MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, F 4-5PM;
W 9-10 AM;
BY APPOINTMENT

(N.B. – STARTING FRIDAY)

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COURSE PAGE: bit.ly/sahay142

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

NOTES / ANNOUNCEMENTS

1. PARTICIPATION → NOT ATTENDANCE !
2. ABLE TO ACCESS NOTES / RECORDINGS ?
3. CONFLICTS WITH EXAM TIMES ?
4. ACADEMIC HONESTY QUIZ.
5. ANY QUESTIONS ?

QUESTIONS ON EXTRA NOTES?

§ 4.7 OPTIMIZATION PROBLEMS
(AKA CALC FOR REAL LIFE)

GUIDING STEPS:

1. UNDERSTAND THE PROBLEM → e.g. "Q" FOR THE QUANTITY THAT NEEDS TO BE OPTIMIZE
2. DRAW A DIAGRAM
3. INTRODUCE NOTATION / SYMBOLS → 3? 6?
4. REWRITE GIVEN INFORMATION IN THE FORM OF EQUATIONS (ESPECIALLY INVOLVING Q)
5. USE DIFFERENTIATION TO FIND GLOBAL EXTREMA.

141 REVIEW

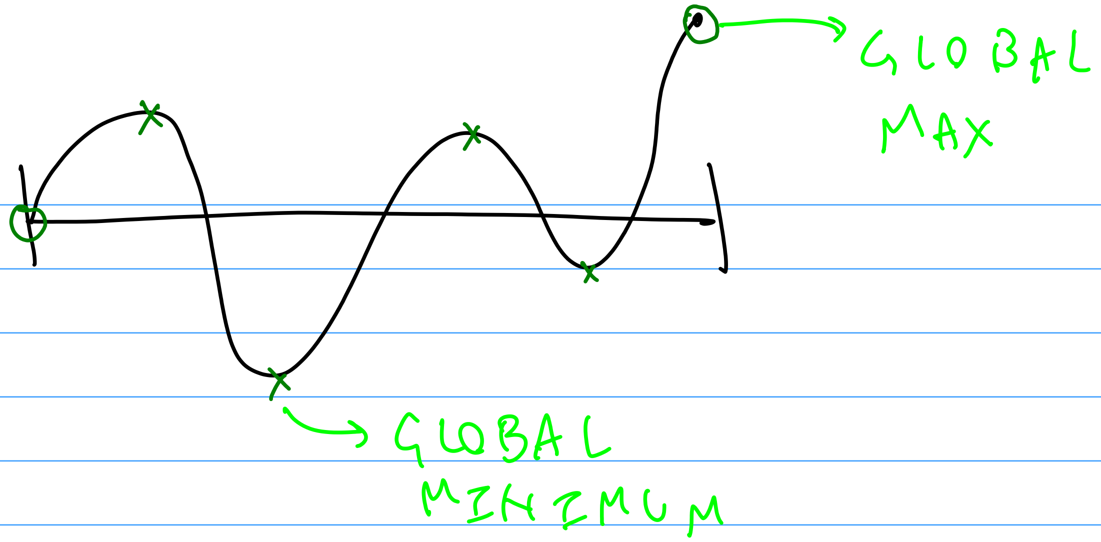
FINDING GLOBAL MAX /MIN

DEFN : x_0 IS A GLOBAL/ ABSOLUTE MAX OF f

IF $f(x) \leq f(x_0)$ FOR ALL x

— — — — — — (MIN) — —

— — $f(x) \geq f(x_0)$ — — — —



① USE 1st / 2nd DERIVATIVE TEST TO FIND LOCAL MAX / MIN

② COMPARE VALUES AT END-POINTS $\rightarrow f(c)$ & ALL EXTREMA TO FIND LARGEST / SMALLEST

(N.B.: BOOK CALLS THIS "CLOSED INTERVAL METHOD"

$$f(x) = x^4 + 5x^3 - 2x^2 + 1 \quad \text{on } \boxed{[-5, 1]}$$

$$f'(x) = 4x^3 + 5(3x^2) - 2(2x)$$

$$= 4x^3 + 15x^2 - 4x$$

$$= x[4x^2 + 15x - 4] = x(4x-1)(x+4)$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-15 \pm \sqrt{15^2 - (4)(4)(-4)}}{2(4)} = \frac{-15 \pm \sqrt{225 + 64}}{8}$$

