MATH 142 (SUMMER '21, SESH A2)

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OFF HRS: M, F 4-5PM ;
~~W 9-10 AM ;~~
BY APPOINTMENT

(N.B. - STARTING FRIDAY)

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COURSE PAGE : bit.ly/sahay142

LECTURES:
5:45 PM - 7:50 PM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

§ 4.9 ANTIDERIVATIVES

(AKA THE INDEFINITE INTEGRAL)

Q1. SUPPOSE $\boxed{f(t)}$ IS THE SPEED AT WHICH A CAR IS MOVING ON A STRAIGHT ROAD AT TIME "t".

WHERE IS THE CAR AT TIME "t"?

Q2. SUPPOSE THE POPULATION OF THE WORLD GROWS AT THE RATE $f(t)$ AT TIME "t".

WHAT IS THE POPULATION AT TIME "t"?

ANTIDIFFERENTIATON!

DEFN: FOR A FUNCTION $\boxed{f(x)}$, WE SAY $\boxed{F(x)}$ IS AN ANTIDERIVATIVE OF $f(x)$ ON THE $\underbrace{\text{INTERVAL } I}$

IF

$$\boxed{F'(x) = f(x)} \quad \text{FOR ALL } x \in I$$

e.g. $f(x) = x^2$ $\qquad F(x) = \dfrac{x^3}{3}$

"POWER RULE" $\quad x^{n-1} = \dfrac{1}{n}\dfrac{d}{dx}\left(x^n\right) \rightarrow \dfrac{x^n}{n}$

$$F'(x) = \frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{3x^2}{3} = x^2 = f(x)$$

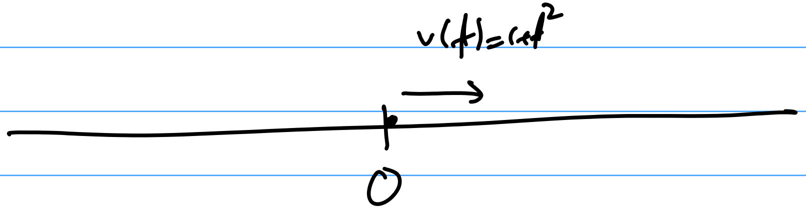
v = VELOCITY

e.g. 1

$$\boxed{v(t) = 1 + t^2}$$

$$\boxed{s(0) = 0}$$

$$v(t) = s'(t)$$



$$\boxed{s(t) = t + \frac{t^3}{3}}$$

$$s(0) = 0 + \frac{0^3}{3} = 0$$

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}\left(t + \frac{t^3}{3}\right)$$

$$= 1 + \frac{3t^2}{3} = 1 + t^2$$

eg 2. $f(x) = e^x$

$F(x) = e^x$ $\qquad F'(x) = f(x)$

$\frac{d}{dx}(e^x) = e^x \Rightarrow$ e^x IS ITS OWN ANTIDERIVATIVE

$F_2(x) = e^x + 10^5$ $\qquad F_2$ IS ALSO AN ANTIDERIVATIVE

$F_2'(x) = e^x = f$

$\boxed{\text{!! ANTIDERIVATIVES ARE NOT UNIQUE !!}}$

N.B.: FOR REASONS THAT WILL BE CLEAR LATER IN THE COURSE, ANTIDERIVATIVES ARE ALSO CALLED "INDEFINITE INTEGRALS"

NOTE: IF $F \longrightarrow$ ANTIDERIVATIVE OF f

& $G(x) = F(x) + \underbrace{C}_{\text{CONSTANT INDEPENDENT OF } x}$

$$G'(x) = F'(x) + \cancel{\frac{d}{dx}(C)} = f$$

Thm: IF F IS AN ANTIDERIVATIVE OF f ON AN INTERVAL I

THEN EVERY ANTIDERIVATIVE OF f IS OF THE FORM

$$F(x) + C$$

FOR SOME CONSTANT C

$\underline{\text{Pf}}$ $\quad f, \quad F, \quad F_2$ CONNECTED INTERVAL

$$\boxed{F'(x) = F_2'(x) = f(x)} \longrightarrow \text{FOR } x \in I$$

$$g(x) = F_2(x) - F(x)$$

$$g'(x) = F_2'(x) - F'(x) = f(x) - f(x) = 0$$

$$\Rightarrow g(x) = C$$

[MEAN-VALUE THEOREM]

