MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, T, F 4-5PM ; BY APPOINTMENT

LECTURES :
5:45 PM – 7:50 PM (ET)
M, T, W, R

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COURSE PAGE : bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK 9 → DUE ~~TOMORROW~~ THURSDAY AT 11 PM

   " 10 → " FRIDAY " "

2. MIDTERM 2 + OTHER SCORES TO BE UPDATED SOON.

§ 7.2 TRIGONOMETRIC INTEGRALS (CONTD.)

LAST TIME:

$$\int \sin^n\theta \cos^m\theta \, d\theta$$

USED: $\sin^2\theta + \cos^2\theta = 1$ (IF n OR m ODD)

HALF/DOUBLE ANGLE FORMULAE

$$\left.\begin{cases} \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \\ \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \end{cases}\right\}$$

IF n AND m EVEN

RECALL

$$\sec^2\theta = 1 + \tan^2\theta$$

GOAL: LEVERAGE TO FIND

$$\int \sec^n\theta \tan^m\theta \, d\theta$$

$\int \sec^n x \tan^m x \, dx$

EXAMPLE 5 Evaluate $\int \tan^6 x \sec^4 x \, dx$.

EVEN

$d(\tan x)$

$n = 4$
$m = 6$

$$\sec^4 x = \underbrace{\sec^2 x}\ \overbrace{\boxed{\sec^2 x}\ dx}$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\int \overbrace{\tan^6 x \ (1+\tan^2 x)}^{f(u)} \ \underbrace{(\sec^2 x \, dx)}_{du}$$

$u = \tan x$

$$= \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$u = \tan x$

$$\int \tan^6 x \sec^4 x \, dx = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

EVEN

$$\sec^{n} x = \underbrace{\left(\sec^{n-2} x\right)}_{\text{USE } \sec^2 x = 1 + \tan^2 x \text{ TO WRITE IN TERMS OF } \tan x} \left(\sec^2 x\right)$$

COMBINE WITH dx TO MAKE

$du = d(\tan x) = \sec^2 x \, dx$

$\boxed{u = \tan x}$

EXAMPLE 6 Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

ODD

NOT EVEN

$$\frac{d}{d\theta}(\sec\theta) = \sec\theta \tan\theta$$

$$\frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right) = \left(-\sin\theta\right)\left(-\frac{1}{\cos^2\theta}\right) = \tan\theta \sec\theta$$

[∵ CHAIN RULE]

KEEP $\tan\theta \sec\theta$

REWRITE THE REST IN TERMS OF $\sec\theta$

$$\int (\tan^4\theta \sec^6\theta)(\tan\theta \sec\theta \, d\theta)$$

$\tan^2\theta = \sec^2\theta - 1$

$$= \int \underbrace{(\sec^2\theta - 1)^2 \sec^6\theta}_{f(u)} \underbrace{(\tan\theta \sec\theta \, d\theta)}_{du}$$

$u = \sec\theta$

$$\int (u^2-1)^2 \, u^6 \, du \qquad (u = \sec\theta)$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11}\theta}{11} - \frac{2\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

ODD POWER OF $\tan\theta$

1. COMBINE ONE COPY OF $\tan\theta$ WITH ONE COPY OF $\sec\theta$

$$du = d(\sec\theta) = (\sec\theta\tan\theta\,d\theta)$$

2. USE $\tan^2\theta = \sec^2\theta - 1$ TO REWRITE THE REMAINING EVEN POWER OF $\tan\theta$ IN TERMS OF $\sec\theta$

3. $u = \sec\theta$

$$\int \sec^n x \ \tan^m x \, dx$$

(a) n EVEN, $n \geq 2$ → SAVE $\sec^2 x\,dx = d(\tan x)$
→ REWRITE IN TERMS OF $\tan x$

(b) m ODD → SAVE $\tan x \sec x\,dx = d(\sec x)$
→ REWRITE IN TERMS OF $\sec x$

(c) n ODD, m EVEN

(d) $n = 0$, m ARBITRARY

WHO KNOWS?

