Math 142: Calculus II

Sample Final Exam

June 30th, 2021

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- The exam will be 75 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- Part A is worth 50 points. Part B is worth 130 points. Every question is labelled with one of these parts.
- There are 11 pages.
- A formula sheet is provided.
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required.
- In several questions, you do NOT need to evaluate the integral. Please read the questions carefully to ensure you do not waste time trying to compute an integral you do not need to.

QUESTION	VALUE	SCORE
1	0	
2	10	
3	30	
4	20	
5	15	
6	45	
7	30	
8	30	
TOTAL	180	

(50) PART A

PART B (130)

Formulas

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

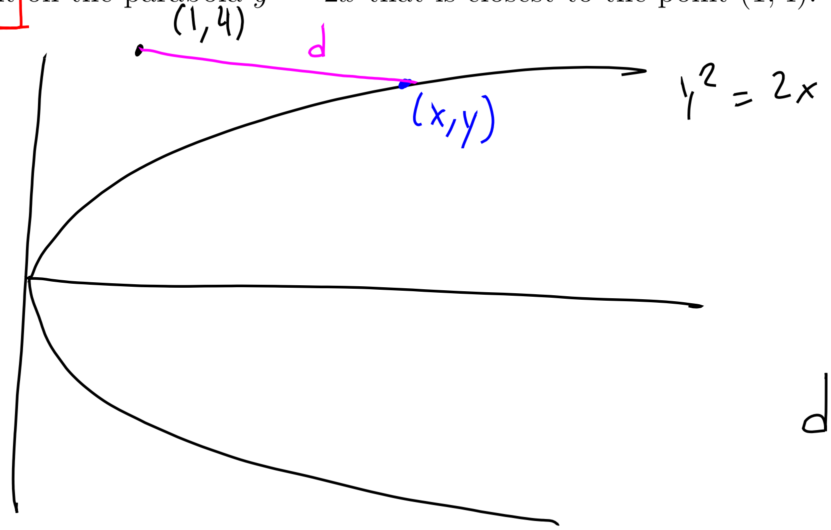
Part A

MCH

$(x,y) \leftrightarrow \boxed{(2,2)}$

2. (10 points)

$\boxed{\text{Find the point}}$ on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.



x, y, d

$y^2 = 2x$

$\Rightarrow x = \frac{y^2}{2}$

$$d = \sqrt{(1-x)^2 + (4-y)^2}$$

$y^2 = 2x$, $\Rightarrow x = y^2/2 = 2$

$$d = \sqrt{(1 - y^2/2)^2 + (4-y)^2}$$

$$d(y) \rightarrow d'(y) = 0 \rightsquigarrow y$$

GOAL: Min $d(y) \Longleftrightarrow$ Min $d(y)^2$

$$f(y) = d(y)^2 = \left(1 - \frac{y^2}{2}\right)^2 + (4-y)^2$$

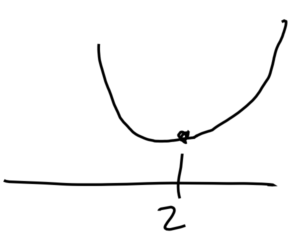
$$f(y) = 17 - 8y + \frac{y^4}{4}$$

$$f'(y) = -8 + y^3 = 0$$

$$\Rightarrow y^3 = 8 \Rightarrow y = 2$$

MAX

MIN



$f'(y) < 0$ $y < 2$

$f'(y) > 0$ $y > 2$

$\Rightarrow$ $y = 2$ IS A GLOBAL MIN

Part A

3. (30 points) Consider the region bounded in the first quadrant by $y = x^3$ and $y = 4x$.

(a) Sketch the curves and shade the region described above.

(b) Write (but do NOT evaluate) an integral that is equal to the area of the region.

(c) Write (but do NOT evaluate) an integral using the shell method for the volume of the solid obtained by revolving the region about the y-axis.

(d) Write (but do NOT evaluate) an integral using the washer method for the volume of the solid obtained by revolving the region about the line $x = 4$.

Part A

4. (20 points) SPHERICAL, CONICAL

A cuboidal tank of height $h = 4\,\mathrm{m}$, width $w = 2\,\mathrm{m}$ and length $l = 5\,\mathrm{m}$ is full of water. The water is pumped out of a hole at the top of the tank over time.

(a) Write (but do NOT evaluate) an integral that represents the work done to empty the tank.

(b) Write (but do NOT evaluate) an integral that represents the work done to empty half the tank.

Recall that the density of water is $1000\,\mathrm{kg\,m^{-3}}$ and that the acceleration due to gravity is $9.8\,\mathrm{m\,s^{-2}}$.

Part B

5. (15 points)

Recall from a previous question the region bounded in the first quadrant by $y = x^3$ and $y = 4x$.

