MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, T, F 4-5PM ; BY APPOINTMENT

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

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COURSE PAGE: bit.ly/sahay142

ANNOUNCEMENTS

WEBWORK DEADLINES:

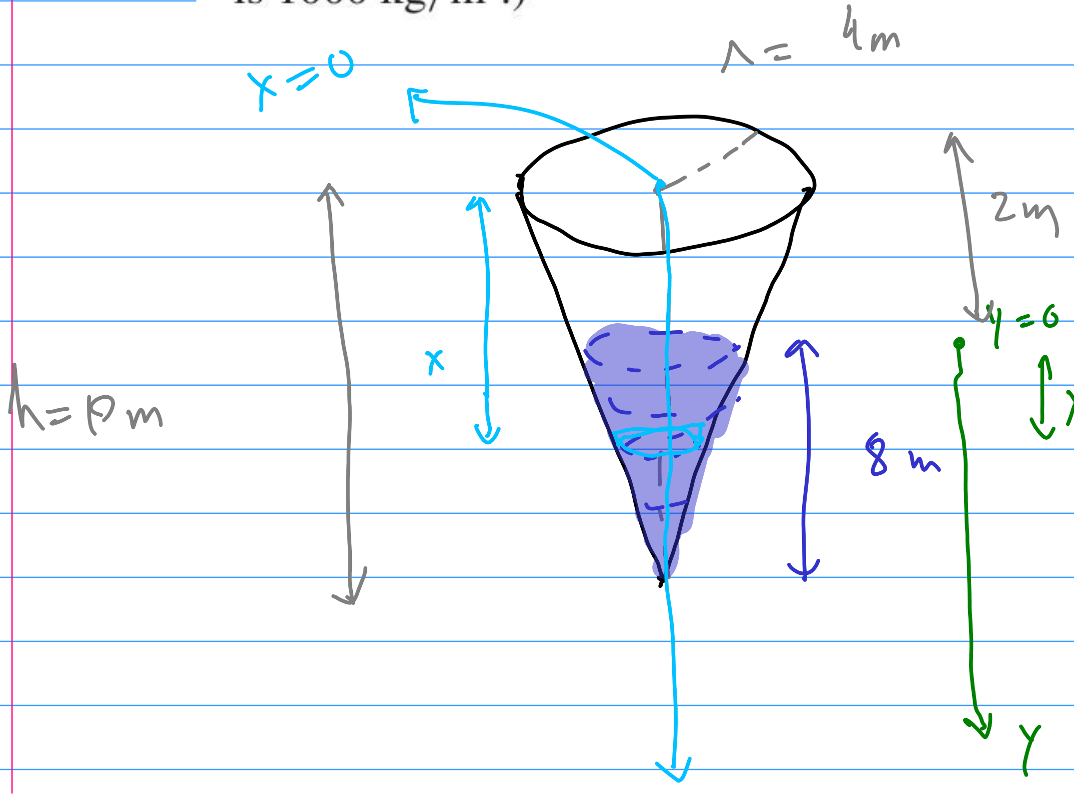
(a) WW 7 → ~~TODAY~~ TOMORROW, 11 PM

(b) WW 8 → FRIDAY, 11 PM

(c) WW 9–12 → SEE SCHEDULE.

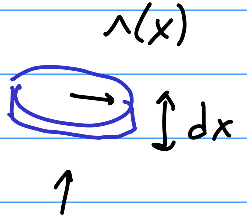
$x = (y+2) \to$ DISTANCE

EXAMPLE 5 A tank has the shape of an $\boxed{\text{inverted}}$ circular cone with height 10 m and base radius 4 m. It is filled with water to a $\boxed{\text{height of 8 m}}$. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)



(DENSITY = MASS/VOLUME)

