MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: ~~M~~, T, F 4-5PM;

$\boxed{\text{BY APPOINTMENT}}$

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

email: anuragsahay@rochester.edu

COURSE PAGE: bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK 9 → DUE TOMORROW AT 11 PM
   " 10 → " FRIDAY ? " "
2. MIDTERM 2 HAS BEEN GRADED. PLEASE CHECK GRADESCOPE.
3. SOLUTIONS WILL BE UP TONIGHT. + MEETING
4. TOTAL GRADES SO FAR UP ON BLACKBOARD.
5. NO OFFICE HOURS ON MONDAY

RECALL:

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$
$$= a\sqrt{\cos^2\theta} = a\cos\theta$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$a\cos\theta$ $\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a\sec\theta$ $\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$a\tan\theta$ $\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

IF YOU SEE THESE

TO MAKE $\sqrt{\cdot}$ POSITIVE

INVERTIBLE

MAKE THESE SUBSTITUTIONS

USE TO SIMPLIFY

EXAMPLE 5 Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

$x = a \sec\theta$

$dx = a \tan\theta \sec\theta \, d\theta$

Table of Trigonometric Substitutions

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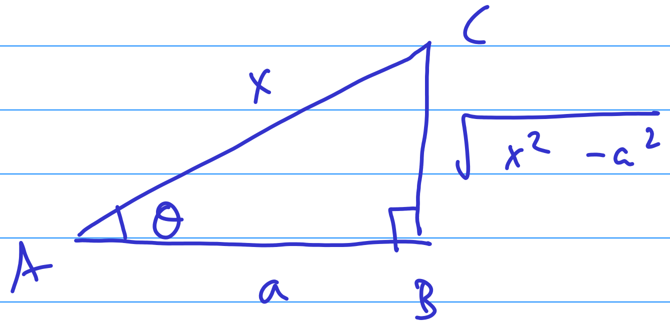
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{(a)(\tan\theta \sec\theta)\, d\theta}{\sqrt{(a\sec\theta)^2 - a^2}} = \int \frac{a\tan\theta \sec\theta}{a\tan\theta}\, d\theta = \int \sec\theta \, d\theta$$

$$\int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$x = a\sec\theta \Rightarrow \theta = \operatorname{Arcsec}\left(\frac{x}{a}\right)$$

TRIANGLE



$$\boxed{\frac{x}{a}} = \sec\theta = \frac{\overset{\text{HYPO}}{AC}}{\underset{\text{BASE}}{AB}}$$

$$\tan\theta = \frac{BC}{AB} = \frac{\sqrt{x^2-a^2}}{a}$$

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C$$

$$= \ln \left| x + \sqrt{x^2 - a^2} \right| - \underbrace{\ln |a|}_{\text{CONSTANT}} + C$$

$$= \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

BREAKOUT ROOM

EXAMPLE 4 Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx.$

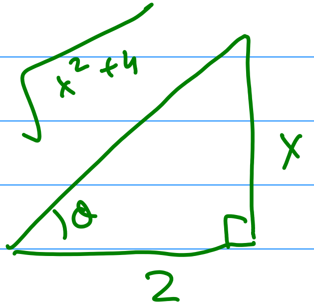
$$\boxed{x = 2\tan\theta}$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4}$$

$$= 2\sqrt{\tan^2\theta+1}$$

$$= 2\sec\theta$$



$$\tan\theta = \frac{x}{2}$$

$$\sec\theta = \frac{\sqrt{x^2+4}}{2}$$

$$\int \frac{(2\tan\theta)(2\sec^2\theta)\,d\theta}{2\sec\theta}$$

$$= \int 2(\tan\theta\sec\theta)\,d\theta$$

$$= 2\sec\theta + C$$

$$= 2\left(\frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \sqrt{x^2+4} + C$$