$$= \frac{-15 \pm \sqrt{289}}{8} = \frac{-15 \pm 17}{8}$$

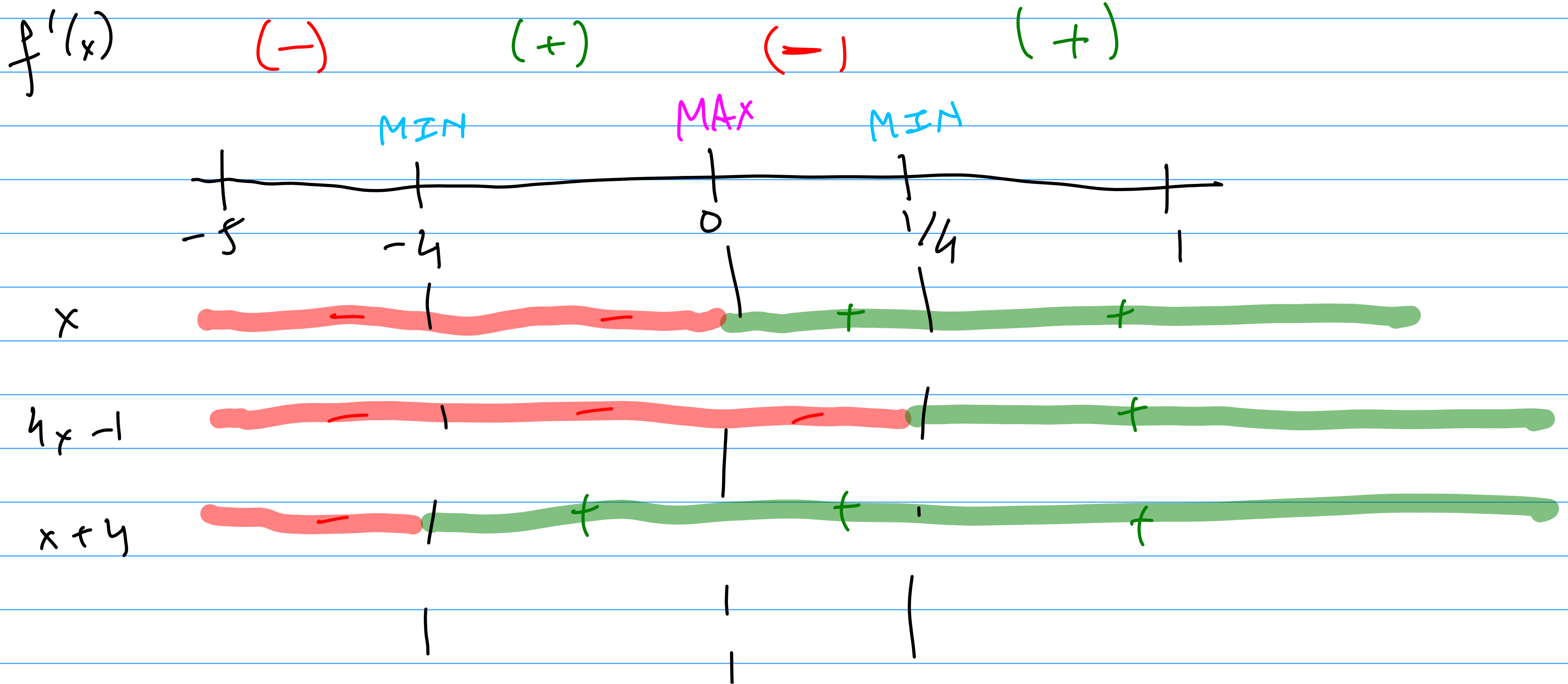
$\to 1/4$

$\to -4$

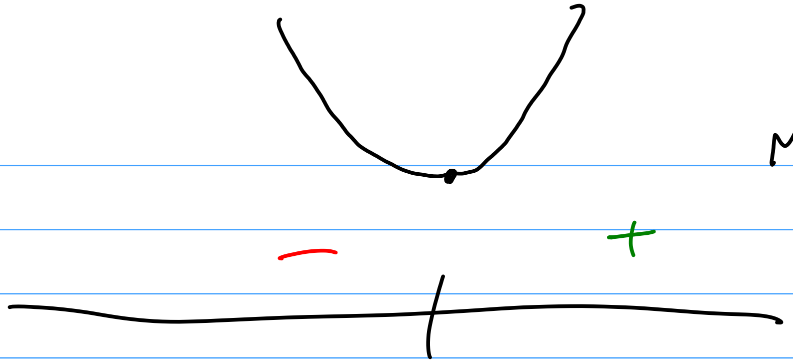
