MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, F 4-5PM ;
~~W 9-10 AM ;~~
BY APPOINTMENT

(N.B. - STARTING FRIDAY)

email: anuragsahay@rochester.edu

COURSE PAGE : bit.ly/sahay142

LECTURES :
5:45 PM - 7:50 PM (ET)
M, T, W, R

Zoom ID :
979-4693-0650

ANNOUNCEMENTS

1. PLEASE CHECK EMAILS
2. NO CLASS ON MONDAY [MEMORIAL DAY WEEKEND]
3. FIRST OFFICE HOUR TOMORROW (4-5PM)
4. WEBWORK 1 & 2 ARE UP

   DUE → (1) SATURDAY
   (2) MONDAY

QUICK REVIEW

POWER RULE (BASE IS VAR. & EXP IS CONSTANT)

x^n VERSUS b^x

$$\left[\underbrace{(\quad)}_{\text{BASE}}{}^{\overbrace{(\quad)}^{\text{EXP}}}\right]$$

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$$

$$\frac{d}{dx}\left(b^x\right) = b^x \cdot \ln b$$

(BASE IS CONSTANT, EXPONENT IS VARIABLE)

SUGGESTED PROBLEM.

$$f(x) = x^2 + 2^x + 2^2$$

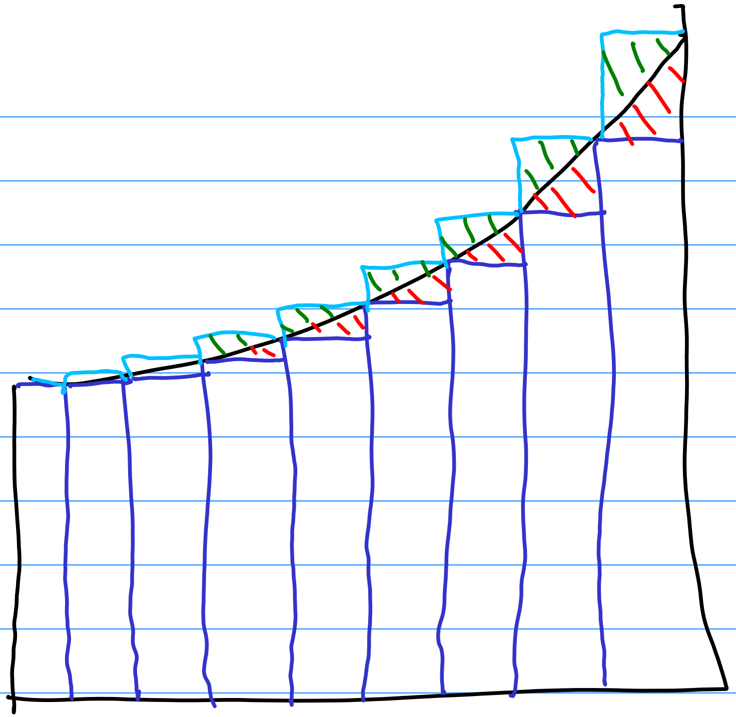
(i) FIND $f'(x)$

(ii) FIND F s.t. $F'(x) = f(x)$

§ 5.1 AREAS & DISTANCES (CON'T'D)

(AKA WHY ARE WE DOING THIS?)

