MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: T, F 4-5PM ; BY APPOINTMENT

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

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COURSE PAGE : bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK 9 → DUE TODAY AT 11 PM

   " 10 → " FRIDAY " "

2. MIDTERM 2 SOLUTIONS ARE UP.
3. MIDTERM REVIEW MEETINGS.

§ 7.4 PARTIAL FRACTIONS

(AKA → HOW TO INTEGRATE RATIONAL FNS.)

[P, Q → POLYNOMIALS]

GOAL : $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

[e.g. $\left(\frac{x+1}{x-1}\right)$, $\left(\frac{x^3-x}{x^2+x+1}\right)$]

HOW? :

1. LONG DIVIDE $P(x)$ BY $Q(x)$
2. FACTORIZE $Q(x)$
3. COMPUTE PARTIAL FRACTIONS DEPENDING ON CASE

   (a) ALL FACTORS ARE LINEAR AND DO **NOT REPEAT**

   (b) ALL FACTORS ARE LINEAR BUT SOME REPEAT.

   (c) SOME FACTORS ARE QUADRATIC, BUT NO FACTOR IS REPEATED

   (d) NONE THE ABOVE

REVIEW:

$$\int \frac{x-12}{x^2-4x}\,dx$$

$$x^2-4x = x(x-4)$$

$$\frac{x-12}{x^2-4x} = \frac{A}{x} + \frac{B}{x-4}$$

$$x-12 = A(x-4) + Bx$$

$$\xrightarrow{x=0} A(-4) = -12 \Rightarrow A = 3$$

$$\xrightarrow{x=4} B(4) = -8 \Rightarrow B = -2$$

$$\int \left(\frac{x - 12}{x^2 - 4x} \right) dx = \int \left(\frac{3}{x} - \frac{2}{x-4} \right) dx$$

$$= 3 \ln x - 2 \ln (x-4) + C$$

$$\int \frac{3x^2+6x+2}{x^2+3x+2}\,dx$$

$$\frac{3x^2+6x+2}{x^2+3x+2} = 3 + \frac{(-3x-4)}{x^2+3x+2}$$

$$(x^2+3x+2) = (x+2)(x+1)$$

$$A(-1) = 2$$

$$\to A = -2$$

$$-\frac{3x-4}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1} \Rightarrow B(1) = -1$$

$$\to B = -1$$

$$\int \frac{3x^2+6x+2}{x^2+3x+2}\,dx = \int 3 - \frac{2}{x+2} - \frac{1}{x+1}\,dx$$

$$= 3 - 2\ln(x+2) - \ln(x+1) + C$$

EXAMPLE 4 Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$

4

3

$$
\begin{array}{r|llll}
 & x & +1 & & \\
\hline
x^3 - x^2 - x + 1 & x^4 & & -2x^2 & +4x & +1 \\
 & x^4 & -x^3 & -x^2 & +x & \\
\hline
 & & x^3 & -x^2 & +3x & +1 \\
 & & x^3 & -x^2 & -x & +1 \\
\hline
 & & & & 4x &
\end{array}
$$

$$x^4 - 2x^2 + 4x + 1 = \underbrace{(x+1)}_{\text{QUOTIENT}}(x^3 - x^2 - x + 1) + \underbrace{(4x)}_{\text{REMAINDER}}$$

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = (x+1) + \frac{4x}{x^3 - x^2 - x + 1}$$