$I \ni a, b$

M.V.T. $[a,b] \subseteq I$

$$\frac{g(b)-g(a)}{b-a} = g'(c) = 0 \qquad c \in (a,b)$$

$$g(b) = g(a)$$

$$\Rightarrow \quad g(x) = g(a) \qquad \forall x \in I$$

$$= C \qquad (a \in I)$$

$$F_2(x) - F(x) = C \Rightarrow \boxed{F_2(x) = F(x) + C} \qquad \blacksquare$$

EXAMPLE 1 Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ (b) $\underbrace{f(x) = 1/x}$ (c) $f(x) = x^n, \quad n \neq -1$

(b) $F(x) = \ln x \rightarrow F'(x) = \frac{1}{x} = f(x)$

GENERAL FORM $F(x) = \ln x + C$

(a) $F(x) = -\cos x \quad F'(x) = -(-\sin x) = \sin x = f(x)$

GENERAL FORM $F(x) = -\cos x \boxed{+C}$ → DON'T FORGET! [CONSTANT OF INTEG.]

* $F' = f$
* $G' = g$

RULES FOR ANTIDIF.

(i) ANTIDERIVATIVE OF $c f(x)$ $= cF(x)$

[$c \rightarrow$ CONSTANT]

$$\frac{d}{dx}\left[cF(x)\right] = c\frac{dF}{dx} = cf(x)$$

(ii) ANTIDERIVATIVE OF $f + g$ $= F + G$

$$\frac{d}{dx}\left[F(x) + G(x)\right] = \frac{dF}{dx} + \frac{dG}{dx} = f + g$$

IMP :

A / D OF f $\neq$ $F \cdot G$

(PRODUCT RULE)

A/D I+I

BLUE

FORMULAE

$$\frac{d}{dx}\left(x^{n+1}\right) = \underbrace{(n+1)}\,x^{n}$$

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^{n}$$

1. x^{n} $\quad (n \neq -1)$ $\quad \boxed{\frac{x^{n+1}}{n+1} + C}$

* 2. $x^{-1} = 1/x$ $\quad \boxed{\ln x + C}$

3. e^x $\qquad$ $\boxed{e^x + C}$

☆ 4. b^x $\qquad$ $\boxed{\frac{b^x}{\ln b} + C}$ $\qquad$ $\boxed{\frac{d}{dx}(b^x) = (b^x \cdot \ln b)}$

$(b \neq 1,\ b > 0)$ $\qquad$ $[\ln e = 1]$

$$F(x) = \frac{b^x}{\ln b} + C \Rightarrow F'(x) = \frac{d}{dx}\left(\frac{b^x}{\ln b}\right) = \frac{1}{\ln b} \cdot \frac{d}{dx}(b^x) = \frac{b^x \ln b}{\ln b}$$

5. $\cos x$ $\boxed{\sin x + C}$

6. $\sin x$ $\boxed{-\cos x + C}$

NOT AS IMPORTANT

FORMULAE (CONTD.)

7. $\sec^2 x \longrightarrow \boxed{\tan x + C}$

8. $\sec x \tan x \longrightarrow \boxed{\sec x + C}$

9. $\dfrac{1}{\sqrt{1-x^2}} \longrightarrow \boxed{\text{Arcsin}\, x + C} \,/\, \boxed{\sin^{-1} x + C}$

10. $\dfrac{1}{1+x^2} \longrightarrow \boxed{\text{Arctan}\, x + C} \,/\, \boxed{\tan^{-1} x + C}$

RECALL

$$\frac{d}{dx}\left[\text{Arcsin}\, x\right] = \frac{1}{\sqrt{1-x^2}}$$

A/D of $x^n \nearrow \dfrac{x^{n+1}}{n+1}$

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

$$4\sin x + \frac{2x^5 - \sqrt{x}}{x} = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x}$$

$$= \underbrace{4\sin x}_{} + \underbrace{2x^4}_{} - \underbrace{x^{-1/2}}_{}$$

$4\sin x \rightsquigarrow -4\cos x$

$2x^4 \rightsquigarrow 2\dfrac{x^5}{5}$

$-x^{-1/2} \rightsquigarrow -\dfrac{x^{1/2}}{1/2}$

$= -2x^{1/2}$

$$g(x) = -4\cos x + \frac{2x^5}{5} - 2x^{4/2} + C$$

EXAMPLE 3 Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $\boxed{f(0) = -2}$.