$$f'(x) = x(4x-1)(x+4)$$

$x = 0, \quad x = 1/4$

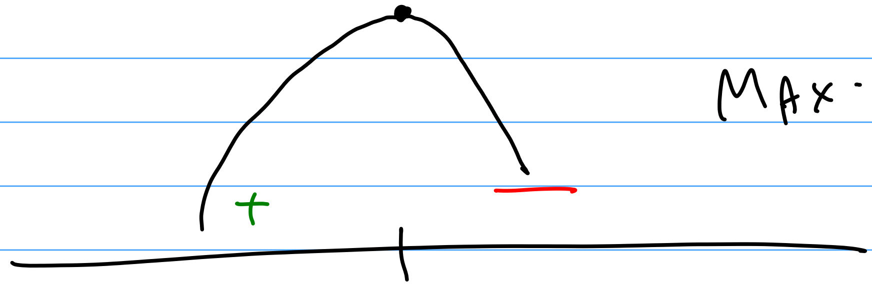
$x = -4$



RECALL



MINIMUM



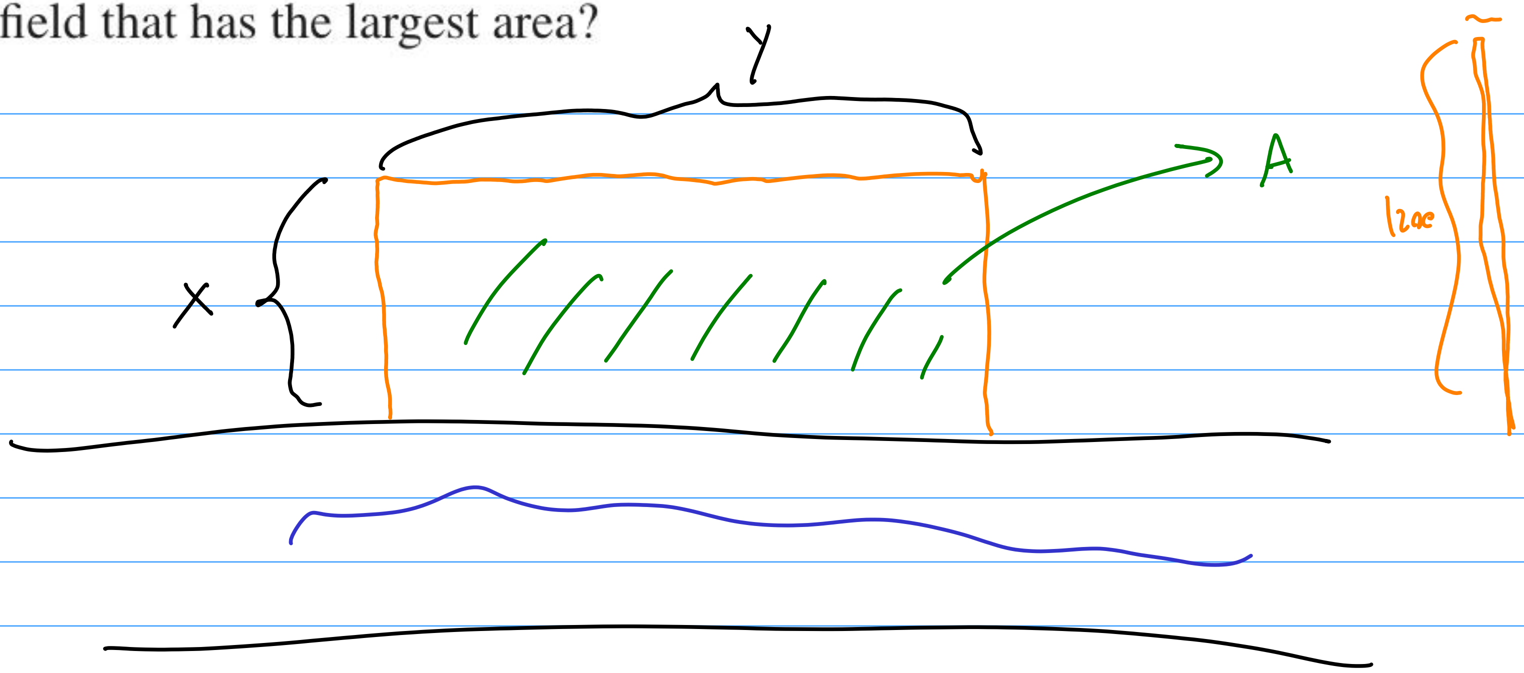
MAX.