$\rho = 1000\, kg/m^3$



VOLUME OF ○ SLAB $= \pi\, r(x)^2\, dx$

MASS OF SLAB $= 1000\pi\, r^2 dx$

MASS OF SLAB $= 1000\pi r^2 dx$ $\quad (F = mg)$

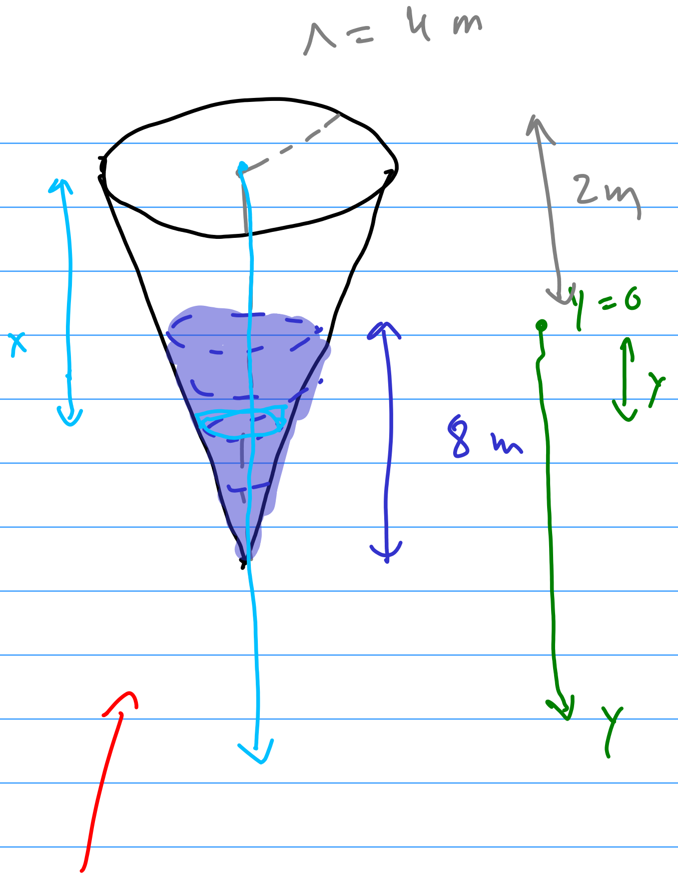
FORCE ON SLAB $= (1000\pi r^2 dx)(9.8)$

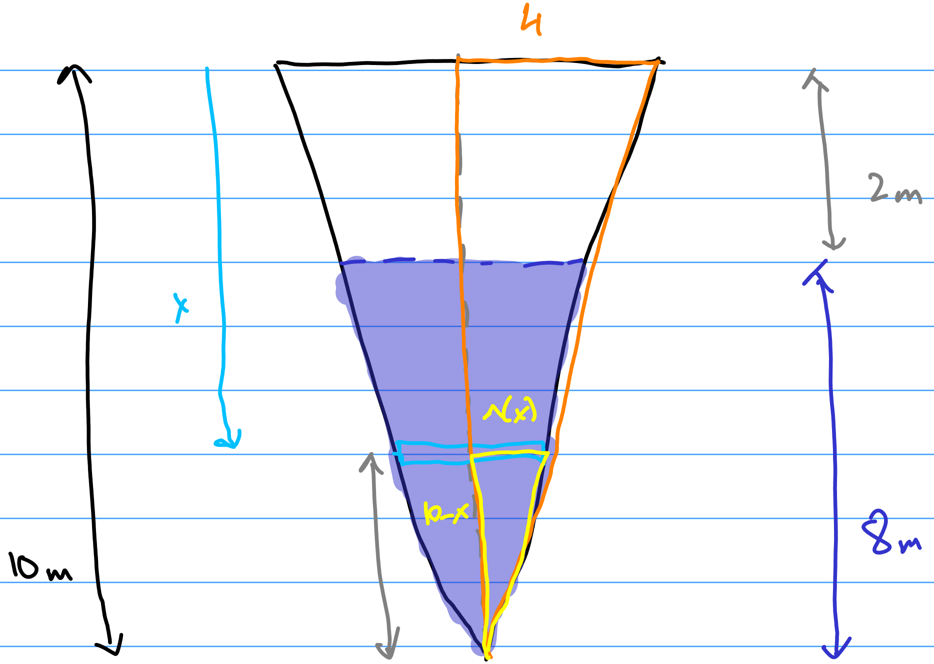
$= 9800\pi r^2 dx$

WORK DONE ON SLAB $= [9800\pi r^2 dx][x]$

$= 9800\pi r^2 x\, dx$ $\quad (W = Fd)$

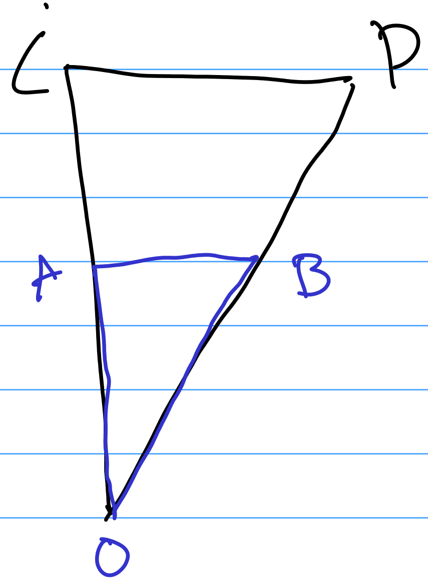
WORK DONE ON THE FULL TANK $= \int_2^{10} 9800\pi r^2 x\, dx$





$$\frac{r(x)}{4} = \frac{10-x}{10} \qquad \left[\because \text{SIMILAR } \Delta s\right]$$

$$\Rightarrow \boxed{r(x) = 4 - 2x/5}$$