EASY

$$\int \tan x\,dx = \ln|\sec x| + C$$

HINT : $\tan x = \dfrac{\sin x}{\cos x}$

$$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du$$

$$= -\ln u + C$$

$u = \cos x \qquad du = -\sin x\,dx$

$$= -\ln \cos x + C$$

$$= \ln\left(\frac{1}{\cos x}\right) + C$$

HARD UNLESS YOU ALREADY KNOW THE ANSWER

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\frac{d}{dx}\ln(\sec x + \tan x) = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$$

$u = \sec x + \tan x \qquad \dfrac{d \ln u}{dx} = \dfrac{du}{dx}\cdot\dfrac{1}{u}$

$$= \sec x$$

EXAMPLE 7 Find $\int \tan^3 x\, dx$.

$$\tan^3 x = (\tan x)\underbrace{(\tan^2 x)}_{\sec^2 x - 1} = (\tan x)(\sec^2 x - 1)$$

$$\int \tan x(\sec^2 x - 1)\, dx = \boxed{\int \overset{u}{\tan x}\ \overset{du}{\sec^2 x\, dx}} - \boxed{\int \tan x\, dx}$$

$u = \tan x \Rightarrow du = \sec^2 x\, dx$

$\int \tan x \sec^2 x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$

$$= \frac{\tan^2 x}{2} - \ln \sec x + C$$

EXAMPLE 8 Find $\overbrace{\int \sec^3 x\,dx}^{I}$.

DIFF $\qquad$ INTEGRATE

$$\sec^3 x = \underbrace{(\sec x)}_{\text{DIFF}}\underbrace{(\sec^2 x)}_{\text{INTEGRATE}}$$

INTEGRATION BY PARTS

$u = \sec x \Rightarrow du = \sec x \tan x\, dx$

$dv = \sec^2 x\, dx \Rightarrow v = \tan x$

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 - 1}\, dx$$

$$= \sec x \tan x - \underbrace{\boxed{\int \sec^3 x}}_{I} + \int \sec x\, dx$$

$$I = \sec x \tan x - I + \int \sec x \, dx$$

$$2I = \sec x \tan x + \ln(\sec x + \tan x) + C$$

$$I = \frac{1}{2}\left[\sec x \tan x + \ln\left|\sec x + \tan x\right|\right] + C$$

N.B.

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

CAN BE USED FOR

$$\boxed{\int \operatorname{cosec}^n\theta \cot^m\theta \, d\theta}$$

$$\frac{d}{d\theta}(\cot\theta) = -\operatorname{cosec}^2\theta$$

$$\frac{d}{d\theta}(\operatorname{cosec}\theta) = -\cot\theta \operatorname{cosec}\theta$$

PRODUCT $\to$ SUM IDENTITIES

1. $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

2. $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

3. $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

GOAL : LEVERAGE TO FIND $\int \sin n\theta \cos m\theta \, d\theta$

$\sin, \cos$, $n\theta$ & $m\theta$

$\int \sin n\theta \sin m\theta \ldots$

$\int \sin nx \quad \cos mx$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x \, dx$.

$n = 4$

$m = 5$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\left.\begin{aligned} A &= 4x \\ B &= 5x \end{aligned}\right] \to \sin 4x \cos 5x = \frac{1}{2}\left[\sin(4x-5x) + \sin(4x+5x)\right]$$

$$= \frac{1}{2}\left[\sin(-x) + \sin 9x\right] = \frac{\sin 9x - \sin x}{2}$$

$$\int \sin 4x \cos 5x \, dx = \int \frac{\sin 9x - \sin x}{2} \, dx = \frac{1}{2}\int \sin 9x \, dx - \frac{1}{2}\int \sin x \, dx$$

$$\frac{1}{2}\int \sin 9x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{-\cos 9x}{(2)(9)} + \frac{\cos x}{2} + C$$

$$= \frac{9\cos x - \cos 9x}{18} + C \qquad \blacksquare$$

$$\int \sin n\theta \cos m\theta \, d\theta$$

$$\int \sin n\theta \sin m\theta \, d\theta$$

$$\int \cos n\theta \cos m\theta \, d\theta$$

APPLY A SUM-PRODUCT IDENTITY!

