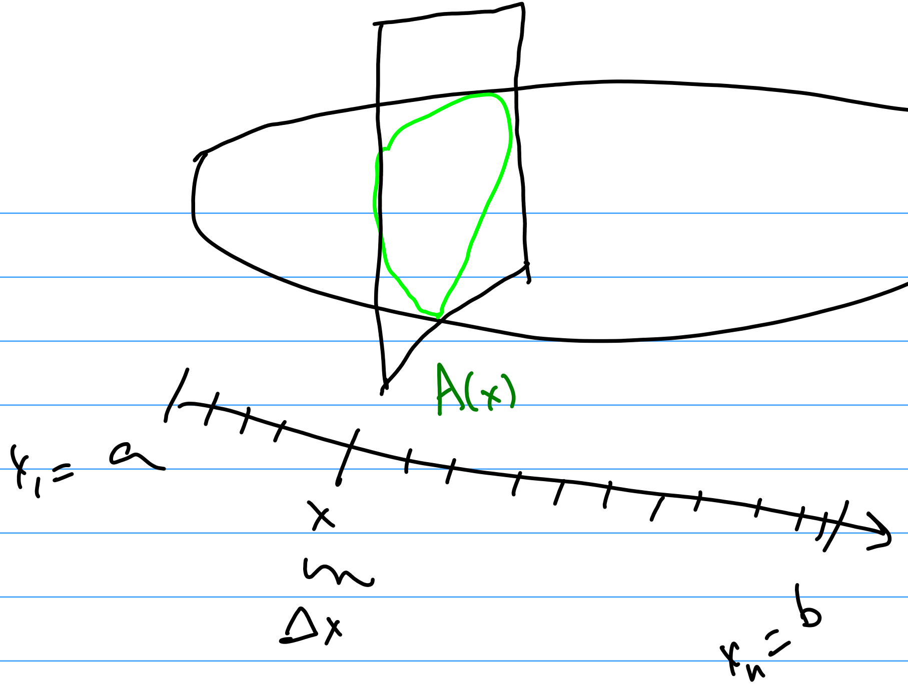


DISTANCE / SPEED

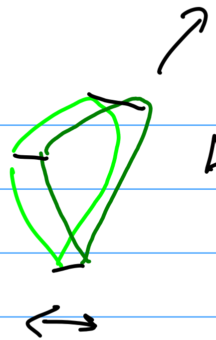
↓ ↑

x ✓

$x = vt$

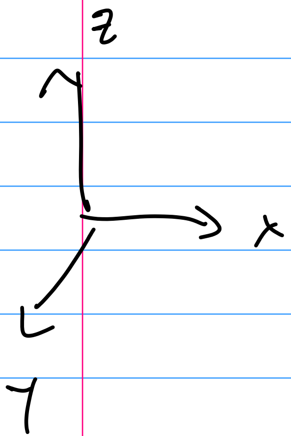


(APPROXIMATE) CYLINDRICAL OBJECT



$A \approx A(x)$

$h = \Delta x$



VOLUME OF A CUT? $= \underbrace{A(x)}_{A}\,\overbrace{\Delta x}^{h}$

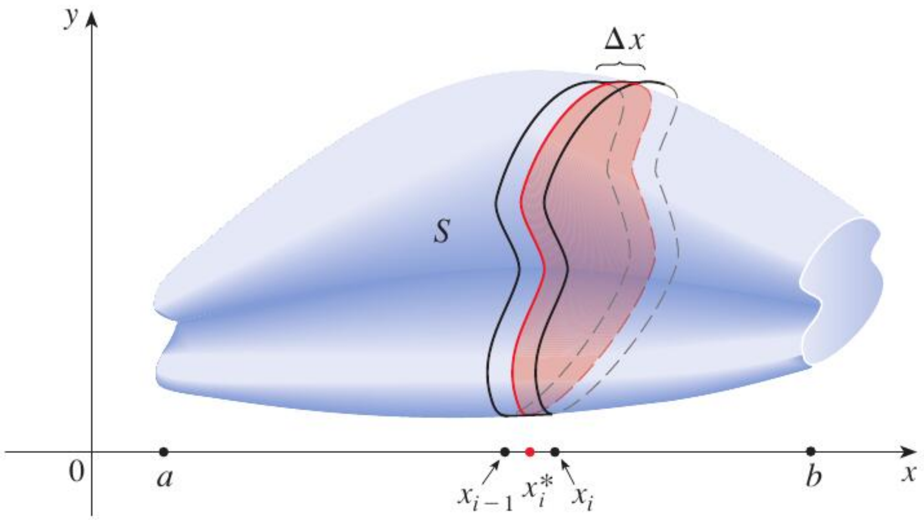
$$V = \sum_{j=0}^{n-1} A(x_j)\,\Delta x$$

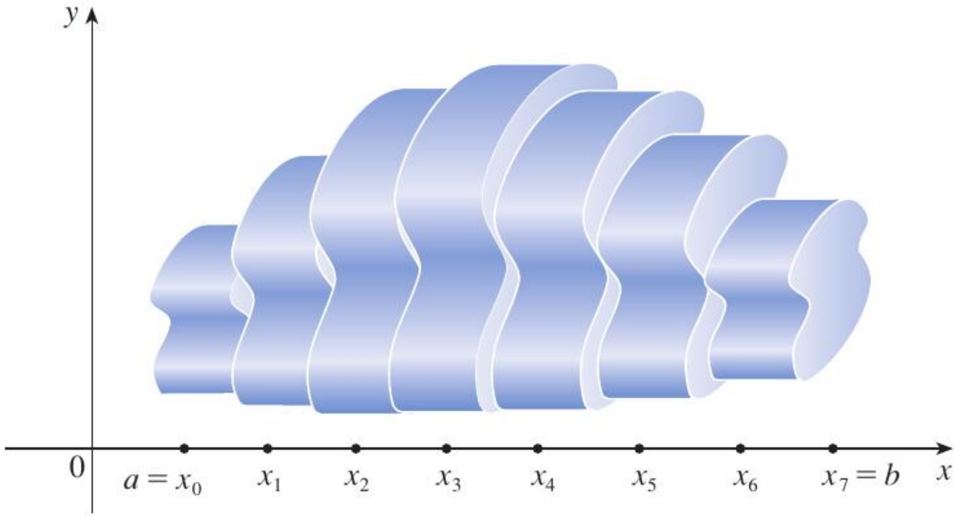
$$V \approx \sum_{j=0}^{n-1} A(x_j)\,\Delta x \xrightarrow[n \to \infty]{\Delta x \to 0} \int_a^b A(x)\,dx$$