EXAMPLE 4 Find $\int \frac{x}{\sqrt{x^2+4}}\,dx.$

$\frac{du}{2}$

$u = x^2 + 4$

$$\int \frac{du}{2\sqrt{u}}$$

TRIG SUBS DON'T APPLY DIRECTLY

$$\boxed{u = 4x^2 + 9}$$

$x = a\tan\theta$

$= \frac{3}{2}\tan\theta$

$\boxed{\sqrt{x^2 + a^2}}$

$\sqrt{x^2 - a^2}$

$\sqrt{a^2 - x^2}$

EXAMPLE 6 Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx.$

$$= \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{8\left[x^2 + 9/4\right]^{3/2}}\, dx$$

$$\left(\sqrt{4x^2 + 9}\right)^3$$

$$4x^2 + 9 = 4\left[x^2 + \frac{9}{4}\right]$$

1. $\sqrt{a^2 - x^2}$

$u = 2\sin\theta$

$du = 2\cos\theta\, d\theta$

COMPLETING THE SQUARE

2. $\sqrt{a^2 + x^2}$

$\boxed{x + 1 = 2\sin\theta}$

3. $\sqrt{x^2 - a^2}$

EXAMPLE 7 Evaluate $\displaystyle\int \frac{x}{\sqrt{\underbrace{3 - 2x - x^2}_{\text{QUADRATIC}}}}\,dx.$ $= \displaystyle\int \frac{x\,dx}{\sqrt{4 - (1+x)^2}}$

$x^2 + 2x + 1 = (x+1)^2$

$u = x + 1$

$$\int \frac{(u-1)\,du}{\sqrt{4 - u^2}}$$

$\frac{b^2}{4a} = -1$

$$3 - 2x - x^2 = 4 - 1 - 2x - x^2$$

$$= 4 - (1 + 2x + x^2) = 4 - (1+x)^2$$

$u = x + 1$

$$\int \frac{(u-1)}{\sqrt{4-u^2}}\, du$$

$$u = 2\sin\theta \Rightarrow \sqrt{4-u^2} = 2\cos\theta \;;\; du = 2\cos\theta\, d\theta$$

$$\int \left[\frac{(2\sin\theta) - 1}{(2\cos\theta)} \right] (2\cos\theta\, d\theta)$$

$$= \int 2\sin\theta\, d\theta - \int d\theta = 2\cos\theta - \theta + C$$

$$= 2\cos\theta - \theta + C$$

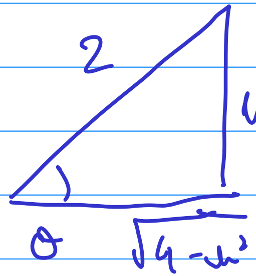
$$u = 2\sin\theta$$

$$\sin\theta = \frac{u}{2}$$

$$= 2\left(\frac{\sqrt{4-u^2}}{2}\right) - \operatorname{Arcsin}\left(\frac{u}{2}\right) + C$$

$$\Rightarrow \theta = \operatorname{Arcsin}\left(\frac{u}{2}\right)$$

$$= \sqrt{4-u^2} - \operatorname{Arcsin}\left(\frac{u}{2}\right) + C$$



$$= \sqrt{4-(x+1)^2} - \operatorname{Arcsin}\left(\frac{x+1}{2}\right) + C$$

$$\cos\theta = \frac{\sqrt{4-u^2}}{2}$$

$$= \sqrt{3-2x-x^2} - \operatorname{Arcsin}\left(\frac{x+1}{2}\right) + C$$

$$u = x+1$$

$$ax^2 + bx + c$$

$$\updownarrow$$

ADD & SUBTRACT $\frac{b^2}{4a}$

$$\underbrace{ax^2 + bx + \frac{b^2}{4a}} + \left(c - \frac{b^2}{4a}\right)$$

$$a\left[\left(x + \frac{b}{2a}\right)^2\right] + \left(c - \frac{b^2}{4a}\right)$$

QUADRATIC FORMULA

GENERAL PRINCIPLE

$$\sqrt{P(x)}$$

$P(x) \rightarrow$ QUADRATIC

$$\left.\begin{matrix} x^2 - a^2 \\ a^2 - x^2 \\ x^2 + a^2 \end{matrix}\right\} \rightarrow \text{STANDARD FORM}$$

BREAK TILL

6:50 PM ET

§ 7.4 PARTIAL FRACTIONS

(AKA → HOW TO INTEGRATE RATIONAL FNS.)