$f(x)$

$$f'(x) = \underbrace{e^x} + \underbrace{20(1+x^2)^{-1}}$$

$$e^x \xrightarrow{A/D} e^x$$

$$20(1+x^2)^{-1} = 20\left(\frac{1}{1+x^2}\right) \xrightarrow{A/D} 20 \operatorname{Arctan} x$$

$$f(x) = e^x + 20 \operatorname{Arctan} x + C$$

$$-2 = f(0) = e^0 + 20 \operatorname{Arctan} 0 + C = 1 + C$$

$1 + C = -2$

$\Rightarrow C = -3$

$$\boxed{f(x) = e^{x} + 2\arctan x - 3}$$

A/D OF

$x^n = \frac{x^{n+1}}{n+1}$

$4 = 4x^0$

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

$$f''(x) = 12x^2 + 6x - 4$$

$$f'(x) = 12 \cdot \frac{x^{2+1}}{2+1} + \frac{6x^{1+1}}{1+1} - \frac{4x^{0+1}}{0+1} + C_1$$

$$= 4x^3 + 3x^2 - 4x + C_1$$

$$f(x) = 4\frac{x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + \frac{C_1 x^1}{1} + C_2$$

$$= x^4 + x^3 - 2x^2 + C_1 x + C_2$$

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

$f(0) = C_2$

$\Rightarrow C_2 = 4$

$f(1) = C_1 + C_2$

$C_1 = 1 - C_2$

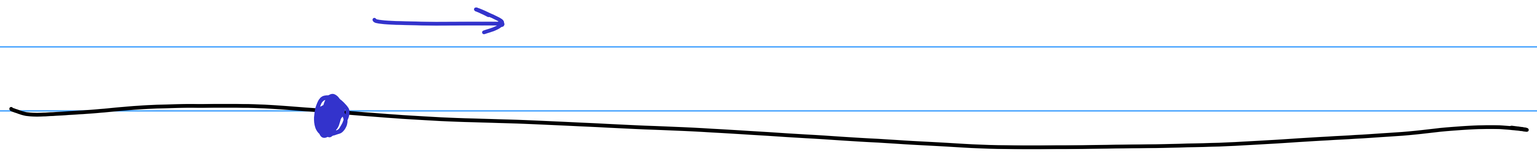
$= 1 - 4 = -3$

EXAMPLE 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

$a(t) = 6t + 4$

$\boxed{v(0) = -6}$

$\boxed{s(0) = 9}$



$6t + 4$

$6\left(\frac{t^2}{2}\right) + 4 \cdot \frac{t^1}{1} + C$

$v(t) = 3t^2 + 4t + C$

$-6 = v(0) = (3)(0^2) + 4(0) + C = C$

A/D of r s t

$v(t) = 3t^2 + 4t - 6$

$s(0) = 9$

$s(t) = A/D$ of $3t^2 + 4t^1 - 6$

$= \frac{3t^3}{3} + \frac{4t^2}{2} - 6t + C$

$= t^3 + 2t^2 - 6t + C$

$9 = s(0) = 0^3 + 2\cdot 0^2 - 6\cdot 0 + C$

$C = 9$

$$\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$$

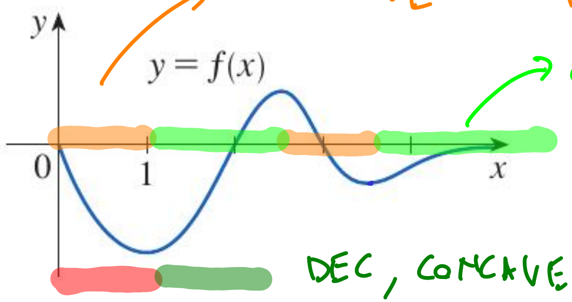
$F'(x) = f(x)$

59. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.