FACTORISE

$$x^3 - x^2 - x + 1 = (x+1)(x^2 - 2x + 1)$$

$$= (x+1)(x-1)^2$$

$\deg P < \deg Q$

$$Q(x) = (a_1 x + b_1)^{\lambda_1} \cdots (a_n x + b_n)^{\lambda_n}$$

FACT

$$\frac{P(x)}{Q(x)} = \frac{C_{1,1}}{a_1 x + b_1} + \frac{C_{1,2}}{(a_1 x + b_1)^2} + \cdots + \frac{C_{1,\lambda_1}}{(a_1 x + b_1)^{\lambda_1}}$$

$$+ \frac{C_{2,1}}{(a_2 x + b_2)} + \frac{C_{2,2}}{(a_2 x + b_2)^2} + \cdots \frac{C_{2,\lambda_2}}{(a_2 x + b_2)^2}$$

$$+ \cdots$$

$$+ \frac{C_{n,1}}{(a_n x + b_n)} + \frac{C_{n,2}}{(a_n x + b_n)^2} + \cdots + \frac{C_{n,\lambda_n}}{(a_n x + b_n)^{\lambda_n}}$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B}{x+1}$$

A_1, A_2, B ? CLEAR DENOMINATORS.

MULTPLY $(x-1)^2(x+1)$

$$4x = A_1(x-1)(x+1) + A_2(x+1) + B(x-1)^2$$

$$= x^2(A_1+B) + x(A_2-2B) + (-A_1+A_2+B)$$

$$4x = x^2(A_1 + B) + x(A_2 - 2B) + (-A_1 + A_2 + B)$$

$$A_1 + B = 0 \Rightarrow A_1 = -B$$

$$A_2 - 2B = 4$$

$$-A_1 + A_2 + B = 0 \Rightarrow A_2 = A_1 - B = -B - B = -2B$$

$$(-2B) - 2B = 4 \Rightarrow B = -1$$

$$A_1 = -(-1) = 1, \quad A_2 = -2(-1) = 2$$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B}{x+1}$$

$$A_1 = 1 \,, \quad A_2 = 2 \,, \quad B = -1$$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = (x+1) + \left[\frac{1}{x+1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right]$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int (x+1)\,dx + \int \frac{dx}{x-1} + \int \frac{2\,dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$= \left(\frac{x^2}{2} + x\right) + \ln(x+1) + \int \frac{2\,dx}{(x-1)^2} - \ln(x+1)$$

$u = x - 1$

$$\int \frac{2\,du}{u^2} = -\frac{2}{u} + C$$

$$= \frac{x^2}{2} + x + \ln(x+1) - \frac{2}{x-1} - \ln(x+1) + C$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{\boxed{x^3 + 4x}}\,dx.$

FACTORISE

$$x^3 + 4x = x(x^2 + 4)$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{(Bx + C)}{x^2 + 4}$$

CLEAR DENOMS

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

① $x = 0$

$$4 = 4A \rightarrow A = 1$$

② $x = 1$

$$5 = 5 + (B+C) \Rightarrow B+C = 0$$

③ $x = -1$ $\quad 7 = 5 + (B-C) \Rightarrow B-C = 2$

$$B = 1, \quad C = -1$$

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{1}{x} + \left(\frac{x-1}{x^2+4}\right) dx$$

$$= \underbrace{\int \frac{dx}{x}}_{\ln x} + \underbrace{\int \frac{x\,dx}{x^2+4}} - \underbrace{\int \frac{dx}{x^2+4}}$$

$u = x^2 + 4$

$du = 2x\,dx$

$$\int \frac{du}{2u} = \frac{\ln u}{2} = \frac{\ln(x^2+4)}{2}$$

$x = 2u$

$$\int \frac{2\,du}{4(u^2+1)}$$

$$= \frac{1}{2}\,\mathrm{Arctan}\,\frac{x}{2}$$

FACT V3: $Q(x) =$ HAS SOME QUADRATIC FACTORS.

$$Q(x) = (a_1 x + b_1) \cdots (a_n x + b_n)(ax^2 + bx + d$$

$$\frac{P(x)}{Q(x)} = \frac{c_1}{a_1 x + b_1} + \cdots + \frac{Ax + B}{ax^2 + bx + c}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$\int \frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}\,dx = \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx + \int \frac{Cx+D}{x^2+x+1}\,dx$$

$$\int \frac{Ex+F}{x^2+1}\,dx + \int \frac{Gx+H}{(x^2+1)^2}\,dx + \int \frac{Ix+J}{(x^2+1)^3}\,dx$$