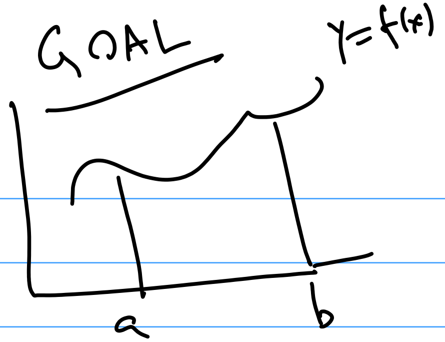


$$\frac{AB}{CD} = \frac{OA}{OC}$$

$$\boxed{a(x) = 4 - \frac{2x}{5}}$$

§ 6.5 AVERAGE VALUE OF A FUNCTION

GOAL



$\{y_1, y_2, \ldots, y_n\} \implies \text{AVERAGE} = \dfrac{y_1 + y_2 + \cdots + y_n}{n}$

MEASURES SIZE "IN AGGREGATE" $\frac{1}{b-a} \sim \frac{1}{n}$ $\qquad \int_a^b f(x)\,dx \sim \sum_{j=1}^{n} y_j$

<u>CLAIM</u>: $\boxed{\text{AVERAGE VALUE OF } f(x) \text{ ON } [a,b] = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx}$

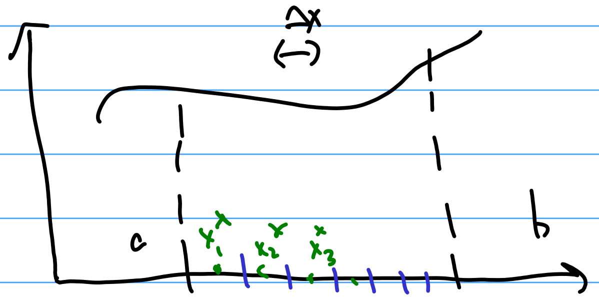
↳ WHAT SHOULD "AVERAGE VALUE" OF f MEAN?

↳ WHY SHOULD THAT BE THE FORMULA?