DEFN:

$$\text{VOLUME} = \int_a^b A(x)\,dx$$

$A(x) \rightarrow$ CROSS-SECTIONAL AREA AT x





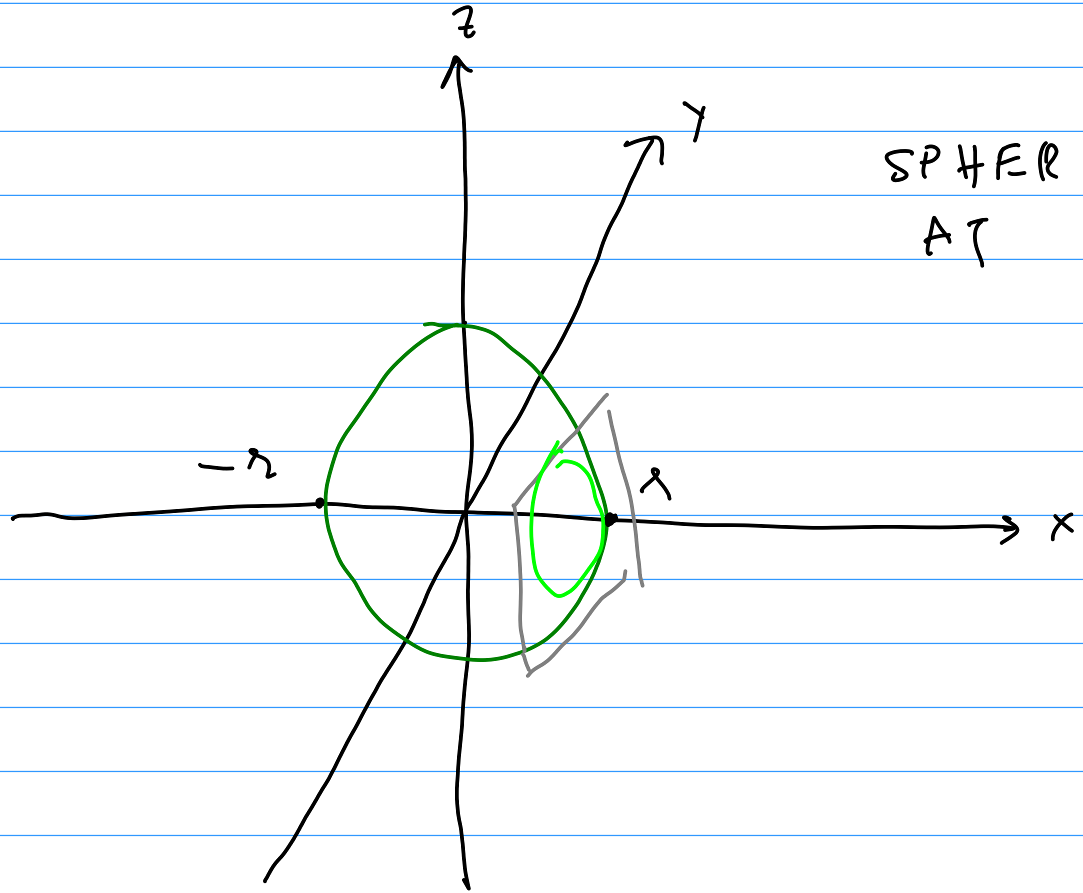
$n = 7$

EXAMPLE 1 Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

$a = -r$

$b = r$

SPHERE CENTERED AT 0,0

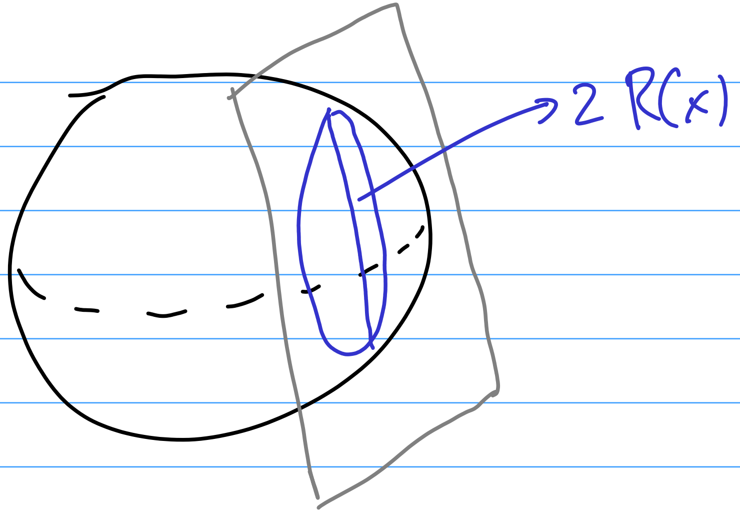


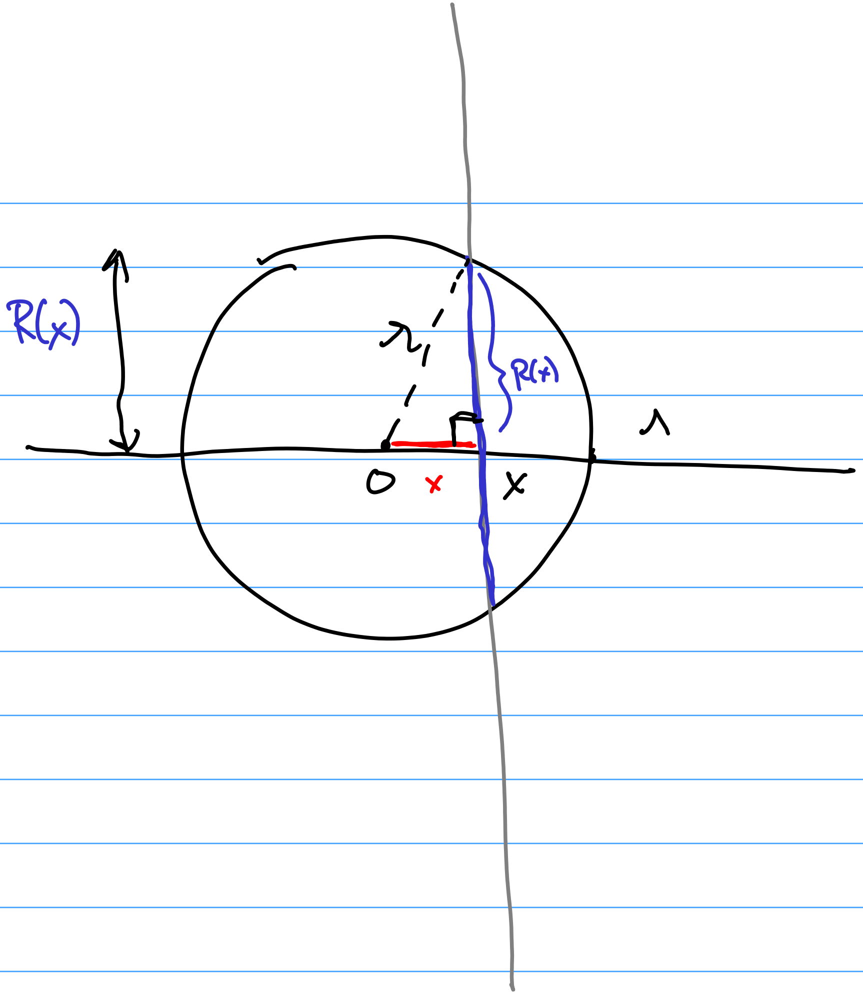
$$\int_a^b A(x)\,dx = \int_{-r}^{r} A(x)\,dx$$