BREAK TILL 6:41 PM ET

§ 7.3 TRIGONOMETRIC SUBSTITUTION

GOAL : USE TRIGONOMETRIC FUNCTIONS & IDENTITIES TO COMPUTE NON-TRIG INTEGRALS.

NEED TO USE INVERSE SUBSTITUTION.

$$\int f(g(x))\,g'(x)\,dx \xrightarrow{\boxed{u=g(x)}} \int f(u)\,du$$

$$\int f(g(x))\,g'(x)\,dx \xrightarrow[\text{SUBSTITUTION}]{\boxed{u=g(x)}} \int f(u)\,du$$

$$\int f(x)\,dx \xrightarrow[dx=g'(t)\,dt]{x=g(t)} \int f(g(t))\,g'(t)\,dt$$

INVERSE SUBSTITUTION

$g \rightarrow$ INVERTIBLE.

WHY? → TO MAKE THE MAP INVERTIBLE

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

IF YOU SEE THESE

MAKE THIS SUBSTITUTION

USE TO SIMPLIFY

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$a = 3$

$a^2 = 9$

$x = 3\sin\theta$

$-\pi/2 \le \theta \le \pi/2$

$$\int_0^a \sqrt{a^2 - x^2}\,dx \; ; \quad x = a\sin\theta$$

$0 \le \theta \le \pi/2$

EXAMPLE 1 Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx.$

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$$

$$\boxed{x = 3\sin\theta} \Rightarrow dx = 3\cos\theta\, d\theta$$

$$\int \frac{\sqrt{9 - (3\sin\theta)^2}}{(3\sin\theta)^2}\,(3\cos\theta\, d\theta)$$

$$= \int \frac{3\sqrt{1 - \sin^2\theta}}{9\sin^2\theta}\,(3\cos\theta)\, d\theta$$

$$-\pi/2 < \theta \le \pi/2$$

$$= \int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}\,(3\cos\theta)\, d\theta = \int \frac{(3\cos\theta)(3\cos\theta)}{9\sin^2\theta}\, d\theta$$

$$\int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

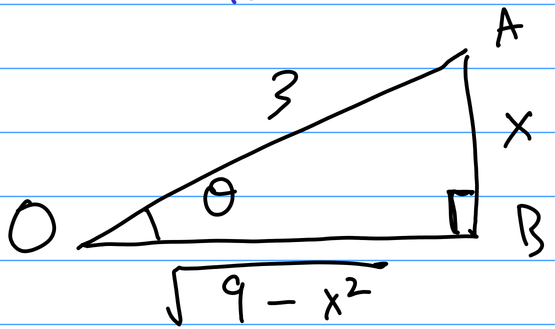
$$= -\cot\theta - \theta + C$$

$$x = 3\sin\theta \Rightarrow \sin\theta = \boxed{\frac{x}{3}}$$

$$\Updownarrow$$

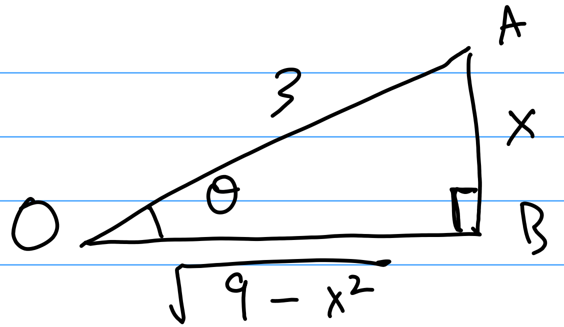
$$\theta = \text{Arcsin}\left(\frac{x}{3}\right)$$