MAX: $f(-5)$, $f(0)$, $f(1)$

MIN: $f(-5)$, $f(-4)$, $f(1/4)$, $f(1)$

EXAMPLE 1 A farmer has $\boxed{2400 \text{ ft}}$ of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

(FENCING)



I — $2x + y = 2400$ [FENCING IS LIMITED]

II — $A = xy$

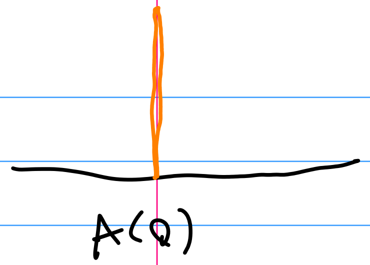
$$2x + y = 2400 \Rightarrow y = 2400 - 2x$$

$$A = xy = x(2400 - 2x)$$

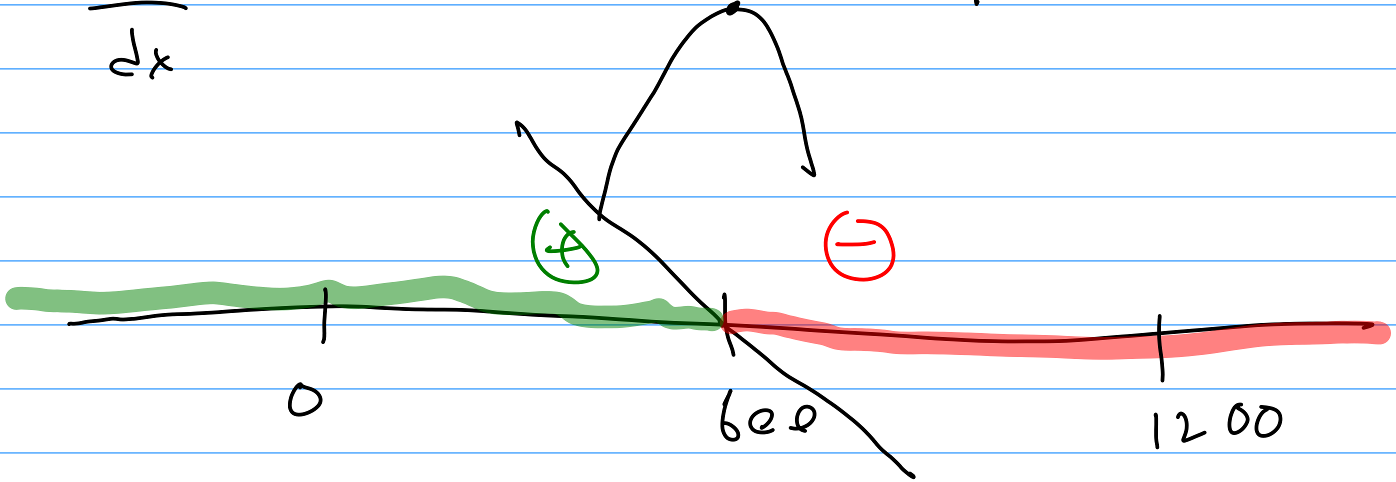
$$A(x) = x(2400 - 2x) \qquad 0 \le x \le 1200$$

$$\frac{dA}{dx} = \frac{d}{dx}\left[2400x - 2x^2\right] = 2400 - 4x$$

$$2400 - 4x = 0 \Rightarrow x = \frac{2400}{4} = 600$$

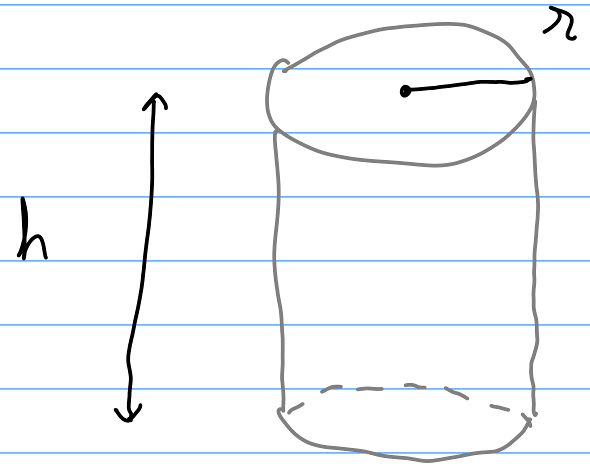


$$\frac{dA}{dx} = 2400 - 4x \longrightarrow \text{LOCAL MAXIMUM}$$



$A(600)$	$A(x) = x(2400 - 2x)$
$A(0) = 0$	$A(600) = (600)(2400 - 2 \times 600)$
$A(1200) = 0$	$= (600)(1200) = 720000 \text{ ft}^2$

EXAMPLE 2 A cylindrical can is to be made to hold $\boxed{1 \text{ L of oil.}}$ Find the dimensions that will minimize the cost of the metal to manufacture the can.