CROSS-SECTION AT x

→ CIRCLE

$A(x) = \pi R(x)$





BY PYTHAGORAS

$$R(x)^2 + x^2 = r^2$$

$$\Rightarrow \boxed{R(x)^2 = r^2 - x^2}$$

(VIEW ALONG GREEN LINE)

CROSS-SECTIONAL AREA $= \pi R(x)^2$

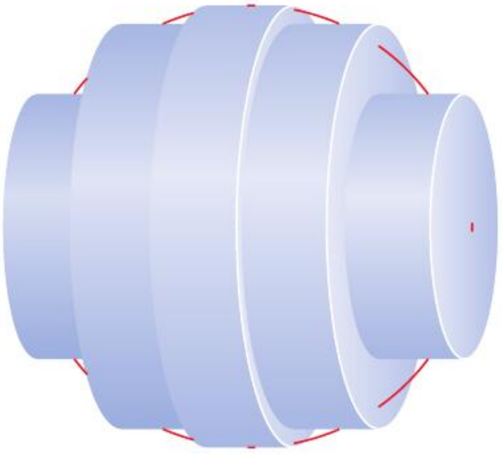
$$A(x) = \pi (r^2 - x^2)$$

$\Rightarrow V$ of $\bigcirc$ $= \displaystyle\int_{-r}^{r} A(x)\,dx$

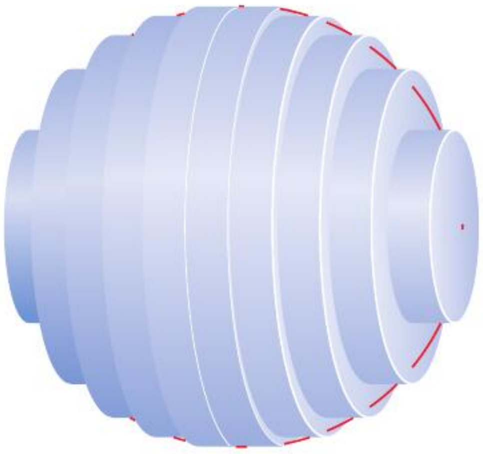
$$= \int_{-r}^{r} \pi (r^2 - x^2)\,dx = \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{x=-r}^{x=r}$$

(N.B., r IS CONSTANT)

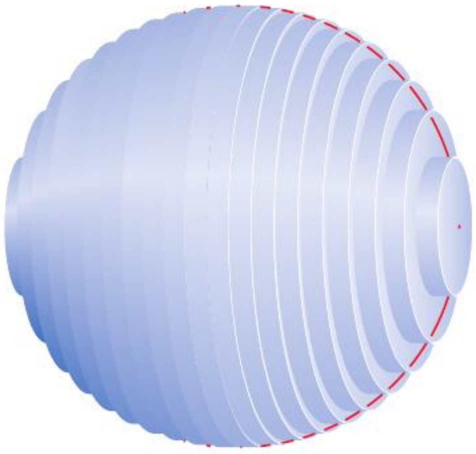
$$= \pi\left[r^3 - r^3/3\right] - \pi\left[-r^3 + r^3/3\right] = \frac{4}{3}\pi r^3$$