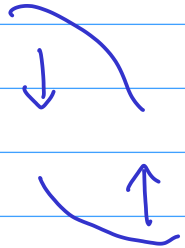
$(F'' < 0)$

$F'(x)$

CONCAVE DOWN



CONCAVE UP $(F'' > 0)$



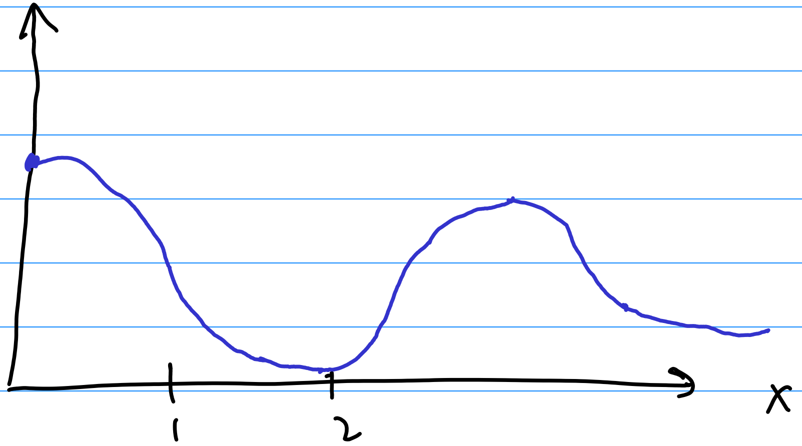
DEC, CONCAVE UP

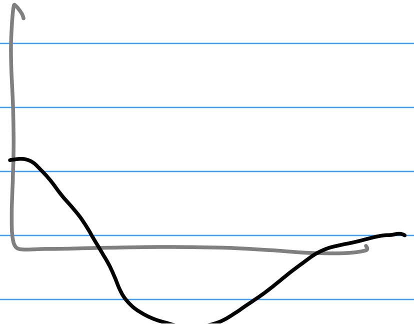
DECREASING & CONCAVE DOWN

$y = F(x)$

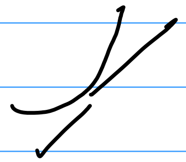
(DRAWN IN BLUE)

$(0,1)$

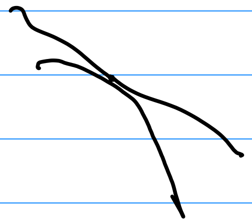




$F''(x) > 0 \rightarrow$ CONCAVE UP



$F''(x) < 0 \rightarrow$ CONCAVE DOWN

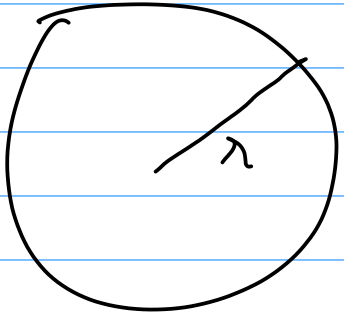


§ 5.1 AREAS & DISTANCES

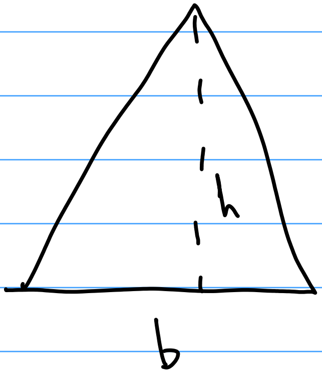
(AKA WHY ARE WE DOING THIS?)

(INTEGRATION)

