MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: T, F 4-5PM;

$\boxed{\text{BY APPOINTMENT}}$

LECTURES:

5:45 PM – 7:50 PM (ET)

M, T, W, R

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COURSE PAGE: bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK 11 → DUE ~~TODAY~~ TOMORROW AT 11 PM

   WEBWORK 12 → " WEDNESDAY " "

2. FINAL EXAM ON THURSDAY (IN CLASS)

3. PRACTICE EXAM WILL BE UPLOADED TONIGHT.

4. EXTRA OFFICE HOURS : TW 4 - 5 PM

   AFTER CLASS

   BY APPOINTMENT

COURSE REVIEW FORMS

DUE : 1st JULY, 2021

IMPROPER

EXAMPLE 3 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

$(-\infty, \infty)$

$a = 0$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx \qquad \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{0}^{t} \frac{1}{1+x^2} = \arctan x \Big]_{0}^{t} = \arctan t$$

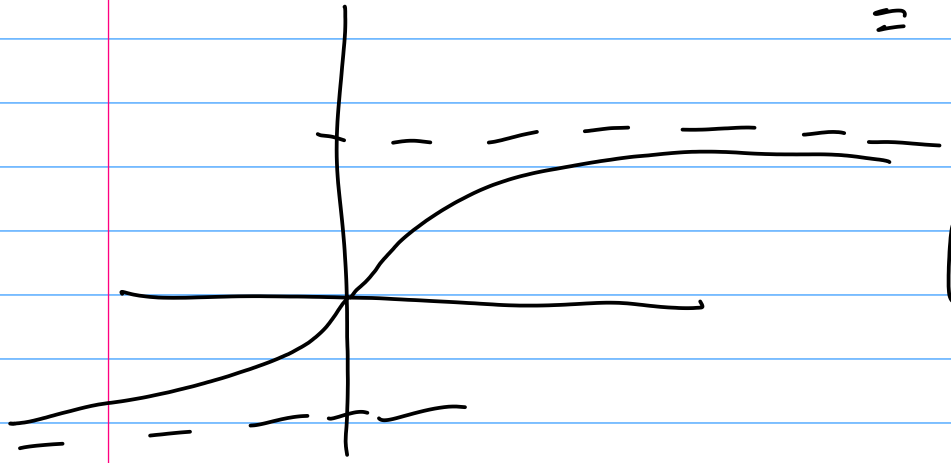
$$\int_{s}^{0} \frac{1}{1+x^2} = \arctan x \Big]_{s}^{0} = -\arctan s$$

1. FIND THE SYMMETRY
2. COMPUTE
3. TAKE LIMITS

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \text{Arctan } t$$

$$= \frac{\pi}{2}$$



$[-\pi/2, \pi/2]$

In $x \to \infty$

As $x \to \pi/2$

In $x \to -\infty$

As $x \to -\pi/2$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{s \to -\infty} \int_{s}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{s \to -\infty} \left(- \operatorname{Arctan} s \right)$$

$$= -\left(-\pi/2\right) = \pi/2$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{0}^{\infty} + \int_{-\infty}^{0} = \pi/2 + \pi/2 = \pi$$

IMPORTANT THEORETICAL EXAMPLE

EXAMPLE 4 For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

$p = 1$, DIVERGES (THURSDAY)

$$\int_1^t \frac{1}{x^p}\,dx$$

$$\int \frac{dx}{x^p} = \begin{cases} \dfrac{x^{-p+1}}{-p+1} + C & p \neq 1 \\ \ln x + C & p = 1 \end{cases}$$

$$= \left.\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^p} = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

$p = 2 \qquad t^{1-p} \rightsquigarrow t^{-1}$

$$\lim_{t\to\infty} t^{-1} = 0$$

[CONVERGES]

$t^{r} \to 0$

IF $r < 0$

$t^{r} \to \infty$

IF $r > 0$

$p = 1/2$

$t^{1-1/2} = t^{1/2}$

$$\lim_{t\to\infty} t^{1/2} = \infty$$

[DIVERGES]