$V \rightarrow$ VOLUME

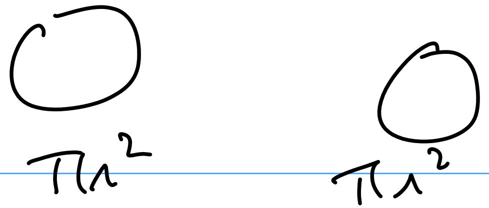
$A \rightarrow$ SURFACE AREA

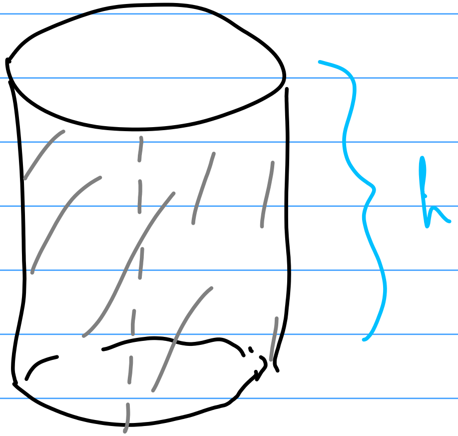
h, r, V, A

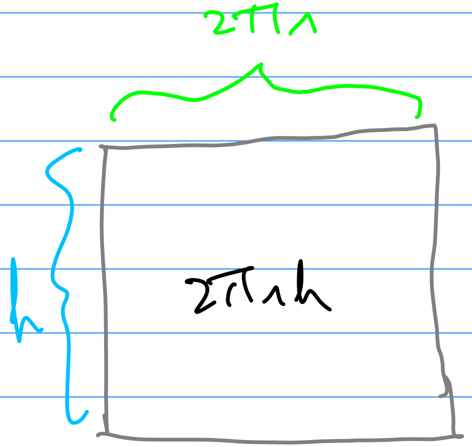
$V = \pi r^2 h \quad - \text{I}$

$A = 2\pi r h + 2\pi r^2 \quad - \text{II}$

$V = 1 \quad - \text{III}$


πr^2
πr^2


h


$2\pi r$
h
$2\pi r h$

A, r, h

$(\mathrm{I}, \mathrm{II})$

$$\pi r^2 h = 1$$

$r \rightarrow$ INDEPENDENT

$$h = \frac{1}{\pi r^2} \longrightarrow$$ PLUG INTO I

$$A = 2\pi r h + 2\pi r^2$$

$$= (2\pi r)\left(\frac{1}{\pi r^2}\right) + 2\pi r^2$$

$$\boxed{A(r) = \frac{2}{r} + 2\pi r^2 \quad \Big| \quad r \geq 0}$$

$$A(r) = \frac{2}{r} + 2\pi r^2$$

$$\frac{d}{dr} \cdot \frac{2}{r} = 2 \frac{d}{dr}\left(\frac{1}{r}\right)$$

$$\frac{d(r^{-1})}{dr} = -r^{-2}$$

$$A'(r) = -\frac{2}{r^2} + 4\pi r = 0$$

$$-\frac{2}{r^2} + 4\pi r = 0 \Rightarrow 4\pi r = \frac{2}{r^2} \Rightarrow 4\pi r^3 = 2$$