[P, Q → POLYNOMIALS]

GOAL : $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

[e.g. $\left(\frac{x+1}{x-1}\right)$, $\left(\frac{x^3-x}{x^2+x+1}\right)$]

HOW? :

1. LONG DIVIDE $P(x)$ BY $Q(x)$

2. FACTORIZE $Q(x)$, COMPUTE PARTIAL

3. CASES

   (a) ALL FACTORS ARE LINEAR AND DO $\boxed{\text{NOT REPEAT}}$

(FINAL ANSWER ONLY CONTAINS LOG)

EXAMPLE 1 Find $\int \frac{x^3 + x}{x - 1}\, dx.$

CUBIC

LINEAR

DEG NUMERATOR > DEG OF DENOMIN.

$$\frac{x^3 + x}{x - 1}$$

$$x^3 + x = x(x^2 + 1)$$

$$\frac{x^3}{x-1} + \frac{x}{x-1}$$

$x^3 + x$ DIVIDED $x - 1$

$$p(x) = q(x)\, b(x) + r(x)$$

$$(x^3 + x) = (x^2 + x + 2)(x - 1) + 2$$

$$
\begin{array}{r|l}
 & x^2 \quad + x \quad + 2 \\
x - 1 & x^3 \qquad\qquad + x \\
 & -[x^3 \quad - x^2] \\
 & + x^2 \quad + x \\
 & x^2 \quad - x \qquad / \; 2x \quad 2x \quad -2
\end{array}
$$

$$\frac{x^3+x}{x-1} = \frac{(x^2+x+2)(x-1)+2}{(x-1)}$$

[∴ LONG DIVISION]

$$= (x^2+x+2) + \frac{2}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int (x^2+x+2) + \left(\frac{2}{x-1}\right) dx$$

$$= \underbrace{\int (x^2+x+2)\,dx}_{\text{POWER RULE}} + \overbrace{2\int \frac{dx}{x-1}}^{u=x-1,\ \ln u}$$

QUADRATIC

EXAMPLE 2 Evaluate $\int \frac{x^2 + 2x - 1}{\boxed{2x^3 + 3x^2 - 2x}}\,dx$. → FACTOR

CUBIC

1. LONG DIVISION DOESN'T CHANGE ANYTHING

$$2x^3 + 3x^2 - 2x = x\left[2x^2 + 3x - 2\right]$$

$$= x\left[2x^2 + 4x - x - 2\right]$$

$$= x\left[(x+2)(2x) - (x+2)\right]$$

$$= x(x+2)(2x-1)$$

$$\frac{x^2 + 2x - 1}{x(x+2)(2x-1)}$$

LINEAR?

$\deg P(x) < \deg Q(x)$

FACT : $Q(x)$ FACTORIZES TO CASE a

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$$

$$\frac{P(x)}{Q(x)} = \frac{C_1}{a_1 x + b_1} + \frac{C_2}{a_2 x + b_2} + \cdots + \frac{C_n}{a_n x + b_n}$$

FOR SOME CONSTANTS $C_1, \ldots, C_n$

$$\frac{C_1}{\cancel{x}}\left[\cancel{x}(x+2)(2x-1)\right]$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{C_1}{x} + \frac{C_2}{x+2} + \frac{C_3}{2x-1}$$