$\lambda = 1 - p > 0 \longrightarrow$ DIVERGENT

$\lambda = 1 - p < 0 \longrightarrow$ CONVERGENT

$p > 1 \Rightarrow$ CONVERGENT

$p \leq 1 \Rightarrow$ DIVERGENT

p-TEST FOR INTEGRALS

TYPE II: DISCONTINUOUS INTEGRAND

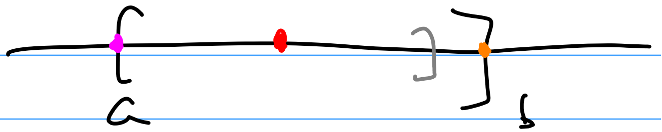
$f \rightarrow$ HAS A DISCONTINUITY IN $[a,b]$

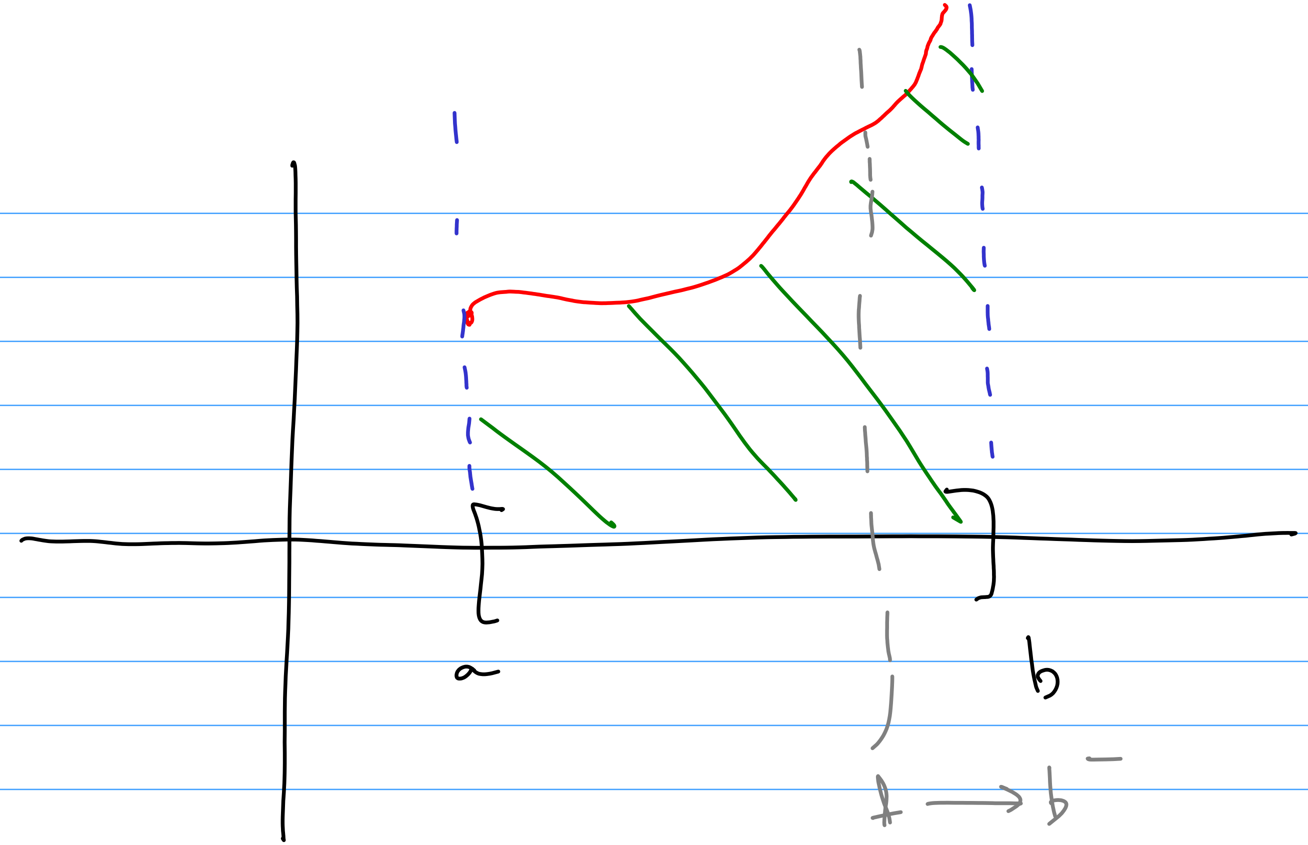
FINITE INTERVAL

(1) $f \rightarrow$ NOT CONT. AT $x=b$

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

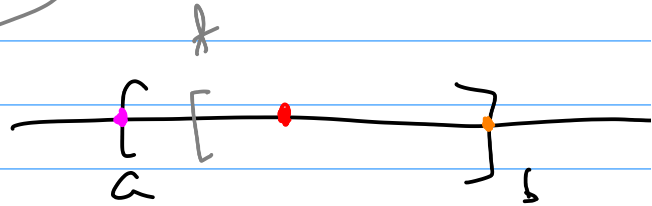
$t < b$




a
b
t → b⁻

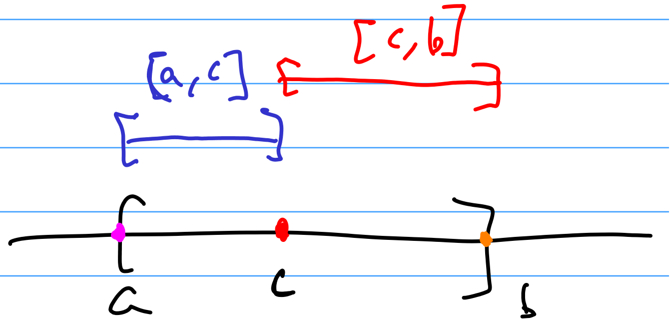
(2) $f \rightarrow$ NOT CONT. AT $x = a$

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$



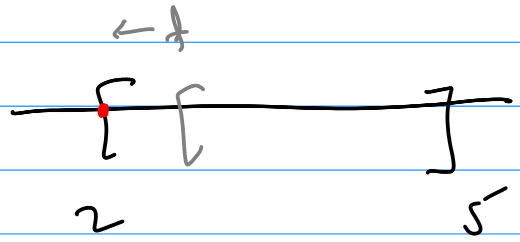
(3) $f \rightarrow$ NOT CONT. AT $c \in (a,b)$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



EXAMPLE 5 Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$



$t > 2$