TRIANGE



$$\sin\theta = \boxed{\frac{AB}{OA}}$$

$$OB^2 + AB^2 = OA^2$$



$\left(\frac{\text{BASE}}{\text{PERP}}\right)$

$$\cot\theta = \frac{OB}{AB} = \frac{\sqrt{9-x^2}}{x}$$

$$\boxed{-\cot\theta} - \theta + C$$

$\arcsin\left(\frac{x}{3}\right)$

CHECK THIS AT HOME

$$\boxed{\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C}$$

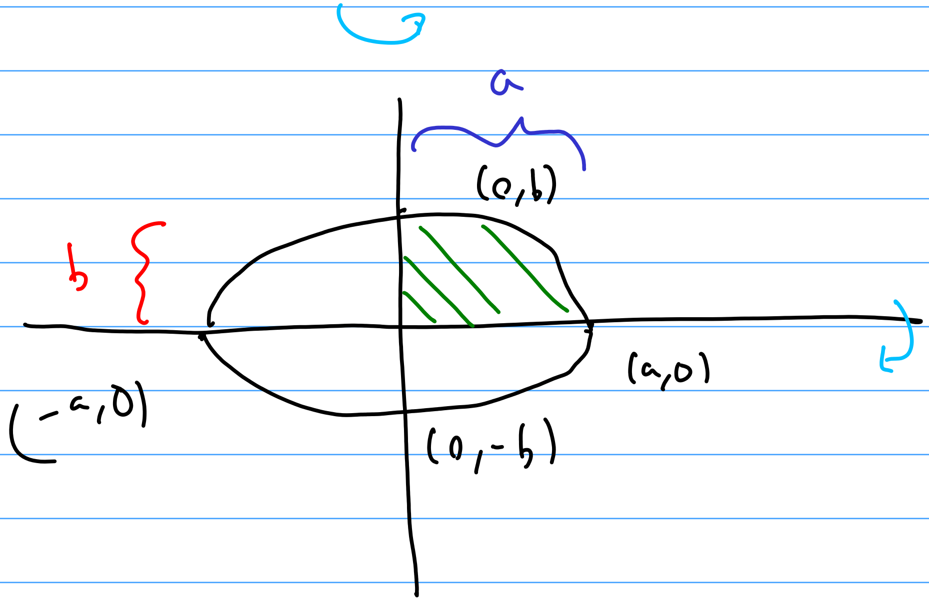
EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$$

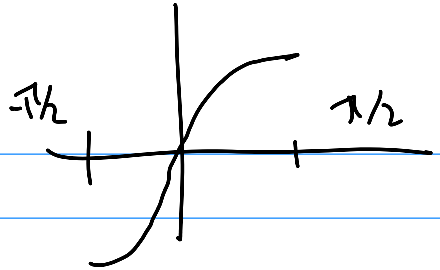


A = 4

WRITE INTEGRAL FOR GREEN AREA.

$$\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$



$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

; $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$$= \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2-(a\sin\theta)^2}\,(a\cos\theta\,d\theta)$$

$x=0$; $a\sin\theta = 0$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x=a$; $a\sin\theta = a$

$\Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

$$\frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-(a\sin\theta)^2}\,(a\cos\theta\,d\theta) = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta)\,d\theta$$

$$= \frac{4b\,a}{a}\int_0^{\pi/2}\sqrt{a^2\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}}\,\cos\theta\,d\theta$$

$$= \left(\frac{4b}{a}\right)a^2\int_0^{\pi/2}\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{2}\left(1+\cos 2\theta\right)d\theta$$