$y = x^3$ $\qquad$ $y = 4x \iff x = y/4$

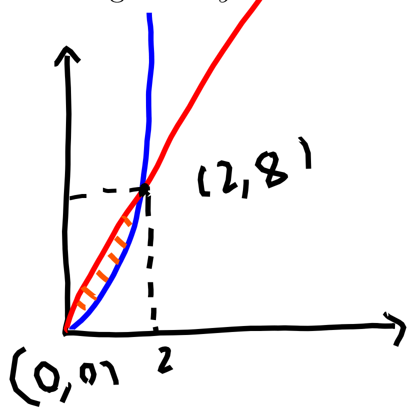
Express the perimeter of the region as the sum of two integrals, one of which is against dx and the other of which is against dy. Do NOT evaluate the integrals.

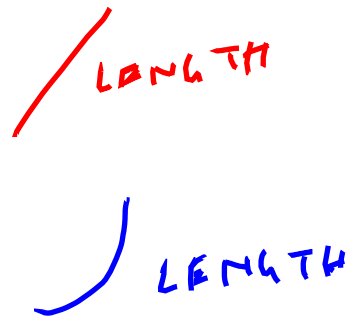
$$\int_c^d \sqrt{1 + g'(y)^2}\, dy$$

c, d, g ?

$c = 0,\ d = 8$

$g(y) = y/4$





BLUE $\to dx$

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

a, b, f ?

$a = 0,\ b = 2$ $\qquad$ $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$$\text{BLUE} = \int_0^2 \sqrt{1 + (3x^2)^2}\, dx$$

$$+$$

$$\text{RED} = \int_0^8 \sqrt{1 + (1/4)^2}\, dy$$

Part B

6. (45 points)

Compute the following indefinite integrals:

(a)

$$\int \frac{\sin 2\theta}{(1-\sin\theta)^2(1+\sin^2\theta)}\,d\theta$$

(b)

$$\int \sin(\ln x)\,dx$$

(c)

$$\int \frac{e^{2u}}{\sqrt{3-2e^u-e^{2u}}}\,du$$

SEE XOURNAL NOTES

Part B

7. (30 points)

Compute the following definite integrals:

(a)

$$\int_1^2 \frac{dv}{v^2(v^2+1)}$$

(b)

$$\int_0^{2\pi} (\sin x)^4 (\cos x)^2\, dx$$

(c)

$$\boxed{\int_0^1 \arctan x\, dx}$$

REFER TO MIDTERM II SOLNS.

$$\frac{1}{v^2(v^2+1)} = \frac{a}{v} + \frac{b}{v^2} + \frac{cv+d}{v^2+1}$$

$$1 = av(v^2+1) + b(v^2+1) + (cv+d)v^2$$

$(v=0 \Rightarrow b=1)$

↑ CONSTANT)

$v \rightarrow 0 = a$

$v^2 \rightarrow 0 = b + d$

$\Rightarrow d = -b = -1$

$v^3 \rightarrow 0 = a + c$

$\Rightarrow c = 0$

$$\int_1^2 \frac{dv}{v^2(v^2+1)} = \int_1^2 \frac{dv}{v^2} - \int_1^2 \frac{dv}{v^2+1}$$

$$= \left.\frac{-1}{v}\right]_1^2 - \arctan v\Big]_1^2$$

$$= \frac{1}{2} - \arctan 2 + \arctan 1$$

Part B

YES!

8. (30 points)

(a) Is the following integral $\boxed{\text{improper?}}$ In either case, $\boxed{\text{compute the value}}$ of the integral. IT DOESN'T EXIST!

INFINITE DISC. AT $x = 1/2 \in [0,2]$!

$$\int_0^2 \frac{dx}{(2x-1)^2} = \int_0^{1/2} + \int_{1/2}^{2}$$

(b) Determine if the following integral $\boxed{\text{converges}}$ or diverges.

$[1, \infty)$

$$\int_1^\infty \left(\frac{\sin x}{x^2}\right)^2 dx.$$

(c) Determine if the following integral converges or $\boxed{\text{diverges.}}$

$$\int_1^\infty \frac{e^x}{x}\, dx.$$

IMPORTANT!

COMPARISON TEST

$$\boxed{0 \le} \left(\frac{\sin x}{x^2}\right)^2 \le \frac{1}{x^4}$$

$$0 \le \sin^2 x \le 1$$

$$\int_1^\infty \left(\frac{\sin x}{x^2}\right)^2 dx \le \int_1^\infty \frac{dx}{x^4} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^4}$$

$$= \lim_{t\to\infty} \left.\frac{-1}{3x^3}\right]_1^t = \lim_{t\to\infty}\left(\frac{1}{3} - \cancel{\frac{1}{3t^3}}\right)$$

$$= 1/3 < \infty$$

$$\int_0^{1/2} = \lim_{t\to 1/2^-} \int_0^t \frac{dx}{(2x-1)^2}$$

$$= \left.\frac{1}{2}\left(\frac{-1}{2x-1}\right)\right]_0^t$$

$$= \frac{-1}{2(2t-1)} - \frac{1}{2}$$

$$\lim_{t\to 1/2^-}\left[\frac{-1}{2(2t-1)} - \frac{1}{2}\right]$$

$= +\infty$ (DOESN'T EXIST)