$$\int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \int_{t-2}^3 \frac{du}{\sqrt{u}}$$

$u = x - 2 \Rightarrow du = dx$

$x = t$	$u = t - 2$
$x = 5$	$u = 3$

$$\int_{t-2}^{3} \frac{du}{\sqrt{u}} = \int_{t-2}^{3} u^{-1/2}\, du = \frac{u^{-1/2+1}}{-1/2+1}$$

$$= 2\sqrt{u}\,\Big]_{t-2}^{3}$$

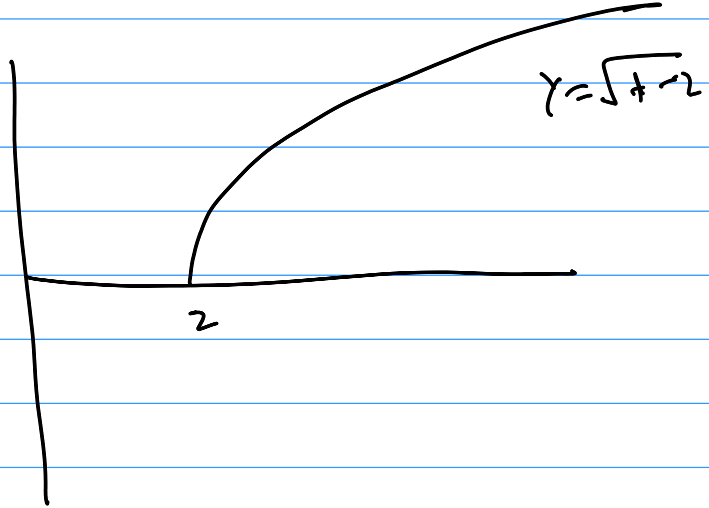
$$\int_{t}^{5} \frac{dx}{\sqrt{x-2}} = 2\sqrt{3} - 2\sqrt{t-2}$$

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \left[2\sqrt{3} - 2\sqrt{t-2} \right]$$

$$= \underbrace{2\sqrt{3}}_{}$$

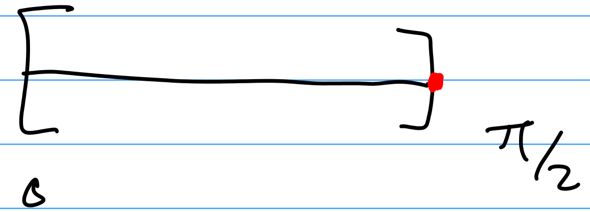
EXISTS, FINITE

$\Rightarrow$ CONVERGENT



EXAMPLE 6 Determine whether $\int_0^{\pi/2} \sec x \, dx$ ~~converges~~ or $\boxed{\text{diverges.}}$

$$\int_0^{\pi/2} = \lim_{t \to \pi/2^-} \int_0^t \sec x \, dx$$



$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\sec x = \frac{1}{\cos x}$$

$$\int_0^t = \ln|\sec t + \tan t| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sec t + \tan t|$$