(a) Using 5 disks, $V \approx 4.2726$



(b) Using 10 disks, $V \approx 4.2097$

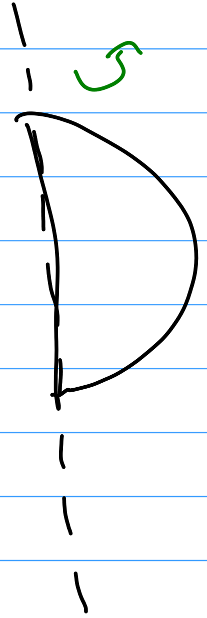


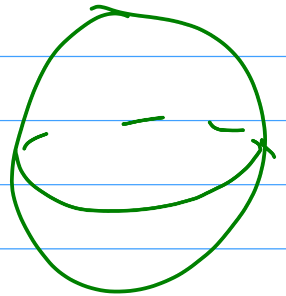
(c) Using 20 disks, $V \approx 4.1940$

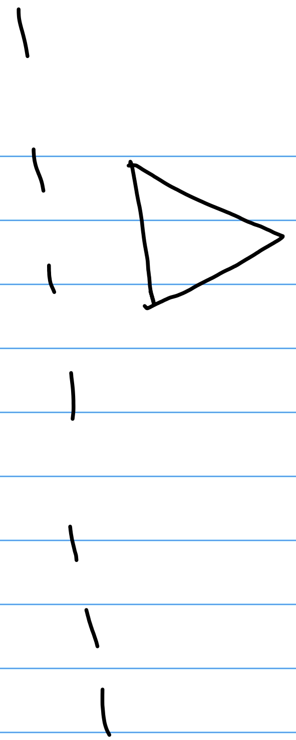
$r = 1$

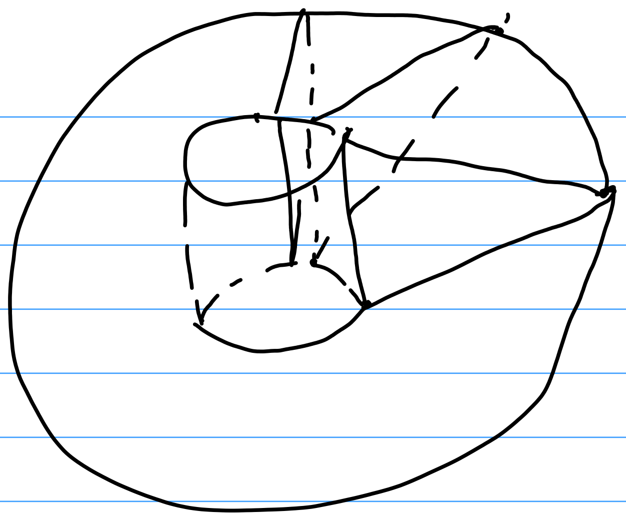
SOLIDS OF REVOLUTION

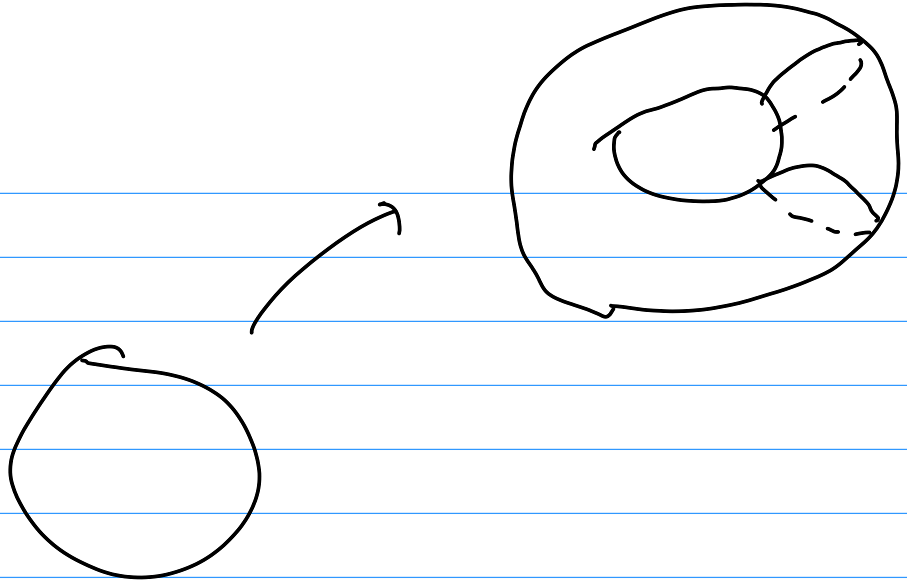
ROTATIONALLY SYMMETRIC AROUND AN
AN AXIS

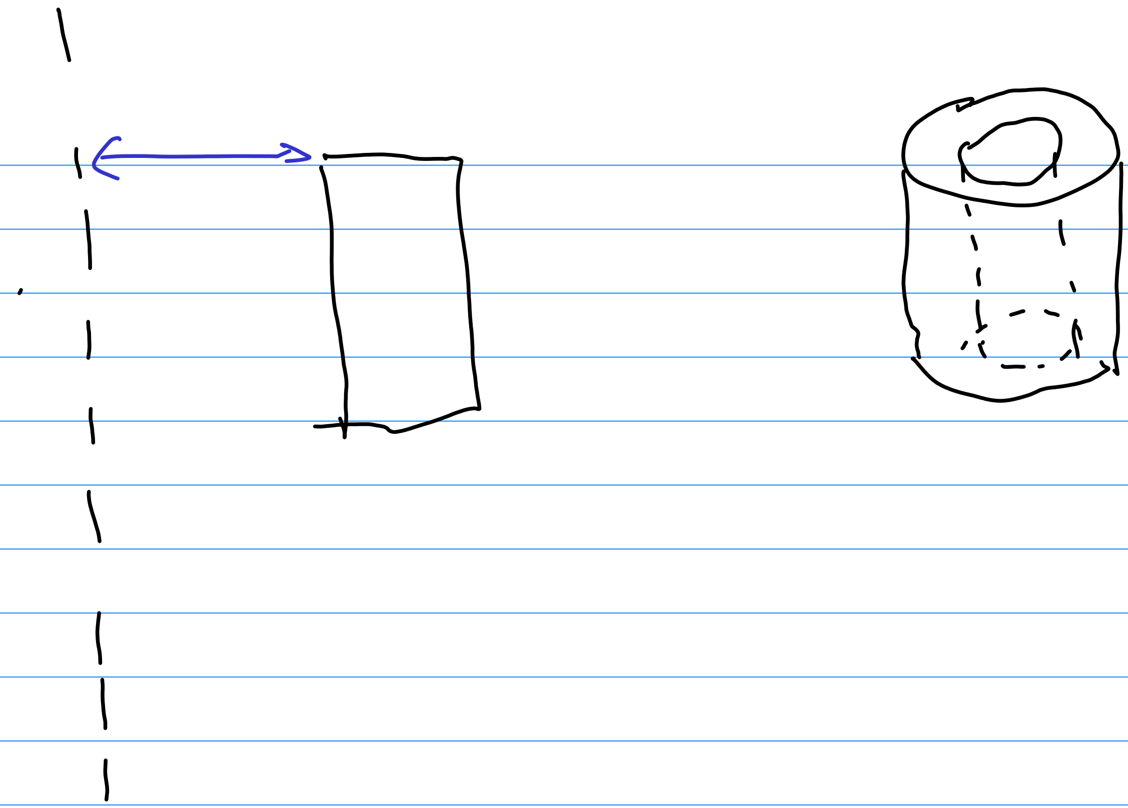






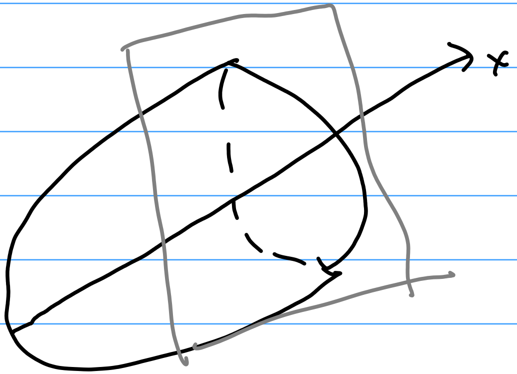






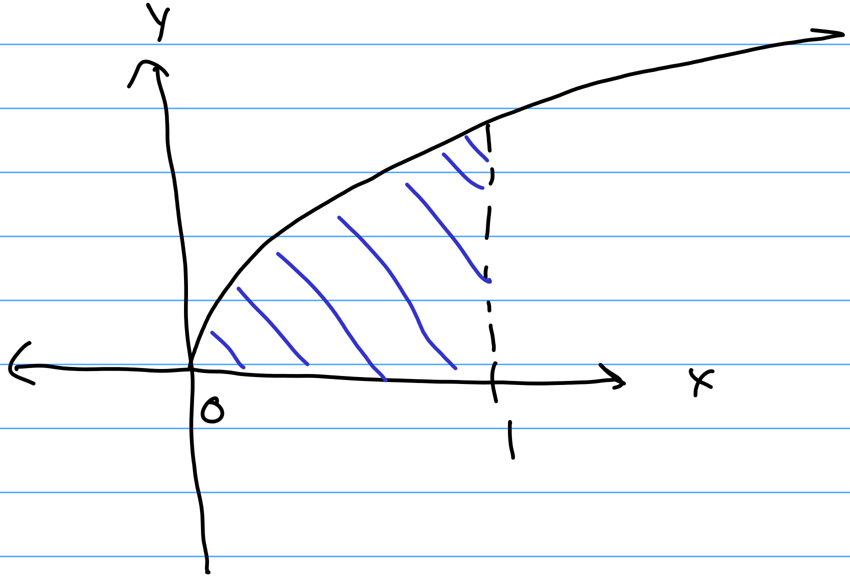
EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