EXAMPLE 8 Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx.$

$\to$ SMALLER

$\to$ LARGER DEGREE

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$

CLEAR DENOMS

$$1 - x + 2x^2 - x^3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$$1 - x + 2x^2 - x^3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$\hookrightarrow x = 0$

$A = 1, \quad B = -1, \quad C = -1, \quad D = 1, \quad E = 0$

$u = x^2 + 1$

$x\,dx = \frac{du}{2}$

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx = \int\left(\frac{1}{x} + -\frac{(x+1)}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$\int \frac{1}{x}\,dx \qquad - \int \frac{x\,dx}{x^2+1} \qquad - \int \frac{dx}{x^2+1} \qquad \int \frac{x\,dx}{(x^2+1)^2}$$

$\sqrt{P(x)}$

$\uparrow$

$P(x) \to$ QUADRATIC

BREAKOUT ROOM

$u = x(x^2+3) = x^3+3x$

$du = 3(x^2+1)\,dx$

$$\int \frac{x^2+1}{x(x^2+3)}\,dx$$

$\frac{du}{3}$

u

1. WRITE DOWN PARTIAL FRACTION DECOMPOSITION?

2. WAS THAT THE RIGHT THING TO DO?

$$\frac{x^2+1}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

(1) $x = 0$

$1 = 3A \Rightarrow A = 1/3$

$$x^2+1 = A(x^2+3) + (Bx+C)x$$

$$x^2+1 = A(x^2+3) + (Bx+C)x$$

$(x=0)$

$A = 1/3$

$$= (A+B)x^2 + Cx + 3A$$

$0 = C$

$3A = 1$

$A+B = 1 \Rightarrow B = 1 - \frac{1}{3} = \frac{2}{3}$

$x = u\sqrt{3}$

$\frac{1}{3(u^2+1)}$

$$\frac{x^2+1}{x(x^2+3)} = \frac{1}{3x} + \frac{2}{3}\left(\frac{1}{x^2+3}\right)$$

BREAK TILL

7:10 PM

EXAMPLE 9 Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx.$ → NOT A POLYNOMIAL

$$\int \frac{\sqrt{x+4}}{x}\,dx$$

$\sqrt{x+4}$ → SQUARE

$$x + 4 = t^2$$

$$dx = 2t\,dt \qquad \sqrt{x+4} = t$$

$$x = t^2 - 4$$

RATIONAL

$$\int \frac{t\,(2t)}{t^2 - 4}\,dt$$

§ 7.8 IMPROPER INTEGRALS

RECALL :

$f \rightarrow$ CONTINUOUS

$[a,b] \rightarrow$ FINITE CLOSED INTERVAL

$\int_a^b f(x)\,dx \rightarrow$ EXISTS! (RIEMANN SUMS)

Q2 WHAT IF f DISCONTINUOUS AT A POINT?

Q1. WHAT IF $[a,b]$ IS REPLACED BY AN INFINITE INTERVAL?

$$\left.\begin{array}{l}[b,\infty) \\ (-\infty,a] \\ (-\infty,\infty)\end{array}\right\} \text{INFINITE INTERVAL}$$

TYPE I : INFINITE INTERVAL.

e.g $y = 1/x^2$ $x \geq 1$ $\rightarrow$ $[1, \infty)$

$t \geq 1$ $\displaystyle\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = \frac{-1}{t} - \left(-\frac{1}{1}\right)$

$= 1 - \frac{1}{t}$

$$\int_1^t \frac{dx}{x^2} = 1 - \frac{1}{t} \leq 1$$

$$\lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = \lim_{t \to \infty} \left(1 - \frac{1}{t}\right)$$

$$= 1$$

DEFN:

(i) IF $\int_a^t f(x)\,dx$ EXISTS FOR ALL $t \geq a$

(DEFN.)

THEN $$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx$$

[IF IT $\boxed{\text{EXISTS}}$]

EXISTS & IS FINITE!