$\ln|\sec 0 + \tan 0|$

$= \ln 1 = 0$

$\sec 0 = 1/\cos 0 = 1$

$\tan 0 = 0$

$$\lim_{t \to \pi/2^-} \int_0^t \sec x \, dx = \lim_{t \to \pi/2^-} \ln \left| \sec t + \tan t \right| = +\infty$$

$$\sec = \frac{1}{\cos}$$

$$t \to \pi/2^- \qquad \sec t \xrightarrow{?} +\infty \qquad (\cos > 0,\ \cos \to 0)$$

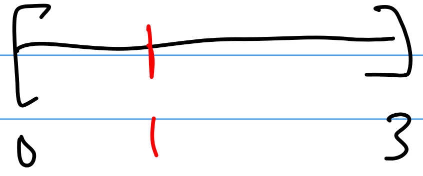
$$\tan t \xrightarrow{?} +\infty \qquad \left(\begin{array}{l} \sin,\ \cos > 0 \\ \cos \to 0 \\ \sin \to 1 \end{array} \right)$$

$$\sec t + \tan t \xrightarrow{?} +\infty$$

$$\ln \left| \sec t + \tan t \right| \xrightarrow{?} +\infty$$

SKIP TILL LATER

BREAKOUT ROOM



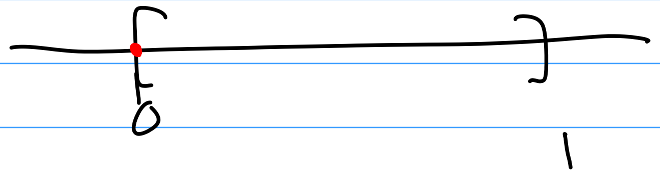
EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

$$\int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

EXAMPLE 8 Evaluate $\int_0^1 \ln x \, dx$.

$[0,1]$

WHERE IS THE DISCONUITY·



$$\boxed{\lim_{t\to 0^+} \int_t^1 \ln x \, dx}$$

$$\int_1^\infty \frac{\sin^4 x}{x^7}\,dx$$

CONVERGENT ?

DIVERGENT ?

NO! , $\frac{\sin^4 x}{x^7} \to$ HOW TO INTEGRATE?

Comparison Theorem Suppose that f and g are continuous functions with $\boxed{f(x) \geqslant g(x) \geqslant 0}$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

$$0 \le \int_a^t g(x)\,dx \le \int_a^t f(x)\,dx$$

[∵ COMPARISON OF INT.]

$$\int_a^t g(x)\,dx \leq \int_a^t f(x)\,dx$$

$=$ FINITE

$< +\infty$ (FINITE)

INFINITE $\Rightarrow$ INFINITE

$x \geq 1$

$$0 < \frac{\ln^{-4} x}{x^7} \leq \frac{1}{x^7}$$

$$\left[\int_1^\infty \frac{dx}{x^p} \quad \begin{array}{l} p > 1 \Rightarrow \text{CONVERGES} \\ p \leq 1 \Rightarrow \text{DIVERGES} \end{array} \right]$$

$$\int_1^\infty \frac{dx}{x^7} \quad (p = 7 \text{ IN THE } p\text{-TEST})$$

CONVERGENT $(p = 7 > 1)$

$\Rightarrow$ COMPARISON TELLS YOU THAT $\int_1^\infty \frac{\ln^{-4} x}{x^7}\, dx$ IS ALSO CONVERGENT

EXAMPLE 9 Show that $\int_0^\infty e^{-x^2}dx$ is convergent.

$$e^{-x^2} \rightsquigarrow e^{-x}$$

$$e^{-x^2} \qquad e^{-x} \qquad [0, \infty)$$

$$e^{-x^2} \geq e^{-x} \Leftarrow x^2 \leq x \qquad x \in [0,1]$$

$$\boxed{e^{-x^2} \leq e^{-x} \Leftarrow x^2 \geq x \qquad x \in [1, \infty)}$$

$$\int_0^\infty = \int_0^1 + \int_1^\infty$$

$$\int_0^1 e^{-x^2}\,dx \longrightarrow \text{WELL-DEFINED}$$ (NOT IMPROPER)

$e^{-x^2} \to$ CONT.