<u>CLEAR DENOMINATORS:</u>

MULTIPLY BY $x(x+2)(2x-1)$

$$x^2+2x-1 = C_1(x+2)(2x-1) + C_2(x)(2x-1) + C_3\,x(x+2)$$

$$= x^2\left[2C_1 + 2C_2 + C_3\right] + x\left[3C_1 - C_2 + 2C_3\right]$$

$$- 2C_1$$

$\Rightarrow$ COEFFICIENTS ARE EQUAL

$2c_1 + 2c_2 + c_3 = 1 \Rightarrow 2c_2 + c_3 = 0$

$3c_1 - c_2 + 2c_3 = 2 \Rightarrow 2c_3 - c_2 = 1/2$

$-2c_1 = -1 \Rightarrow c_1 = 1/2$

$c_3 = -2c_2$

$2(-2c_2) - c_2 = 1/2 \Rightarrow c_2 = -\dfrac{1}{10}$

$c_3 = 1/5$

$C_1 = 1/2 \qquad C_2 = -1/10 \qquad C_3 = 1/5$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{C_1}{x} + \frac{C_2}{x+2} + \frac{C_3}{2x-1}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{5(2x-1)}$$

$$\int \frac{x^2+2x-1}{x(x+2)(2x-1)} = \boxed{\int \frac{dx}{2x}} - \boxed{\int \frac{dx}{10(x+2)}} + \boxed{\int \frac{dx}{5(2x-1)}}$$

$u = x+2 \qquad u = 2x-1$

$$\frac{1}{2}\ln x - \frac{1}{10}\ln(x+2) + \frac{1}{10}\ln(2x-1) + C$$

EXAMPLE 3 Find $\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

$a \to$ FIXED
NON-ZERO

$$x^2 - a^2 = (x+a)(x-a)$$

$a \neq 0$

$$\frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)} = \frac{C_1}{x+a} + \frac{C_2}{x-a}$$

($\because$ FACT)

LAST TIME: (1) CLEARED DENOMS.
(2) EQUATED COEFFICIENTS.

ALTERNATIVE METHOD TO COMPUTE P/F

(1) CLEAR DENOMENATORS

$$\frac{1}{(x+a)(x-a)} = \frac{C_1}{x+a} + \frac{C_2}{x-a}$$

MULTIPLY B $(x+a)(x-a)$

$$1 = C_1(x-a) + C_2(x+a)$$

(2) PLUG IN SMARTLY CHOSEN VALUES OF x.

$$1 = C_1(x-a) + C_2(x+a)$$

PLUG IN $x = a$

$$1 = C_2(a+a) \Rightarrow C_2(2a) = 1 \Rightarrow C_2 = \frac{1}{2a}$$

PLUG IN $x = -a$

$$1 = C_1(-a-a) \Rightarrow C_1(-2a) = 1 \Rightarrow C_1 = \frac{-1}{2a}$$

$$\frac{1}{(x+a)(x-a)} = \frac{C_1}{x+a} + \frac{C_2}{x-a}$$

$$= \frac{-1}{2a}\left(\frac{1}{x+a}\right) + \frac{1}{2a}\left(\frac{1}{x-a}\right)$$

$$\int \frac{dx}{x^2-a^2} = \frac{-1}{2a}\int \frac{dx}{\underbrace{x+a}_{u=x+a}} + \frac{1}{2a}\int \frac{dx}{\underbrace{x-a}_{u=x-a}}$$

$$= \frac{-1}{2a}\ln(x+a) + \frac{1}{2a}\ln(x-a) + C$$

BREAKOUT ROOM

1. $$\int \frac{5}{(x-1)(x+4)}\,dx$$

DO THESE AT HOME

2. $$\int \frac{x-12}{x^2-4x}\,dx$$

3. $$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx$$

$$\frac{5}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

1. LONG DIVISION
2. FACTORIZED
3. PARTIAL FRACTIONS

$$5 = A(x+4) + B(x-1)$$

$$5 = x(A+B) + (4A-B)$$

$$\left.\begin{aligned} A+B &= 0 \\ 4A-B &= 5 \end{aligned}\right\} \begin{aligned} A &= 1 \\ B &= -1 \end{aligned}$$

$$5 = A(x+4) + B(x-1)$$

PLUG IN ① $x = -4$ $\quad 5 = B(-4-1) \Rightarrow B = -1$

② $x = 1$ $\quad 5 = A(1+4) \Rightarrow A = 1$

$$\frac{5}{(x-1)(x+4)} = \frac{1}{x-1} - \frac{1}{x+4}$$