$n=0, m=2$

$$\int \sin^n\theta\cdot\cos^m\theta\,d\theta$$

$n, m \rightarrow$ EVEN

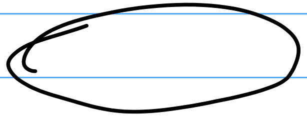
$$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$$

$$\int_0^{\pi/2} \frac{1}{2}\left(1 + \cos 2\theta\right) d\theta = \int_0^{\pi/2} \frac{d\theta}{2} + \int_0^{\pi/2} \frac{\cos 2\theta \, d\theta}{2}$$

$$= \frac{\theta}{2}\Big]_0^{\pi/2} + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\Big]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\left(\sin \pi - \sin 0\right)$$

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \pi/4$$

AREA OF  $= 4ab \int_0^{\pi/2} \cos^2\theta \, d\theta$

$$= (4ab)\left(\frac{\pi}{4}\right)$$

AREA OF AN ELLIPSE $= \pi ab$

SANITY CHECK

$a = b = r$

$\Rightarrow \quad x^2 + y^2 = r^2 \qquad (\pi r^2)$

EXAMPLE 3 Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$. $\longrightarrow \sqrt{x^2+4} = \sqrt{4+x^2}$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\boxed{\sqrt{a^2 + x^2}}$	$x = a\tan\theta,\quad \boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}} \rightarrow$ INVERTIBLE	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$x = 2\tan\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx \qquad -\pi/2 < \theta < \pi/2$$

$$x = 2\tan\theta \ni dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{1}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}\,(2\sec^2\theta\, d\theta)$$

$$\int \frac{1}{(4\tan^2\theta)}\left(\frac{1}{2\sqrt{\sec^2\theta}}\right)(2\sec^2\theta\, d\theta)$$

$$= \frac{1}{4}\int \left(\frac{1}{\tan^2\theta}\right)\left(\frac{1}{\sec\theta}\right)(\sec^2\theta)\,d\theta$$

$$\frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$\boxed{\begin{aligned}\sec\theta &= 1/\cos\theta\\ \tan^2\theta &= \sin^2\theta/\cos^2\theta\end{aligned}}$$

$$= \frac{1}{4}\int \frac{(1/\cos\theta)}{(\sin^2\theta/\cos^2\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{\overbrace{\cos\theta\,d\theta}^{du}}{\underbrace{\sin^2\theta}_{u^2}}$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4u} + C$$

$$-\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C$$

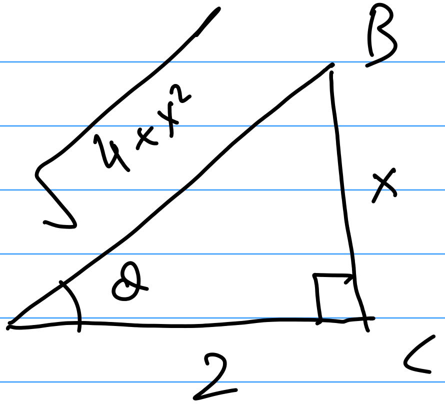
$$= -\frac{\csc\theta}{4} + C$$

$$AB^2 = AC^2 + BC^2$$

$$x = 2\tan\theta$$

$$\Rightarrow \tan\theta = \frac{x}{2}$$

$$\tan\theta = \frac{\overset{\text{PERP}}{BC}}{\underset{\text{BASE}}{AC}}$$



$$\frac{x}{2} \leftrightarrow \frac{BC}{AC}$$

$$\csc\theta = \frac{AB}{BC} = \frac{\sqrt{4+x^2}}{x}$$

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = -\frac{1}{4}\left(\frac{\sqrt{4+x^2}}{x}\right)+C$$

EXAMPLE 4 Find $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

$\boxed{x = 2\tan\theta}$

a IS CONSTANT

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 5 Evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

$\boxed{x = a\sec\theta}$

EXAMPLE 6 Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

$\boxed{x = \frac{3}{2}\tan\theta}$

$$(4x^2+9)^{3/2} = \left(\sqrt{4x^2+9}\right)^3$$

$$= \left(2\sqrt{x^2+9/4}\right)^3$$

EXAMPLE 7 Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

COMPLETING THE SQUARE