$[0,1] \to$ FINITE

$$\int_1^\infty e^{-x^2}\,dx$$ CONV/DIV ?

$$\int_1^\infty \qquad\qquad x \geq 1$$

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx$$

$$= \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t \to \infty} \int_1^t e^{-x}\, dx = \lim_{t \to \infty} \left[-e^{-x} \right]_1^t$$

$$= \lim_{t \to \infty} \left[-e^{-t} - \left(-e^{-1} \right) \right]$$

$$= \frac{1}{e} - \lim_{t \to \infty} e^{-t}$$

$$= \frac{1}{e} < \infty \qquad \boxed{\text{FINITE}}$$

$$e^{-t} = \frac{1}{e^{(\text{LARGE})}} = \frac{1}{\text{LARGE}} \to 0$$

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx < \infty$$

$$\Rightarrow \int_1^\infty e^{-x^2}\,dx \longrightarrow \text{CONVERGES}$$

$$\int_0^\infty e^{-x^2}\,dx \rightarrow \text{CONVERGES.}$$

EXAMPLE 10 The integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\int_1^\infty \left(\frac{1+e^{-x}}{x}\right) dx$$

$$\frac{1+e^{-x}}{x} \geq g(x)$$

$e^{-x} > 0 \Rightarrow \frac{1+e^{-x}}{x} \geq \frac{1}{x}$ $\qquad \int g(x) \to$ EASY

$$\boxed{\int_1^\infty \frac{1+e^{-x}}{x}\,dx} \geq \underbrace{\int_1^\infty \frac{dx}{x}}$$

DIVERGES

$p = 1$

$$\left[\int_1^\infty \frac{dx}{x^p}\right.$$

$p > 1 \Rightarrow$ CONVERGES

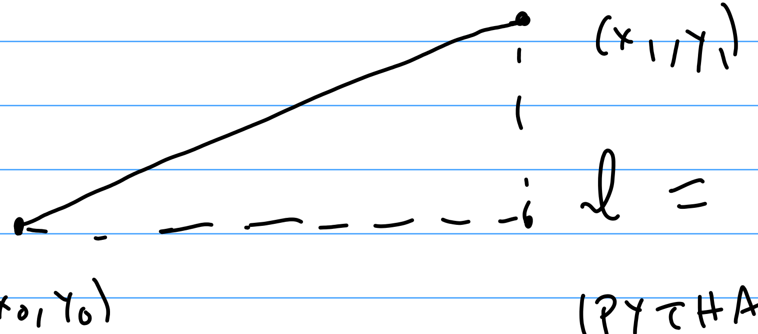
$p \leq 1 \Rightarrow$ DIVERGES]

DIVERGES!

$\boxed{\text{BREAK TILL 7:00 PM ET}}$

§ 8.1 ARC LENGTH

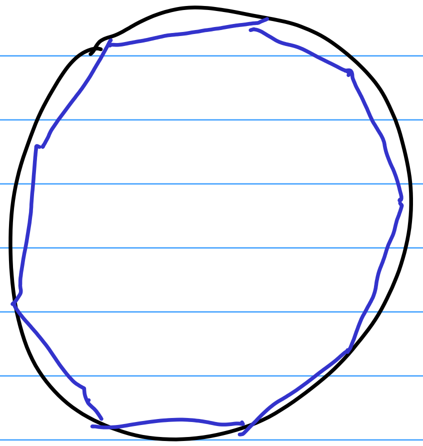
WHAT IS LENGTH?