1. DRAW A PICTURE



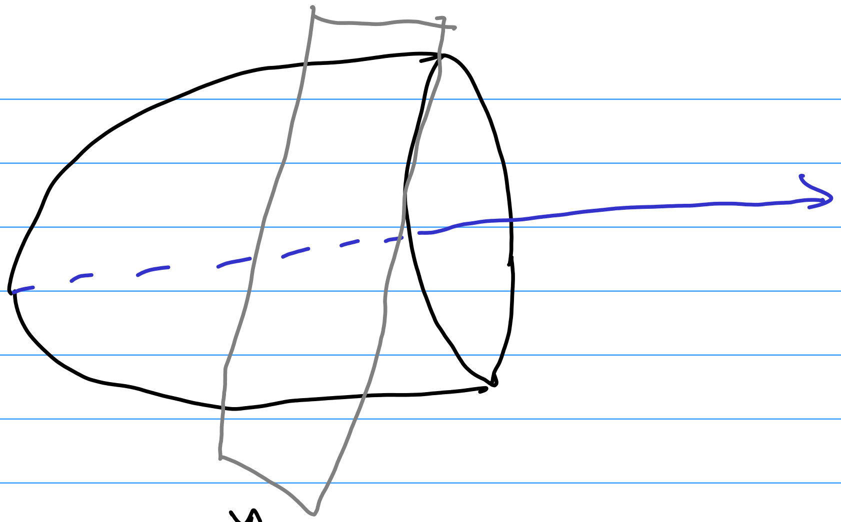
VOLUME

PARABOLOID

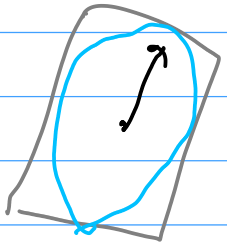


IMP : CROSS-SECTION OF A SOLID OF REVOLUTION WILL ALWAYS BE A CIRCLE OR AN ANNULUS

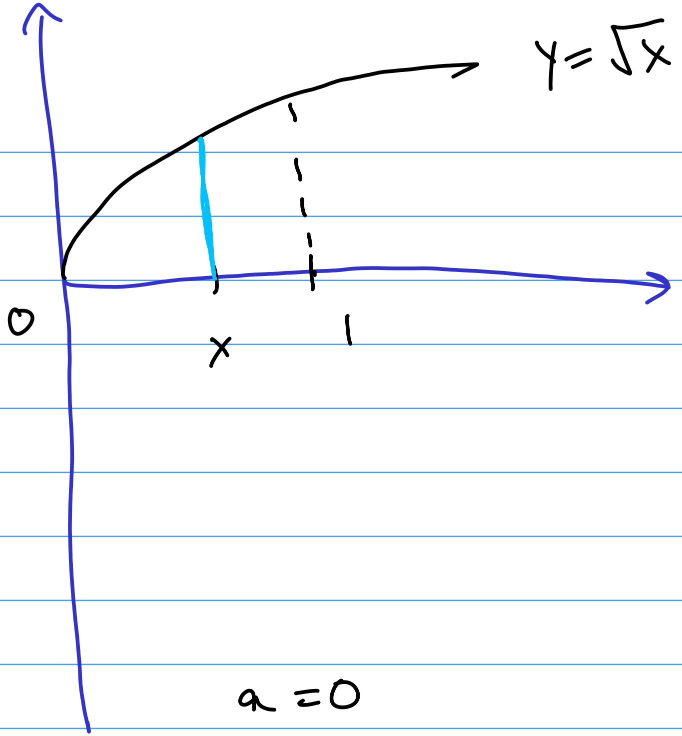
$\perp$ TO THE AXIS OF SYMMETRY



CUT AT x



$$A(x) = \pi r(x)^2$$



$a = 0$

$b = 1$

$$r(x) = y(x) = \sqrt{x}$$

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_0^1 (\pi x)\,dx = \frac{\pi x^2}{2}\Bigg]_0^1 = \boxed{\frac{\pi}{2}}$$

VOLUME

1. DRAW A PICTURE (CHOOSE AXES APPROPRIATELY)
2. TAKE A CROSS-SECTIONAL CUT (ALONG AN AXIS)
3. COMPUTE CROSS-SECTIONAL AREA [A(x)]
4. SET UP & SOLVE THE INTEGRAL