$$\Rightarrow r^3 = \frac{2}{4\pi}$$

$$A'(r) = \frac{4\pi r^3 - 2}{r^2}$$

$$\Rightarrow r = \sqrt[3]{\frac{2}{4\pi}}$$

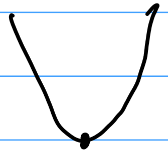
$$A'(r) = -2/r^2 + 4\pi r$$

$$\frac{d}{dr}\left[\frac{1}{r^2}\right] = -2r^{-3}$$

$$1/r^2 = r^{-2}$$

$$A''(r) = \frac{d}{dr}\left[-\frac{2}{r^2} + 4\pi r\right]$$

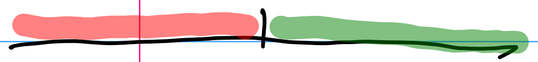
$$= (-2)(-2r^{-3}) + 4\pi$$



MIN

$$= \frac{4}{r^3} + 4\pi$$

$$r_0 = \sqrt[3]{\frac{2}{4\pi}}$$



r_0

$$= \frac{4}{\left[2/4\pi\right]} + 4\pi$$

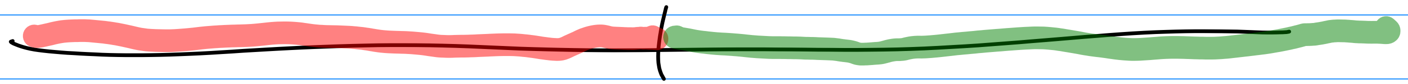
$$r^3 = \frac{2}{4\pi}$$

$$= 8\pi + 4\pi = 12\pi > 0$$

$A'(\lambda) = 0 \qquad A''(\lambda) > 0$

$\Rightarrow$ λ IS A POINT OF MINIMUM

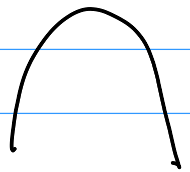
$A'(\lambda) < 0$



$\lambda_0 = \sqrt[3]{}$

N.B. FIRST DERIVATIVE TEST FOR ABSOLUTE / GLOBAL EXTREMA

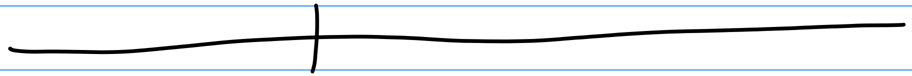
$f'(x) = 0$ $\qquad x_0$



FOR ALL $x < x_0 \Rightarrow f'(x) > 0$

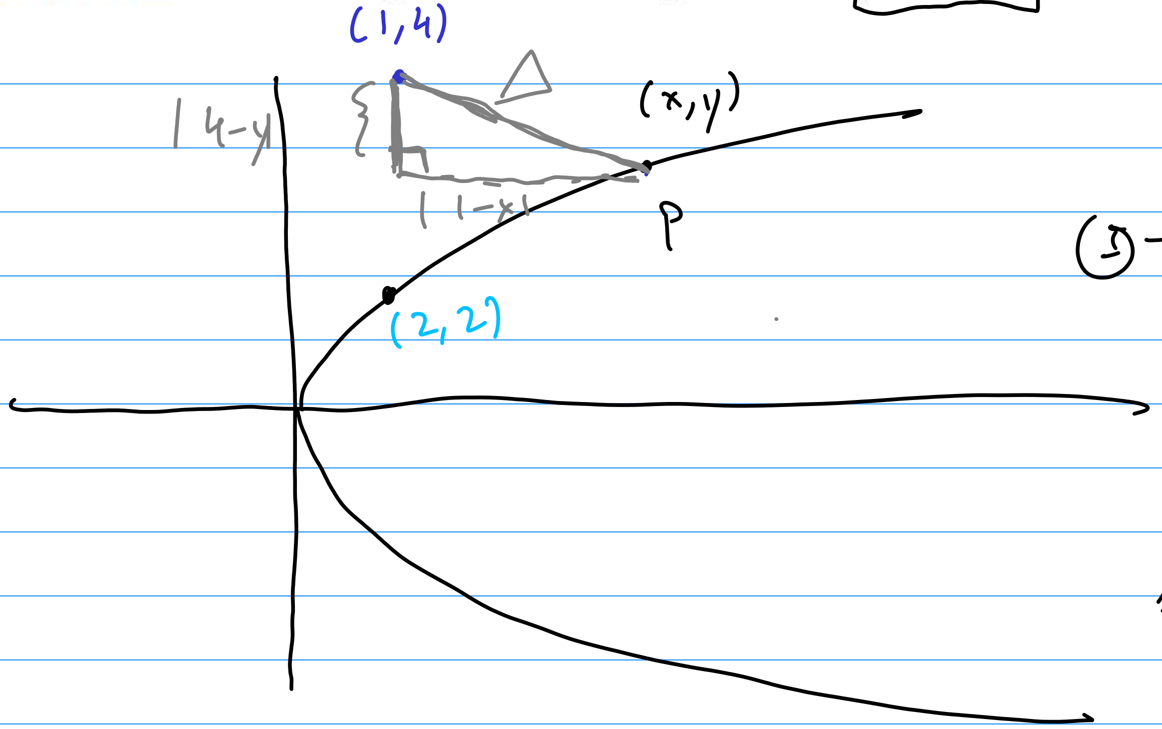
FOR ALL $x > x_0 \Rightarrow f'(x) < 0$

$\Big\}$ GLOBAL MAX



$0 < \wedge$

EXAMPLE 3 Find the point on the parabola $\boxed{y^2 = 2x}$ that is closest to the point $(1, 4)$.



$$\text{(I)} - \Delta^2 = (4-y)^2 + (1-x)^2$$

$$\text{(II)} - y^2 = 2x$$

$$\Delta^2 = (4-y)^2 + \left[1 - \frac{y^2}{2}\right]^2$$

$$\Rightarrow \Delta = \sqrt{(4-y)^2 + \left(1 - \frac{y^2}{2}\right)^2}$$

GOAL: MINIMIZE Δ

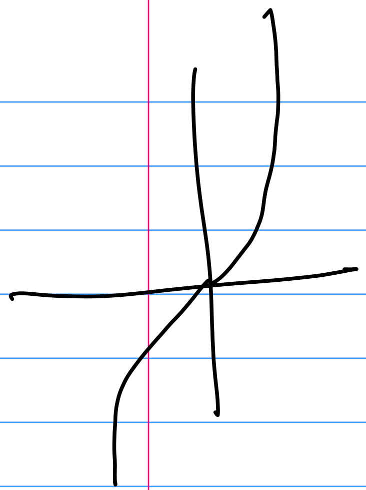
$$\Delta = \sqrt{(4-y)^2 + \left(1 - \frac{y^2}{2}\right)^2}$$