HEURISTIC: SUBDIVIDE $[a,b]$ AND PRETEND THAT IT IS A FINITE SET OF VALUES

$n \Delta x = b-a$

$\Rightarrow n = (b-a/\Delta x)$



n

"n VALUES"

$\sim \{ f(x_1^*), f(x_2^*), \ldots, f(x_n^*) \}$

$$\{ f(x_1^*), \dots, f(x_n^*) \} \to \frac{f(x_1^*) + \dots + f(x_n^*)}{n}$$

$$\text{AVERAGE VALUE OF } f \underset{[\text{DEFN}]}{\simeq} \sum_{j=1}^{n} \frac{f(x_j^*)}{n}$$

$$\simeq \sum_{j=1}^{n} \frac{f(x_j^*)}{\frac{b-a}{\Delta x}}$$

$$= \frac{1}{b-a} \boxed{\sum_{j=1}^{n} f(x_j^*) \Delta x}$$

RIEMANN SUM

$n \longrightarrow \infty$

$\xrightarrow{\hspace{3em}}$

$\Delta x \longrightarrow 0$

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

DEFINITION OF
AVERAGE OF A
FUNCTION.

EXAMPLE 1 Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$\bar{f}_{[a,b]}$ = AVERAGE VALUE OF f ON $[a,b]$

$$\bar{f}_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$b = 2,\ a = -1$
$f(x) = 1 + x^2$

$$= \frac{1}{2-(-1)} \int_{-1}^{2} (1 + x^2)\,dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[2 + \frac{8}{3} - \left(-1 - \frac{1}{3}\right)\right]$$

$$= \frac{6}{3} = 2$$

GEOMETRIC

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\left.f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx\right\} \bar{f}_{[a,b]}$$