BREAK TILL

6:55PM

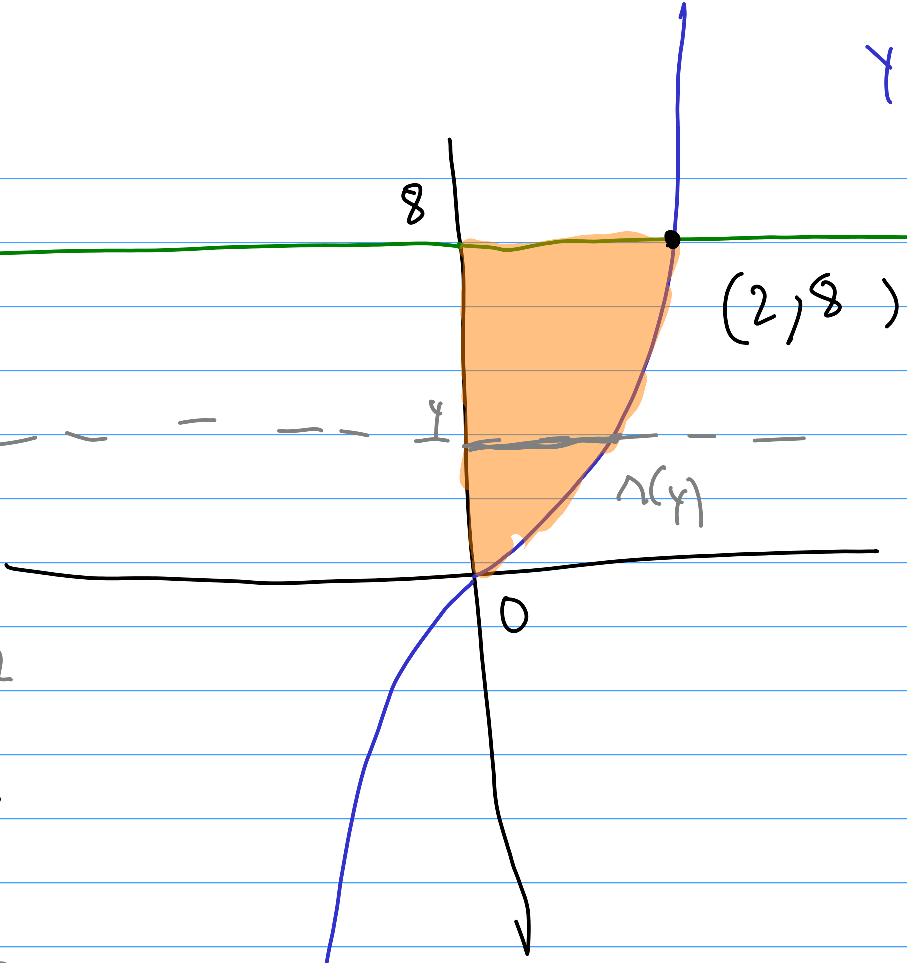
BREAKOUT ROOM

2D - PICTURE
1. ID THE AREA
2. ID THE AXIS

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

$A(y)$

$\int_a^b A(y)\,dy$



$y = x^3$

$y = 8$

$A(y) = \pi\, r(y)^2$

$r(y) = \sqrt[3]{y}$

$V = \int_0^8 \pi \left(\sqrt[3]{y}\right)^2 \, dy$

1. DRAW A ROUGH PICTURE OF SOLID
2. TAKE A CUT AND FIND $r(y)$

$$V = \int_0^8 \pi \left(\sqrt[3]{y}\right)^2 \, dy$$

$$= \int_0^8 \pi y^{2/3} \, dy$$

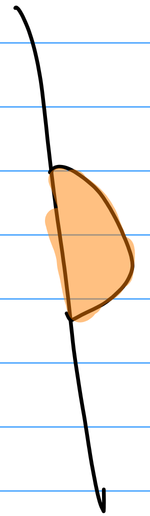
$8 = 2^3$

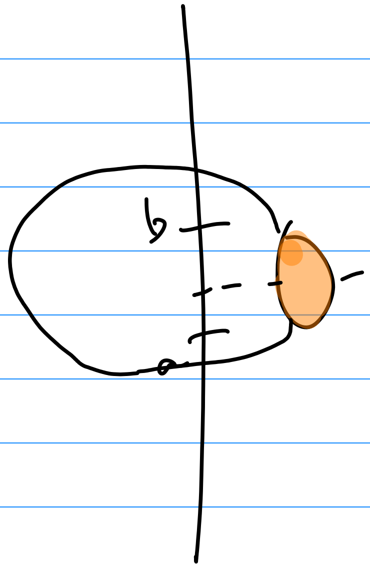
$$= \frac{3}{5} \pi y^{5/3} \Big]_0^8 \quad = \frac{3}{5} \pi \cdot (8)^{5/3}$$

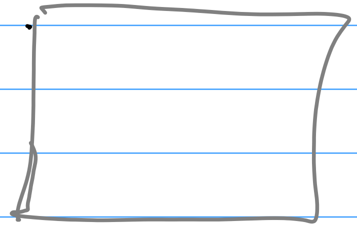
$$= \frac{3}{5} \pi \cdot (2^5)$$

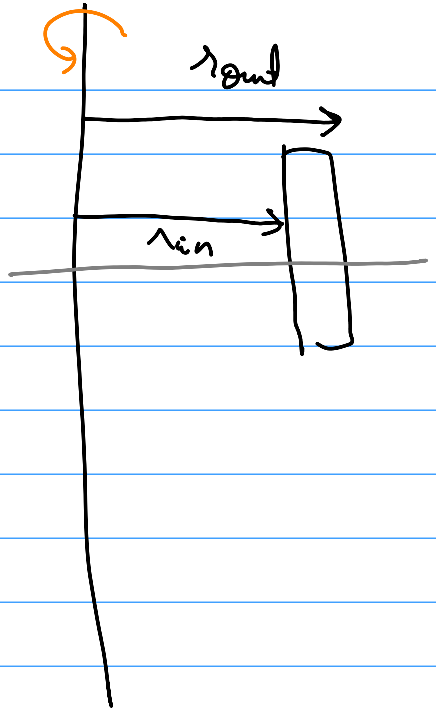
$$= 96\pi/5$$

"WASHER METHOD"

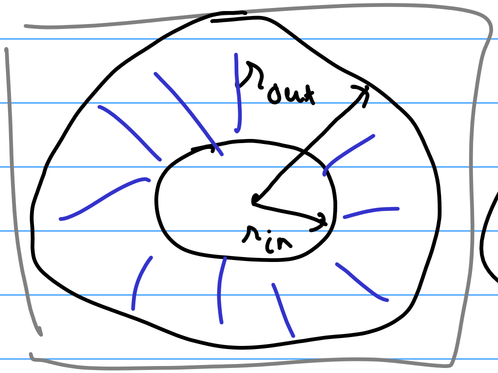








ANNULUS



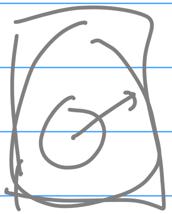
$$\underbrace{\left(\pi r_{out}^2 - \pi r_{in}^2\right)}_{A(x)}$$

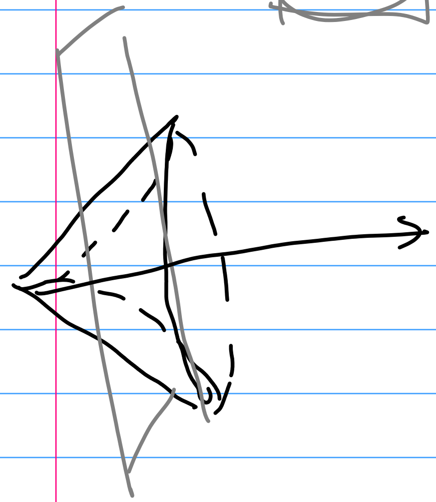
$$\left.\begin{array}{l} r_{out} \\ r_{in} \end{array}\right\} \to x$$