πr^2



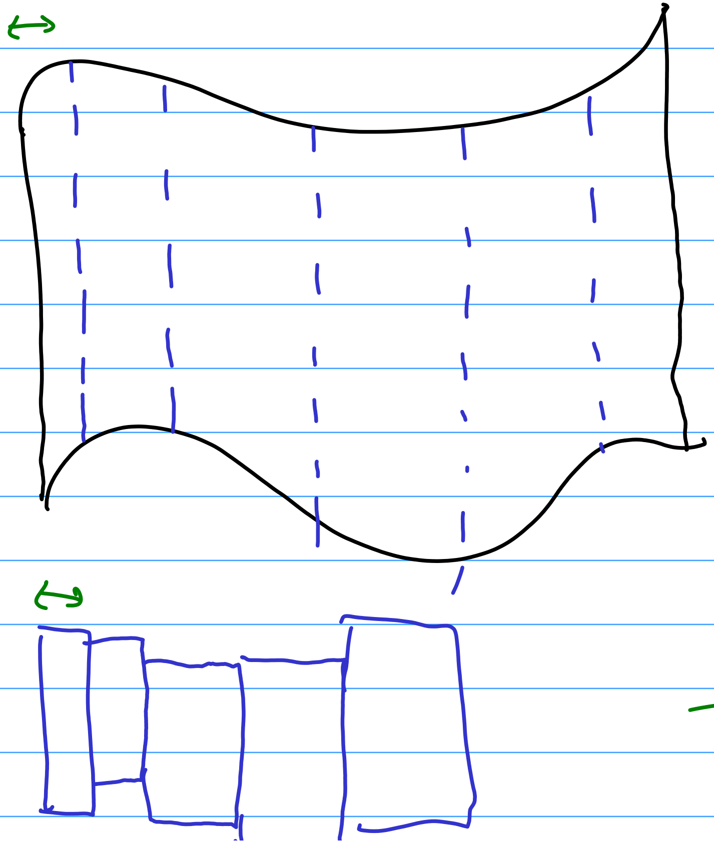
$\frac{1}{2}bh$



bh

SPECIAL

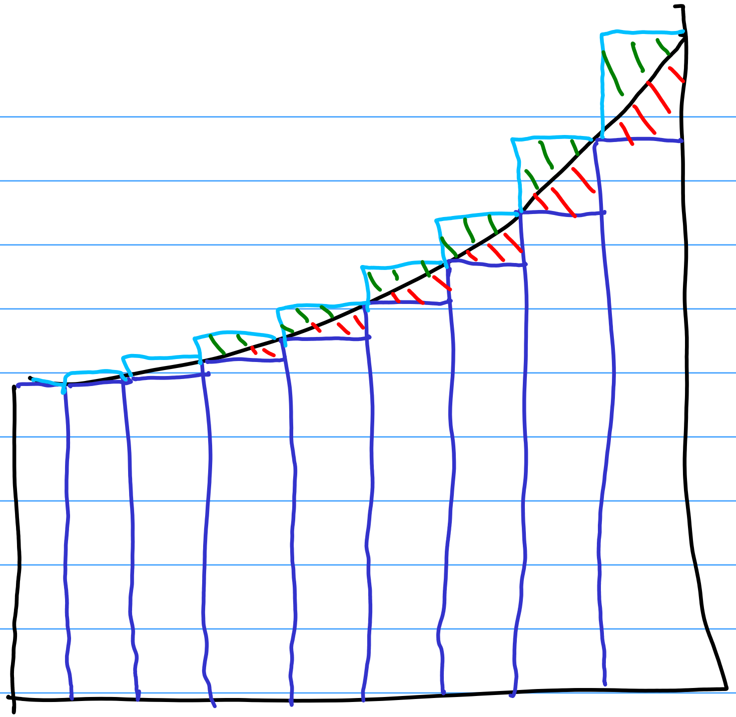
Q. AREA "PROBLEM"



AREA

ADD ALL THE AREAS

$a < \alpha < A$



SUM OF SMALLER []