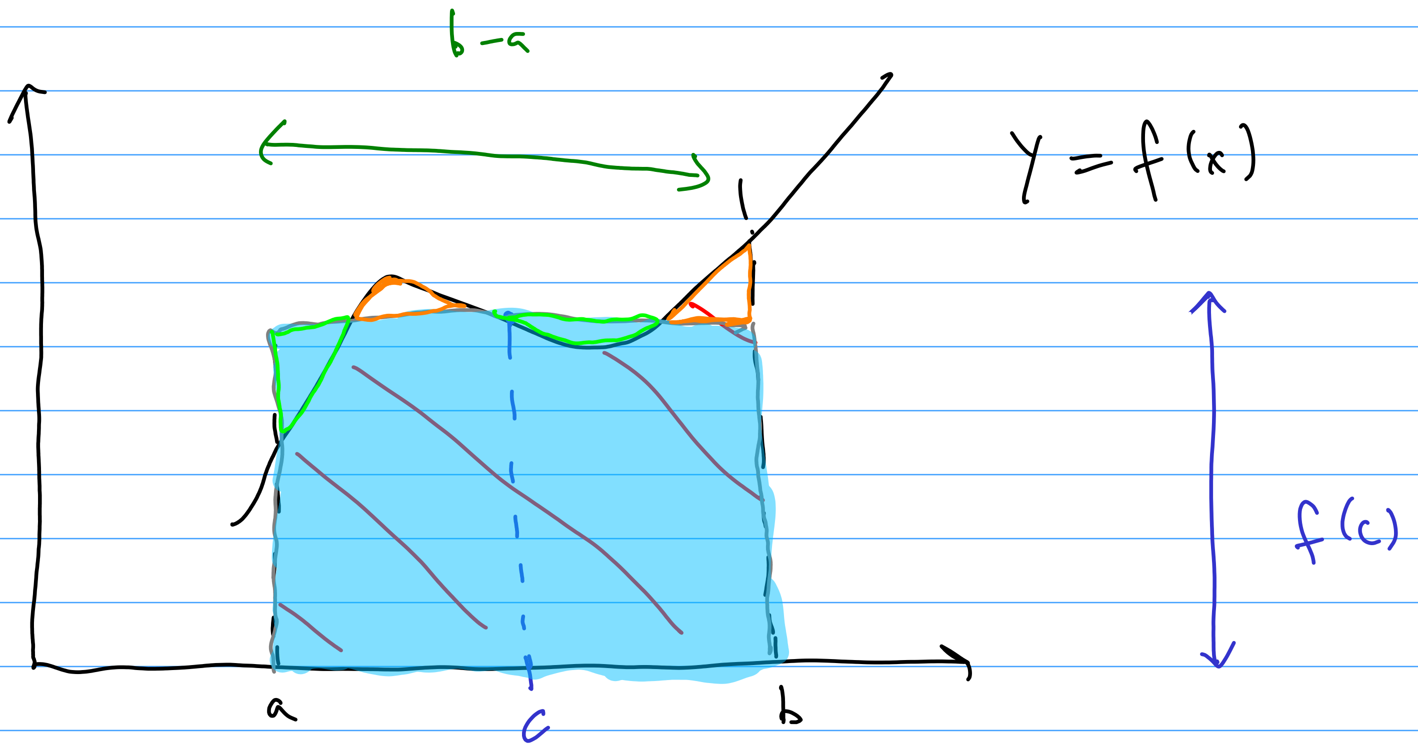
that is,

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

PF IS AN EXERCISE IN THE BOOK

AT SOME POINT, THE VALUE OF f AT THAT POINT IS EQUAL TO THE "AVERAGE VALUE" OF f.

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$



EXAMPLE 2 Since $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$, the Mean Value Theorem for Integrals says there is a number c in $[-1, 2]$ such that

$$\int_{-1}^{2} (1 + x^2)\, dx = f(c)[2 - (-1)]$$

VERIFIED MVT FOR INTEGRALS

$$\bar{f}_{[-1,2]} = 2$$

$\exists c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \bar{f}_{[a,b]}$

$a = -1,\ b = 2$

$\exists c \in [-1,2],\quad f(c) = \bar{f}_{[-1,2]} = 2$

$1 + c^2 = 2 \Rightarrow c^2 = 2 - 1 \Rightarrow c = \pm 1$

$v = \frac{ds}{dt}$

$s(t)$

↑

DISPLACEMENT

EXAMPLE 3 Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

(LAYMAN'S)

AVERAGE VELOCITY OF A CAR $[t_1, t_2]$ $= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$

|| NET CHANGE FUNDAMENTAL THEOREM OF CALC

(CALCULUS)

AVERAGE OF VELOCITY AS A FUNCTION ON $[t_1, t_2]$ $= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt$

$f_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx$

BREAK TILL 6:57 PM.

BREAKOUT ROOM

9–12

(a) Find the average value of f on the given interval.

(b) Find c in the given interval such that $f_{\text{avg}} = f(c)$.

OPTIONAL (c) Sketch the graph of f and a rectangle whose base is the given interval and whose area is the same as the area under the graph of f.

9. $f(t) = 1/t^2$, $\boxed{[1, 3]}$

$$\frac{1}{3-1}\int_1^3 \frac{1}{t^2}\,dt$$

$$= \frac{1}{2}\int_1^3 \frac{1}{t^2}\,dt$$

$$= \frac{1}{2}\left[1 - \frac{1}{3}\right]$$

$$= \frac{1}{3}$$

$$\frac{1}{c^2} = f(c) = 1/3 \rightarrow c = \sqrt{3}$$

MVT $(c \in [1, 3])$

§ 7.1 INTEGRATION BY PARTS

(AKA REVERSE PRODUCT RULE)

[PRODUCT RULE] $$\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + f'(x)\,g(x)$$

FTC $\Rightarrow$ $$f(x)\,g(x) = \int \left[f(x)\,g'(x) + f'(x)\,g(x)\right] dx$$

$$= \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx$$

$$\int \underbrace{f(x)}_{u}\, \underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\, \underbrace{g(x)}_{v} - \int \overbrace{f'(x)\, dx}^{du}\, \underbrace{g(x)}_{v}$$

(INTEGRATION BY PARTS)

$$\int u\, dv = uv - \int v\, du$$

$u \quad v$

$f(x), g(x)$

EXAMPLE 1 Find $\int x \sin x \, dx$.