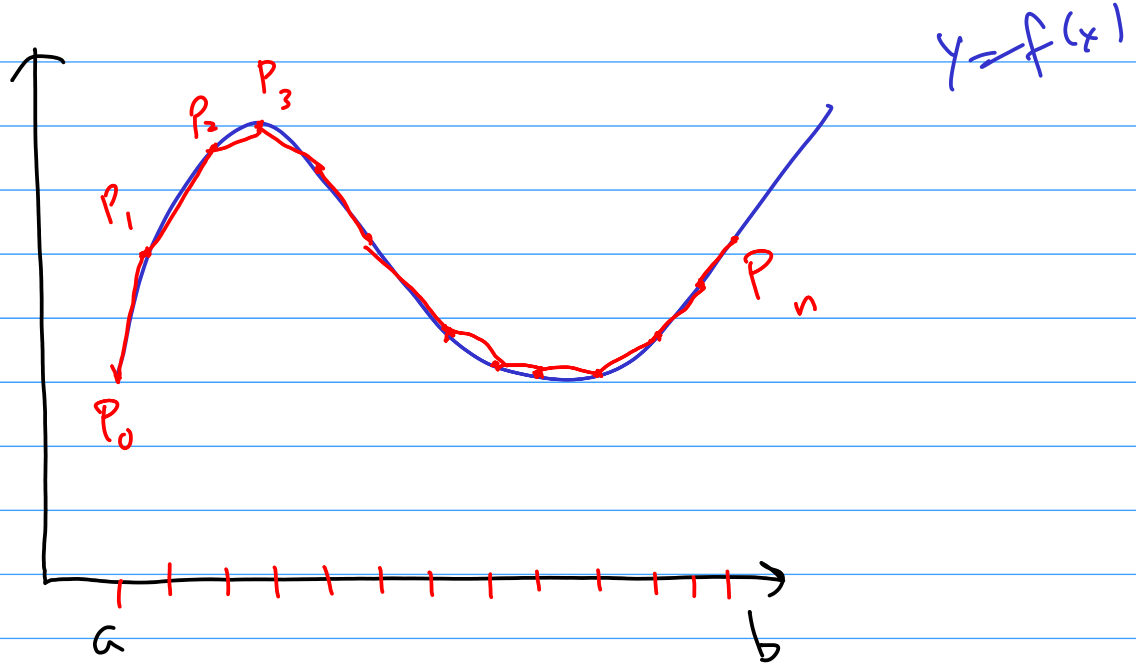
$$\ell = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}$$

(PYTHAGORAS)

PRIMITIVE



CIRC $= 2\pi r$



$$L := \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

(IF IT EXISTS)

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

<u>ASSUMPTION</u>

① f IS DIFFERENTIABLE

② f' IS CONTINUOUS

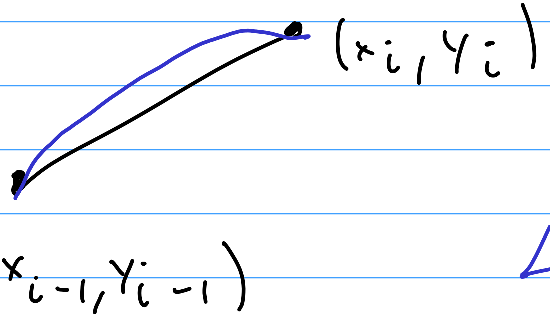
$$P_i = (x_i, y_i)$$

$$|P_{i-1} P_i| = \sqrt{(y_i - y_{i-1})^2 + (x_i - x_{i-1})^2}$$

$$\Delta y_i = y_i - y_{i-1}$$

$$= \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2}$$

$$\Delta x_i = x_i - x_{i-1}$$



$$y_i = f(x_i)$$

$$\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$$

M.V.T. FOR DERIVATIVES

$$\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*)$$

$$\left(\exists\, x_i^* \in [x_{i-1}, x_i]\right)$$

$$\Rightarrow \quad \Delta y_i = f'(x_i^*)\, \Delta x_i$$

$$|P_{i-1} P_i| = \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2} = \sqrt{\left(f'(x_i^*)\, \Delta x_i\right)^2 + (\Delta x_i)^2}$$

$$|P_{i-1}P_i| = \left(\sqrt{1 + f'(x_i^*)^2}\right) \Delta x_i$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \left(\sqrt{1 + f'(x_i^*)^2}\right) \Delta x_i}_{\text{RIEMANN SUM}}$$

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\, dx} \quad \left[\because f' \text{ IS CONT.}\right]$$