↑

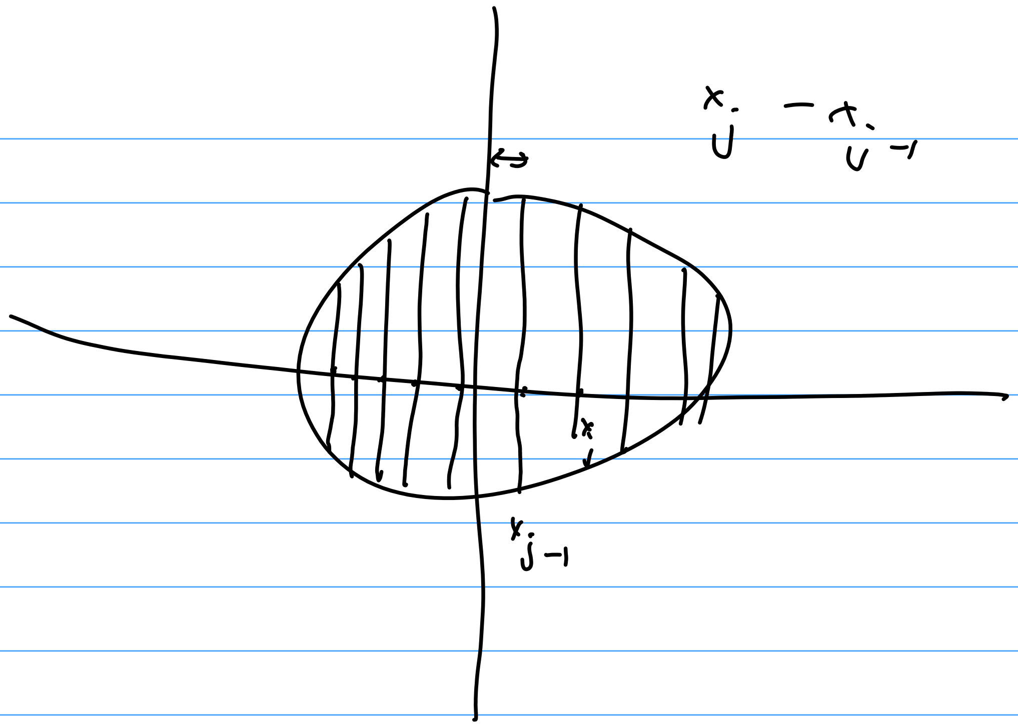
a

α

↑

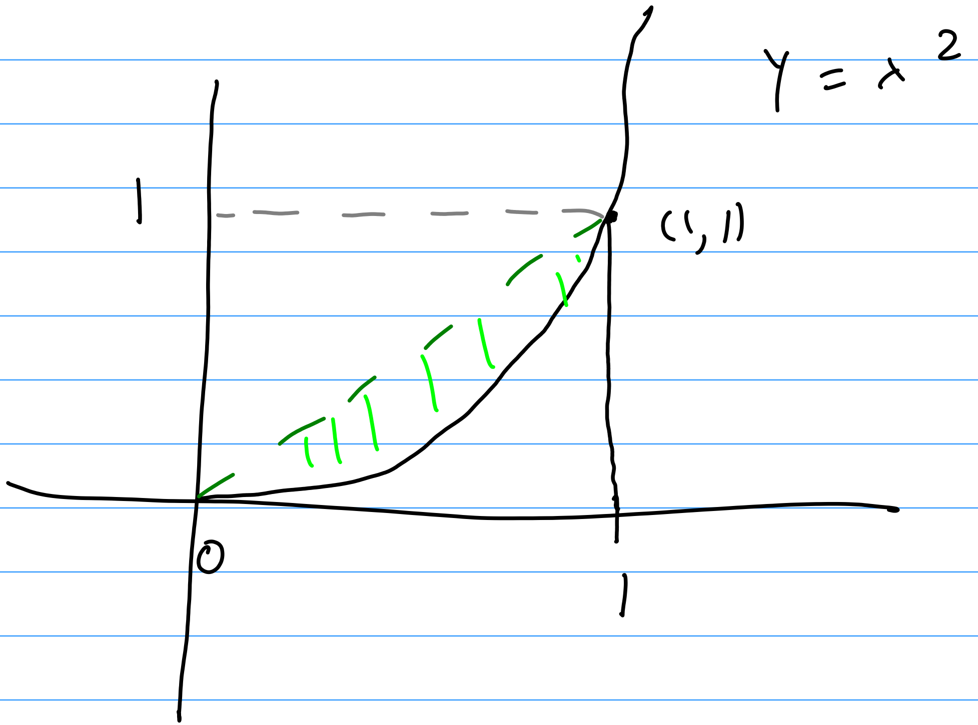
AREA UNDER THE

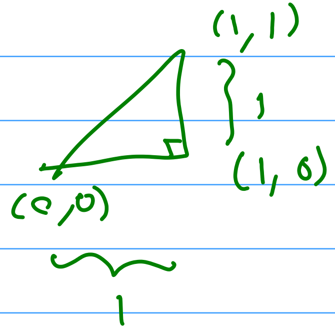
$A \rightarrow$ SUM OF LARGER []