TRICK: Δ IS MINIMAL IF & ONLY IF Δ^2 IS MINIMAL

$$f(y) = [\Delta(y)]^2$$

$$= (4-y)^2 + \left[1 - \frac{y^2}{2}\right]^2$$

$$= 16 - 8y + y^2 + 1 - y^2 + y^4/4$$



$$16 - 8y + y^2 + 1 - y^2 + y^4/4$$

$$f(y) = \frac{y^4}{4} - 8y + 17$$

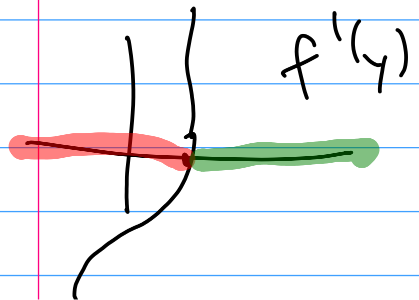
$$f'(y) = y^3 - 8$$

$$y^3 = 8$$

$$y = 2$$

$$f''(y) = 3y^2$$
$$= 12$$

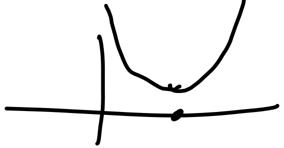
$\boxed{y = 2} \longrightarrow$ EXTREMUM



GLOBAL MINIMUM [BY 1st DER. TEST]