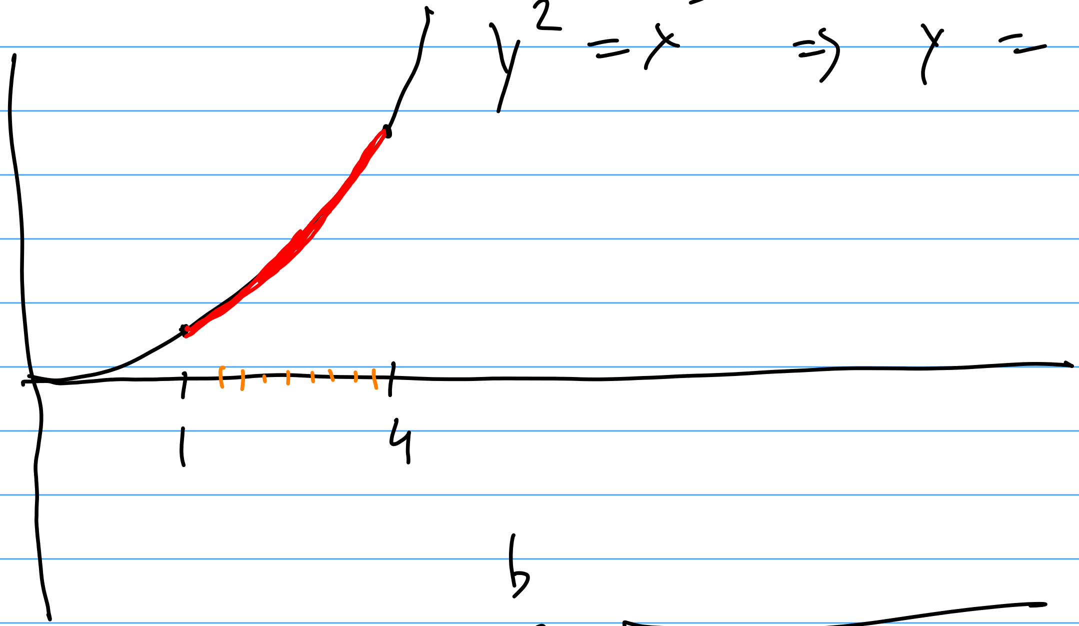
EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)

$$y^2 = x^3 \Rightarrow y = x^{3/2}$$

$a = 1$

$b = 4$

$f(x) = x^{3/2}$



$$\Rightarrow f'(x) = \frac{3}{2} x^{1/2}$$

$$f'(x)^2 = \frac{9}{4} x$$

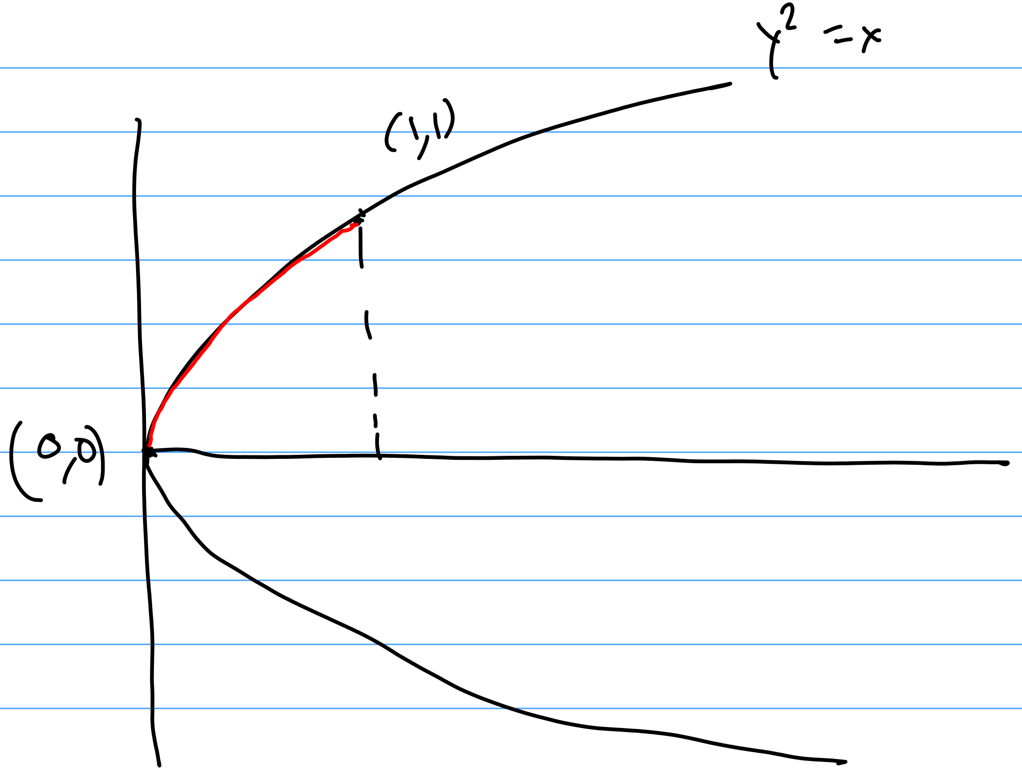
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$L = \int_{1}^{4} \sqrt{1 + \frac{9}{4} x} \, dx$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