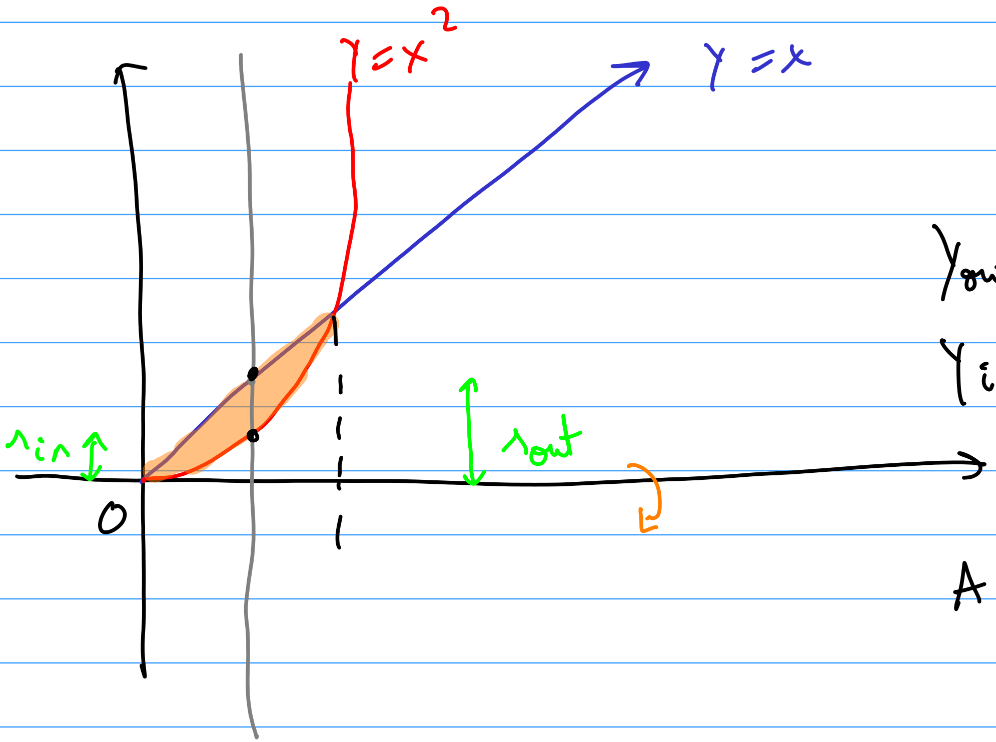
$$A(x) = \pi\left(y_{out}^2 - y_{in}^2\right)$$

ILLUSTRATE VIA AN EXAMPLE

EXAMPLE 4 The region $\mathcal{R}$ enclosed by the curves $\boxed{y = x}$ and $\boxed{y = x^2}$ is rotated about the x-axis. Find the volume of the resulting solid.







$y_{out} = x$

$y_{in} = x^2$

$A(x) = \pi \left(y_{out}^2 - y_{in}^2 \right)$

$= \pi \left(x^2 - x^4 \right)$

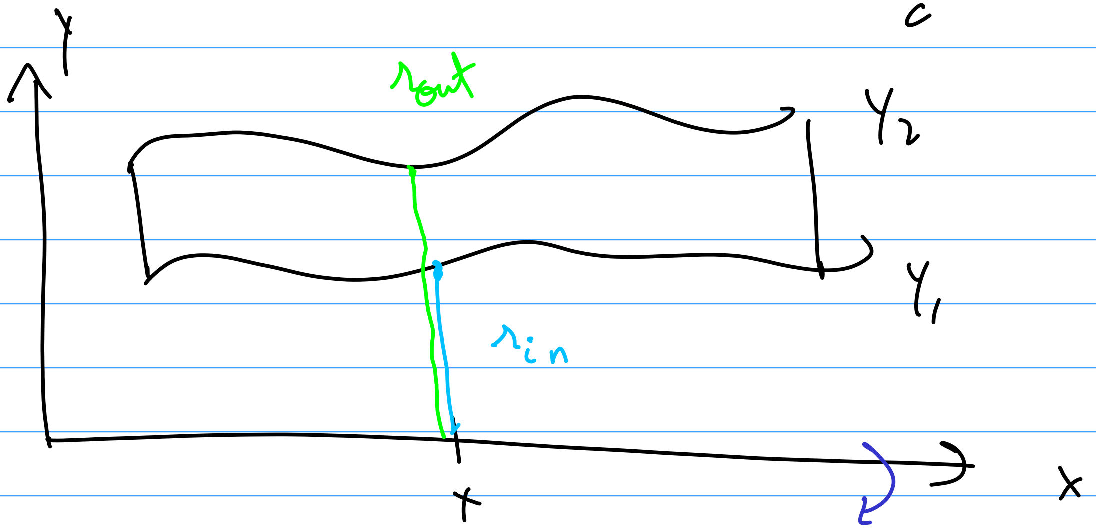
$a = 0, \quad b = 1$

$$V = \int_0^1 \pi \left(x^2 - x^4 \right) dx$$

(DISK) / WASHER METHOD

$$\int_a^b A(x)\,dx \qquad\qquad \int_c^d A(y)\,dy$$

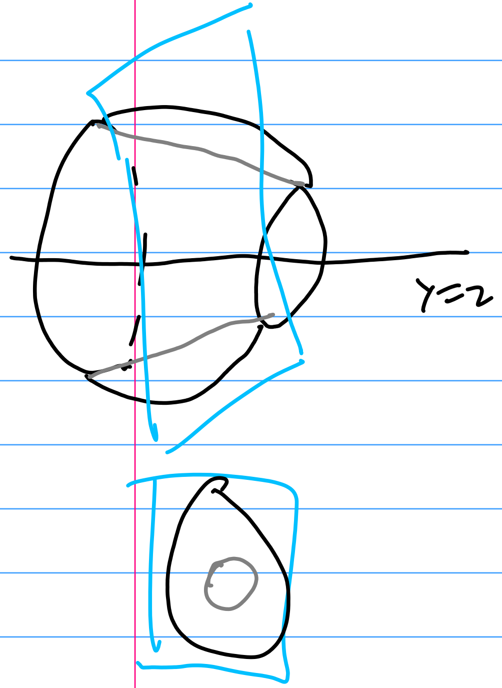
$$\int_a^b \pi\left(r_{out}^2 - r_{in}^2\right)dx$$