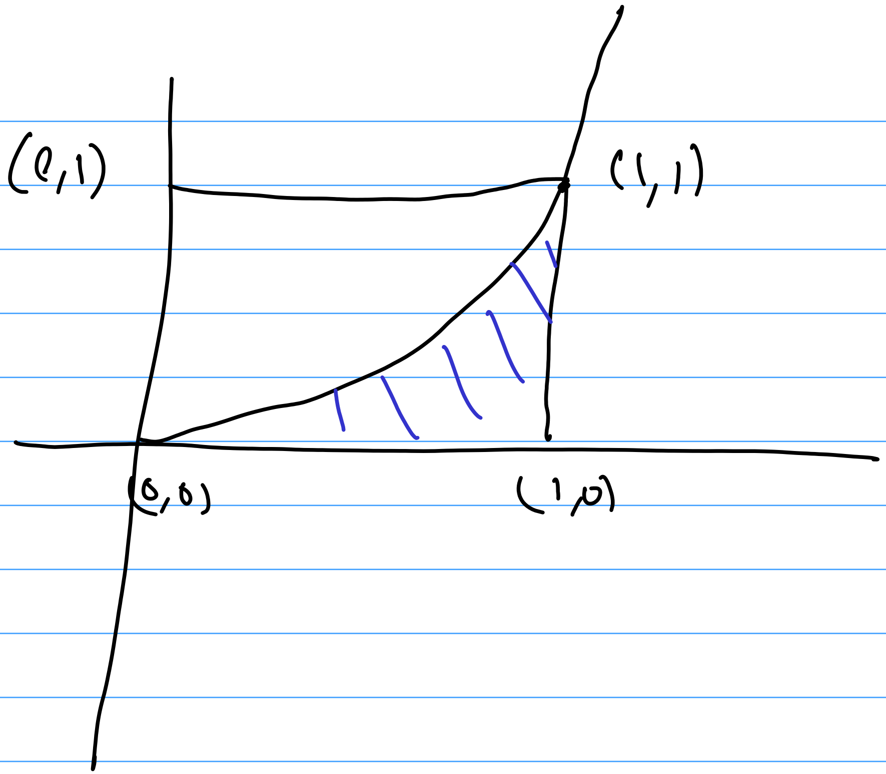
$x_j - x_{j-1}$
x_j
x_{j-1}

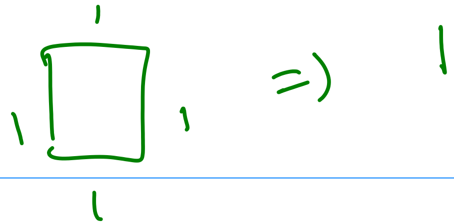
AREA UNDER $y = x^2$ FROM $x = 0$ TO $x = 1$