$\int v \, du$

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} = \overbrace{x(-\cos x)}^{uv} - \int (-\cos x) \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

$u = x$

$dv = \sin x \, dx$

$(v' = \sin x)$

$\Rightarrow v = -\cos x$

$v = -\cos x$

$du = dx$

$\left[u = x \Rightarrow \frac{du}{dx} = 1 \right]$

$$\int x \sin x \, dx = -x\cos x + \sin x + C$$

$$\frac{d}{dx}\left(-x\cos x + \sin x + C\right) = \frac{d}{dx}\left(-x\cos x\right) + \frac{d}{dx}\left(\sin x\right)$$

(∵ PRODUCT RULE)

$$= \left[-\cos x + x\sin x\right] + \cos x$$

$$= x\sin x$$

BREAKOUT ROOM

$\ln x = 1 \cdot \overset{u}{\ln x}$

$dv = 1 \cdot dx$

$\ln x =$ PRODUCT OF TWO FUNCTIONS.

EXAMPLE 2 Evaluate $\boxed{\int \ln x \, dx.}$

$$\boxed{\begin{aligned} u &= \ln x \\ dv &= dx \end{aligned}}$$

$$\int u\,dv = uv - \int v\,du$$

FIND: v, du

WRITE I/B/P FORMULA

$v = x$

$du = \frac{dx}{x}$

$$\int \ln x\,dx = x\ln x - \int \frac{x\,dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + C$$

$$\int f\, g'\, dx = fg - \int f' g\, dx$$

WHEN SHOULD WE PLAY THIS GAMBIT

1. f MUST GET SIMPLER ON DIFFERENTIATING
2. g' MUST NOT GET MUCH MORE COMPLICATED ON INTEGRATING

$$\int F(x)\, dx$$

$F(x)$ = PRODUCT OF FUNCTIONS

1. AS IN BLUE
2. AS IN RED

EXAMPLE 3 Find $\int t^2 e^t\, dt$.

$u = t^2 \Rightarrow du = 2t\,dt$

$dv = e^t\,dt \Rightarrow v = e^t$

$$\int u\,dv = uv - \int v\,du = t^2 e^t - \int e^t\,(2t\,dt)$$

$$= t^2 e^t - \int 2t\,e^t\,dt$$

$$\int 2t\, e^t\, dt = 2te^t - 2e^t + C$$

$$u = 2t \Rightarrow du = 2\,dt$$

$$dv = e^t\, dt \Rightarrow v = e^t$$

$$uv - \int v\,du = (2t)(e^t) - \int e^t\,(2\,dt)$$

$$= 2te^t - \int 2e^t\,dt = 2te^t - 2e^t + C$$

$$\int t^2 e^t \, dt = t^2 e^t - \int 2t e^t \, dt$$

$$= t^2 e^t - \left[2t e^t - 2e^t + C \right]$$

$$= (t^2 - 2t - 2) e^t + C$$

N.B.: SOMETIMES YOU NEED TO INTEGRATE BY PARTS MORE THAN ONCE.

BREAKOUT ROOMS

$u = y$

$dv = e^{-y}\,dy$

$v = -e^{-y}$

6. $\displaystyle\int y e^{-y}\,dy$

$dv = w\,dw \Rightarrow v = \frac{w^2}{2}$

$u = \ln w$

9. $\displaystyle\int w \ln w\,dw$

11. $\displaystyle\int (x^2 + 2x)\cos x\,dx$

$u = x^2 + 2x$

$dv = \cos x\,dx$

22. $\displaystyle\int e^x \sin \pi x\,dx$