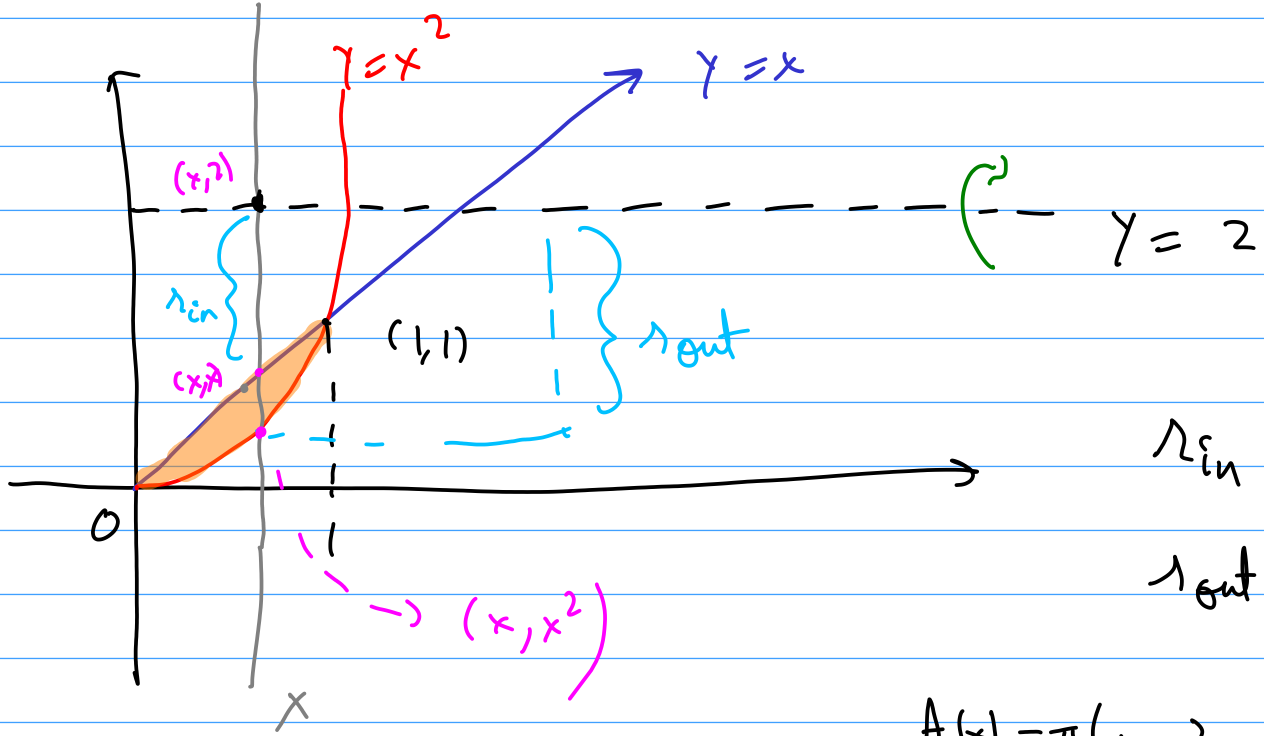


r_{out}

r_{in}

EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $\boxed{y = 2.}$





$$r_{in} = 2 - x$$

$$r_{out} = 2 - x^2$$

$$A(x) = \pi\left(r_{out}^2 - r_{in}^2\right)$$

$$A(x) = \pi\left(r_{out}^{2} - r_{in}^{2}\right)$$

$$= \pi\left(\left(2 - x^{2}\right)^{2} - \left(2 - x\right)^{2}\right)$$

$$= \pi\left[4 - 2x^{2} + x^{4} - 4 + 2x - x^{2}\right]$$

$$A(x) = \pi\left[2x - 3x^{2} + x^{4}\right]$$

$$\int_{0}^{1} \pi\left[2x - 3x^{2} + x^{4}\right] dx$$

1. SOLIDS OF
REV.

2. NON-SOLIDS OF
REV

$A(x)$

EXAMPLE 7 Figure 12 shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

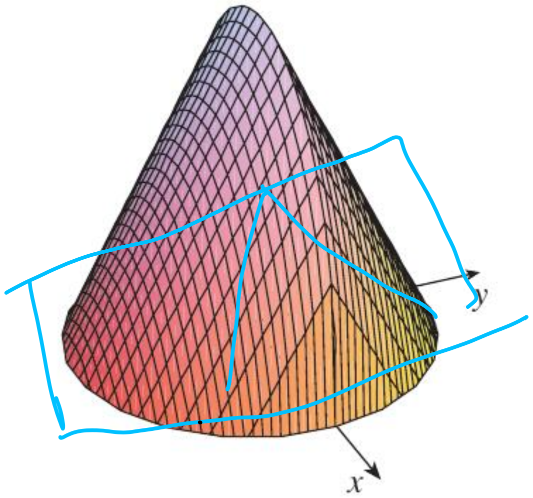
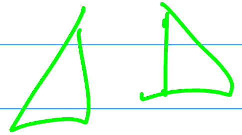
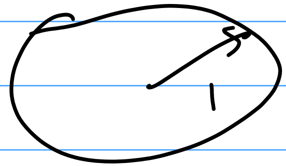
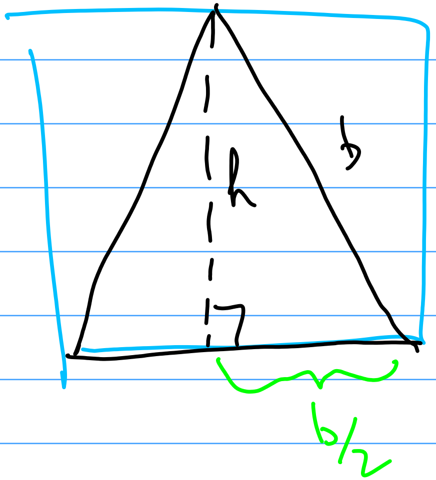


FIGURE 12



$$\boxed{h = \frac{b\sqrt{3}}{2}}$$





AREA OF Δ $= \frac{1}{2} b h$

$$h^2 + \left(\frac{b}{2}\right)^2 = b^2 \Rightarrow h^2 = b^2 - \frac{b^2}{4} = \frac{3b^2}{4}$$

$$\text{AREA} = \frac{1}{2} b h \qquad \left(h = \frac{\sqrt{3}}{2} b \right)$$

$$= \frac{1}{2} b \left(\frac{\sqrt{3}}{2} b \right)$$

$$= \frac{\sqrt{3}}{4} b^2$$

$$\boxed{\text{AREA OF EQUILATERAL } \Delta \text{ OF BASE } b = \frac{\sqrt{3}}{4} b^2}$$