LIMIT EXISTS $\longrightarrow$ CONVERGENT

LIMIT DOESN'T EXIST $\longrightarrow$ DIVERGENT

(ii) IF $\int_t^b f(x)\,dx$ EXISTS FOR ALL $t \leq b$,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dX$$

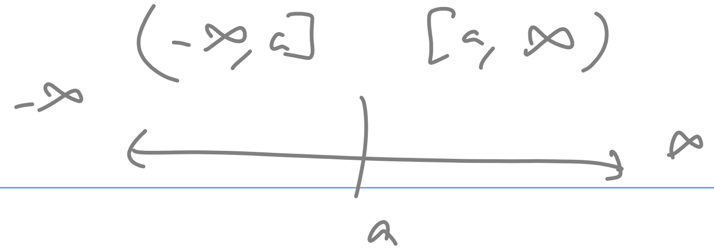
(IF IT EXISTS)

CONVERGENT / DIVERGENT

CONVERGENT → LIMIT EXISTS *

(* → AND IS FINITE)

DIVERGENT → IF IT DOESN'T



FOR ANY $a \in \mathbb{R}$, IF

$$\int_a^\infty f(x)\,dx \quad \& \quad \int_{-\infty}^a f(x)\,dx$$

BOTH EXIST (CONVERGENT), THEN

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or $\boxed{\text{divergent.}}$

$$\int_1^\infty \quad = \lim_{t \to \infty} \int_1^t$$

$$\int_1^t \left(\frac{1}{x}\right) dx = \ln x \Big]_1^t = \ln t$$

$$\lim_{t \to \infty} \int_1^t \frac{dx}{x} = \lim_{t \to \infty} \ln t = +\infty$$

EXAMPLE 2 Evaluate $\int_{-\infty}^{0} xe^x\,dx$. $= -1$

$$-1 \leftarrow \int_t^0 x e^x \, dx = x e^x \Big]_t^0 - \int_t^0 e^x \, dx$$

$u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$$= \Big[x e^x - e^x \Big]_t^0$$

$$= \left[0 e^0 - e^0 \right] - \left[t e^t - e^t \right]$$

$$= -t e^t + e^t - 1$$

$$\int_t^0 x e^x \, dx = e^t(1-t) - 1$$

$$\lim_{t\to-\infty} \int_t^0 x e^x \, dx = \left[\lim_{t\to-\infty} e^t(1-t)\right] - 1$$

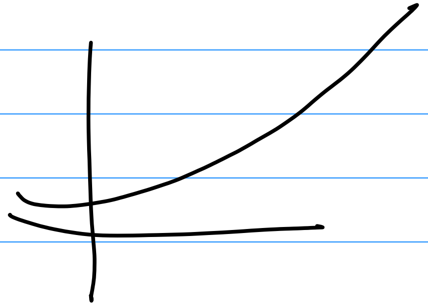
$= 0$

$$e^t(1-t) = \frac{\overbrace{1-t}^{\to\infty}}{\underbrace{e^{-t}}_{\to\infty}}$$

l'Hôpital's Rule

$$\lim_{t \to -\infty} \frac{(1-t)}{e^{-t}} = \lim_{t \to -\infty} \frac{\frac{d}{dt}(1-t)}{\frac{d}{dt}(e^{-t})}$$

$$= \lim_{t \to -\infty} \frac{-1}{-e^{-t}}$$



$$= \lim_{t \to -\infty} e^{t} = 0$$

EXAMPLE 3 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

$(-\infty, \infty)$

$a = 0$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx \qquad \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_0^t \frac{1}{1+x^2} = \operatorname{Arctan} x \Big]_0^t = \operatorname{Arctan} t$$

$$\int_t^0 \frac{1}{1+x^2} = \operatorname{Arctan} x \Big]_t^0 = -\operatorname{Arctan} t$$

1. FIND THE SYMMETRY
2. COMPUTE
3. TAKE LIMITS