$$\int_a^b \sqrt{1+f'^2}\, dx$$

$$\int_c^d \sqrt{1+g'(y)^2}\, dy$$



$a = 0$

$b = 1$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$c = 0$

$d = 1$

$g(y) = y^2$

$g'(y) = 2y$

$$\int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx$$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$$y = f(x) \qquad x = a \longrightarrow x = b$$

$$\Updownarrow$$

$$x = g(y) \qquad y = c \rightarrow y = d$$

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy$$

$$L = \underbrace{\int_0^1 \sqrt{1 + (4y^2)} \, dy}_{\text{TRIGNOMETRIC SUBSTITUTION}}$$

$$y = \frac{1}{2} \tan \theta$$

BREAKOUT ROOM

EXAMPLE 3

(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

$$\int_1^2 \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\,dx$$

1 AGAINST dx

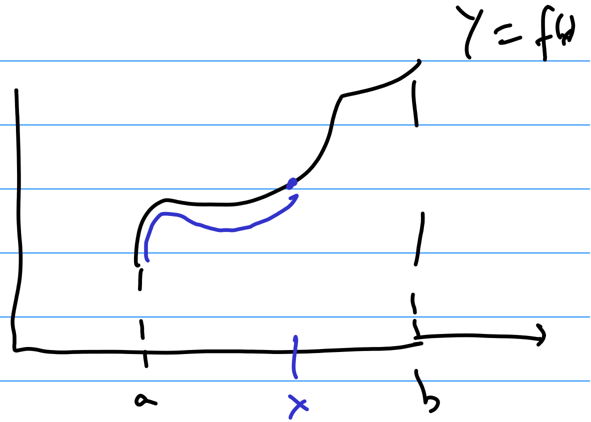
2 AGAINST dy $\rightarrow$ $\displaystyle\int_{1/2}^{1} \sqrt{1 + \left(\frac{-1}{y^2}\right)^2}\,dy$

ARC LENGTH FUNCTION

$y = f(x) \quad ; \quad a \leq x \leq b$

$$s(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt$$

ARC LENGTH FUNCTION



$$s(x) = \int_a^x \sqrt{1+f'(t)^2}\, dt$$

NOTE : $\dfrac{ds}{dx} = \sqrt{1+f'(x)^2}$ ($\because$ FUNDAMENTAL THEOREM)

$$\Rightarrow \quad ds = \left(\sqrt{1+f'(x)^2}\right) dx$$

$$\Rightarrow \quad \int_{s(a)}^{s(b)} dS = \int_a^b \sqrt{1+f'(x)^2}\, dx = L$$

$$\boxed{L = \int ds}$$

(NET CHANGE THEOREM)

EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.

$$f(x) = x^2 - \frac{1}{8} \ln x \qquad a = 1$$

$$f'(x) = 2x - \frac{1}{8} \cdot \frac{1}{x}$$

$$f'(x)^2 = \left(2x - \frac{1}{8x}\right)^2 = 4x^2 + \frac{1}{64x^2} - (2)(2x)\left(\frac{1}{8x}\right)$$

$$f'(x)^2 = 4x^2 + \frac{1}{64x^2} - \frac{1}{2}$$

$$1 + f'(x)^2 = 1 + \left(4x^2 + \frac{1}{64x^2} - \frac{1}{2}\right)$$

$$= 4x^2 + \frac{1}{64x^2} + \frac{1}{2}$$

$$= 4x^2 + \frac{1}{64x^2} + 2(2x)\left(\frac{1}{8x}\right) = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + f'(x)^2} = \sqrt{\left(2x + \frac{1}{8x}\right)^2}$$

$$= 2x + \frac{1}{8x} \qquad x \geq 1$$

$$s(x) \int_a^x \sqrt{1 + f'(t)^2}\, dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt$$

$$s(x) = x^2 + \frac{\ln x}{8} - 1$$