$$x = \frac{y^2}{2} = \frac{2^2}{2} = 2$$

$$P = (2, 2)$$



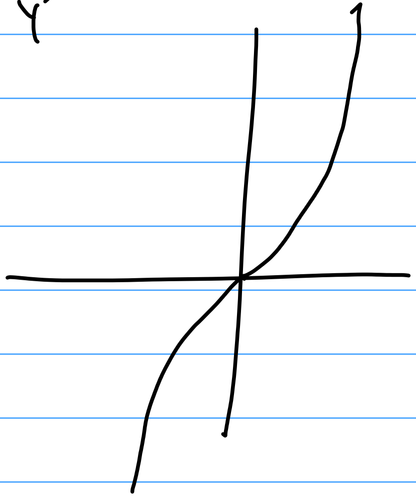
ABSOLUTE MINIMUM

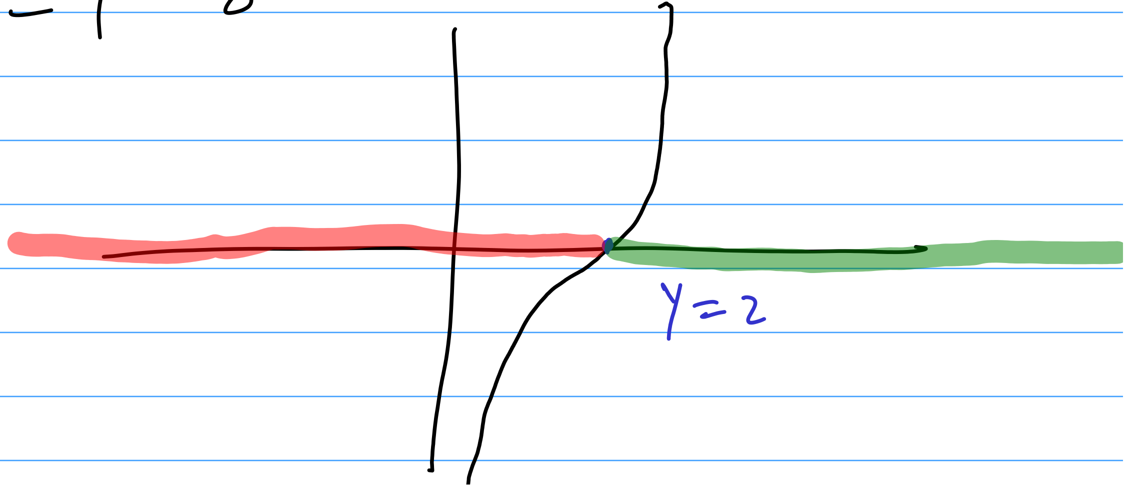
FOR ALL $y < 2$ $\quad f'(y) < 0$

FOR ALL $y > 2$ $\quad f'(y) > 0$

$$f'(y) = y^3 - 8$$

y^3



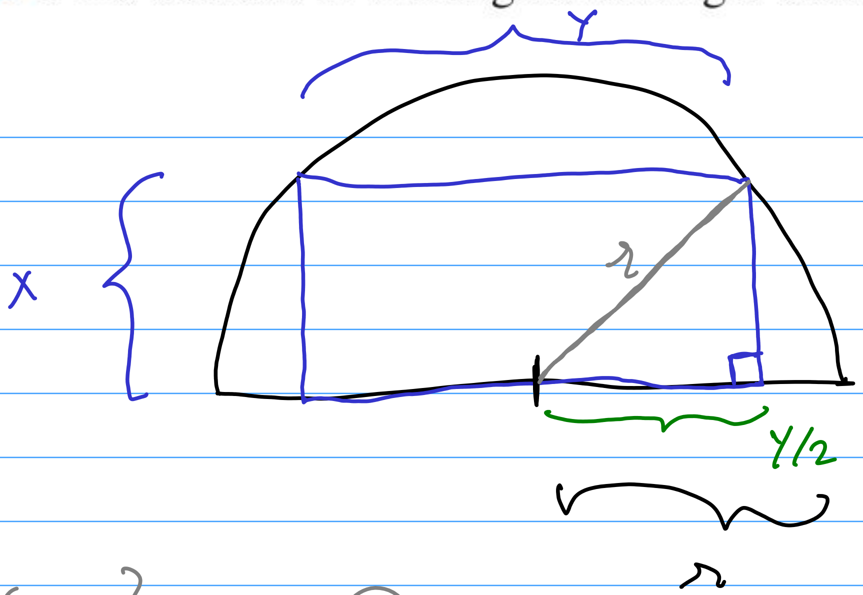


BLUE → VARIABLES

BLACK → CONSTANT

IMPLICIT DIFFERENTIATION

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



A → AREA

$A = xy$ — I

$r^2 = x^2 + \left(\frac{y}{2}\right)^2$ — II

$$A = xy \quad \text{— I}$$

$$x^2 + \left(\frac{y}{2}\right)^2 = r^2 \quad \text{— II}$$

$\left.\begin{matrix} \text{I} \\ \text{II} \end{matrix}\right\} \rightarrow$ diff WRT x

APPROACH 1 : ELIMINATE y & OPTIMIZE $A(x)$

$$\text{I} : \frac{dA}{dx} = \frac{d}{dx}[xy] = \left[\frac{d(x)}{dx}\right]y + x\left[\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} \rightarrow y'$$

$$\boxed{\frac{dA}{dx} = y + xy'}$$

II :

$$\frac{d}{dx}\left[x^2 + \left(\frac{y}{2}\right)^2\right] = \frac{d}{dx}\left[r^2\right] = 0$$

$$2x + \frac{d}{dx}\left[\frac{y^2}{4}\right]$$

$$\underbrace{\left(\frac{dy}{dx}\right)}_{y'} \cdot \underbrace{\left(\frac{d\,(y^2/4)}{dy}\right)}_{2y}$$

[CHAIN RULE]
LIEBNIZ

$$2x + \frac{y\,y'}{2} = 0 \longrightarrow \boxed{y' = -\frac{4x}{y}}$$

$$A = xy$$

$$x^2 + \left(\frac{y}{2}\right)^2 = r^2$$

$$\boxed{\frac{dA}{dx} = y + xy'}$$

$$\boxed{y' = -\frac{4x}{y}}$$

$$\frac{dA}{dx} = y + x\left[-\frac{4x}{y}\right] = \frac{y^2 - 4x^2}{y} = 0$$

$$\boxed{x^2 + \left(\frac{y}{2}\right)^2 = r^2}$$

$$\boxed{y^2 - 4x^2 = 0}$$

$$y^2 = 4x^2$$

$$x^2 + \left(\frac{y^2}{4}\right) = r^2$$

$$y^2 = 4x^2$$

$$x^2 + \left(\frac{y^2}{4}\right) = r^2$$

$$\Rightarrow x^2 + \frac{4x^2}{4} = r^2$$

$$\Rightarrow 2x^2 = r^2 \qquad x > 0$$

$$\Rightarrow x = r/\sqrt{2}$$

$$y^2 = 4x^2 = 4\left(\frac{r^2}{2}\right) = 2r^2$$

$$y = \sqrt{2}\, r$$

$$A = xy = \left(\frac{\lambda}{\sqrt{2}}\right) \cdot \left(\sqrt{2}\,\lambda\right)$$

$$= \lambda^2$$

$\therefore$ OPTIMAL AREA $= \lambda^2$