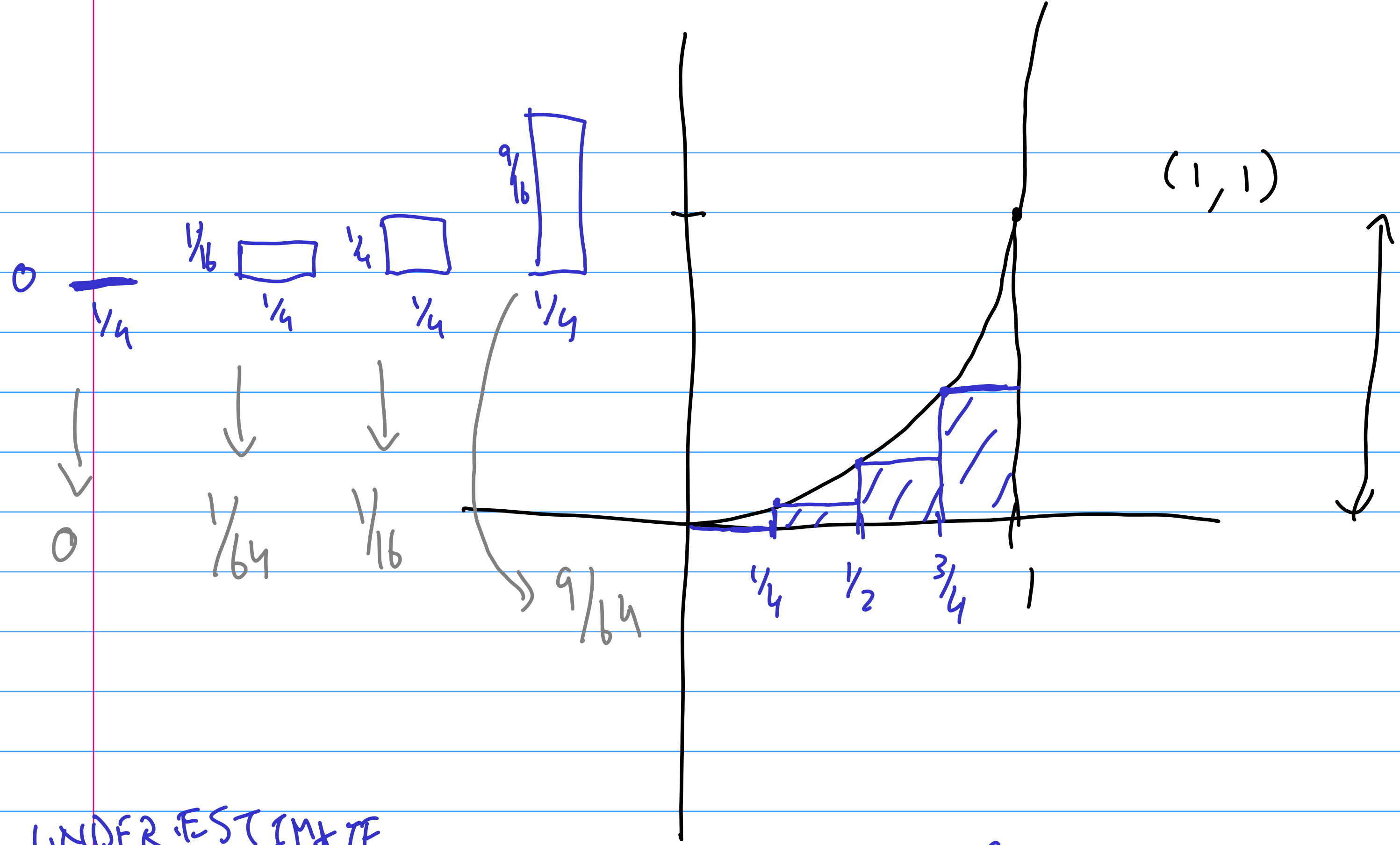


AREA $\approx \frac{1}{2}$


(0,1)
(1,1)
(0,0)
(1,0)


1
1
1
1
=) 1

$f(x) = x^2$



$f(\tfrac{3}{4}) = 9/16$

$f(1/2) = 1/4$

$y = x^2$

$f(1/4) = (1/4)^2 = 1/16$

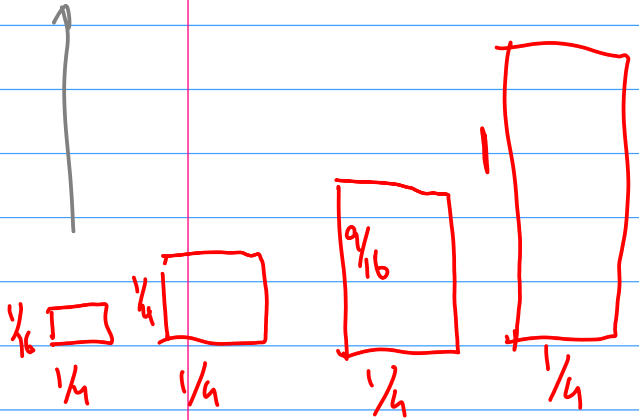
UNDERESTIMATE

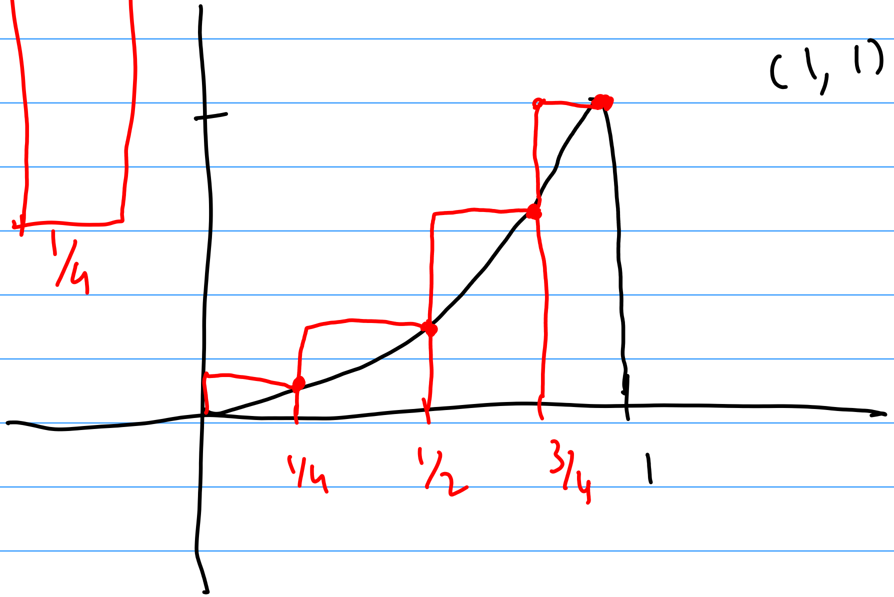
AREA $= 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} = \frac{14}{64} = \frac{7}{32}$

OVERESTIMATE AREA $= \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4}$

$= \frac{30}{64} = \frac{15}{32}$

$\frac{1}{64}$ $\frac{1}{16}$ $\frac{9}{64}$ $\frac{1}{4}$





(1, 1)

$f(1) = 1^2 = 1$

$f(x) = x^2$

$f(1/4) = 1/16$

$f(1/2) = 1/4$

$f(3/4) = 9/16$

100 RECTANGLES

OVERESTIMATE $>$ ACTUAL AREA $>$ UNDERESTIMATE

$= 15/32$ $\qquad\qquad$ $= 7/32$

$$\text{AREA} \sim [7/32, 15/32]$$

WHAT HAPPENS AS # OF CUTS/RECTANGLES $\longrightarrow \infty$