$a < \alpha < A$



SUM OF SMALLER ▯

↑

a

α

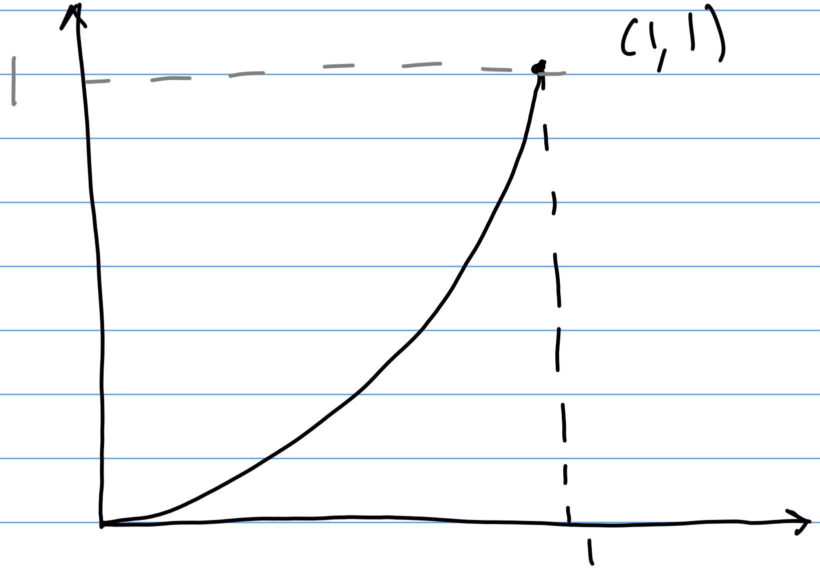
↑

AREA UNDER THE

$A \rightarrow$ SUM OF LARGER ▯

WHAT HAPPENS AS # OF CUTS/RECTANGLES $\longrightarrow \infty$

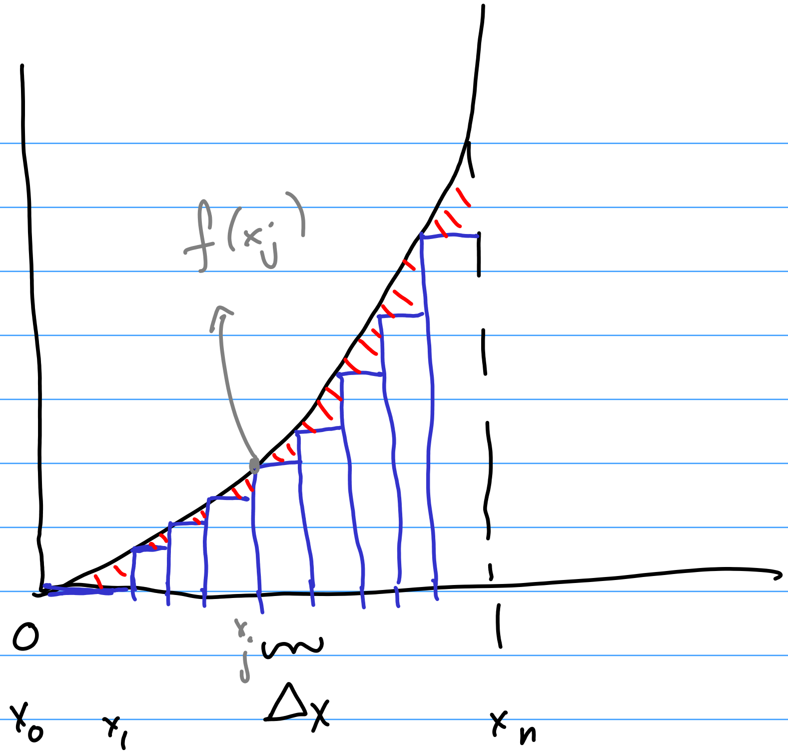
$y = x^2$



$$n\Delta x = 1$$

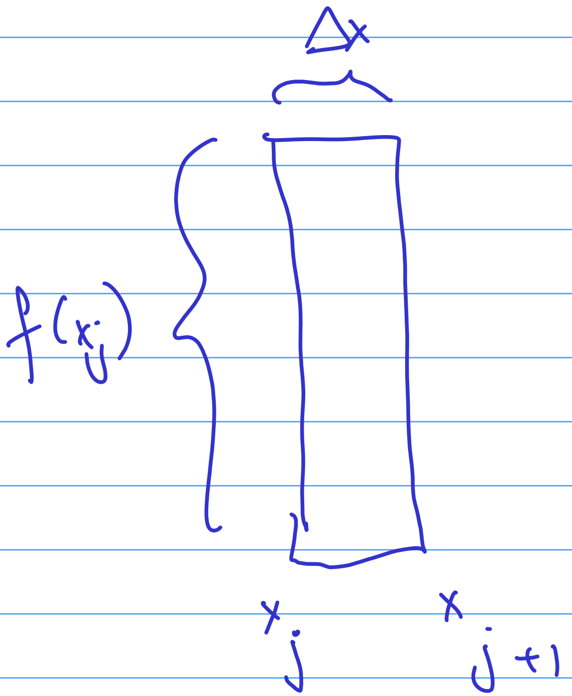
$$\Delta x = \frac{1}{n}$$

$$\boxed{x_j = \frac{j}{n}}$$



$$0 = x_0 < x_1 < x_2 \dots < x_n = 1$$

$f(x) = x^2$



AREA OF j^{th} RECTANGLE

$$f(x_j)\,\Delta x = f\left(\frac{j}{n}\right) \cdot \frac{1}{n}$$

$$= \left(\frac{j}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{j^2}{n^3}$$

TOTAL AREA = SUM OF j^{th} RECTANGLE OF RECT.

$$= \sum_{j=0}^{n-1} \frac{j^2}{n^3}$$

$$\sum_{j=a}^{b} F(j) = F(a) + F(a+1) + \dots + F(b)$$

$\sum \rightarrow$ GREEK ANTECEDENT OF $\int$

$$\sum_{j=0}^{n-1} \frac{j^2}{n^3} = \sum_{j=1}^{n-1} \frac{j^2}{n^3}$$

$$= \frac{1}{n^3} \sum_{j=1}^{n-1} j^2$$

$$\sum_{j=a}^{b} (c f(j)) = c f(a) + c f(a+1) + \cdots + c f(b-1) + c f(b)$$

$$= c \left[f(a) + f(a+1) + \cdots + f(b) \right]$$

$$= c \sum_{j=a}^{b} f(j)$$

NOTE: $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{j=1}^{n-1} j^2 = 1^2 + 2^2 + \dots + (n-1)^2 = \frac{(n-1)(n-1+1)[2(n-1)+1]}{6}$$

$$= \frac{(n-1)\,n\,(2n-1)}{6}$$

$$\boxed{\sum_{j=1}^{n-1} j^2 = \frac{n(n-1)(2n-1)}{6}}$$

AREA OF RECT $= \frac{1}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right]$

AREA OF RECT $= \frac{1}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right]$

$$= \frac{1}{n^2}\left[\frac{(n-1)(2n-1)}{6}\right]$$

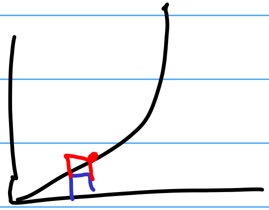
$$= \frac{2n^2-3n+1}{6n^2}$$

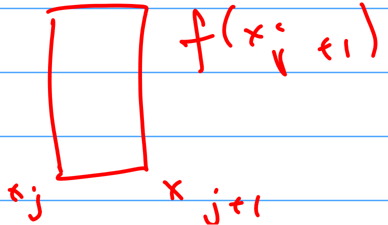
$$\lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} \simeq \frac{2n^2}{6n^2} = \frac{1}{3} \quad ?$$

AREA UNDER THE CURVE $= 1/3$

SUGGESTED PROBLEM

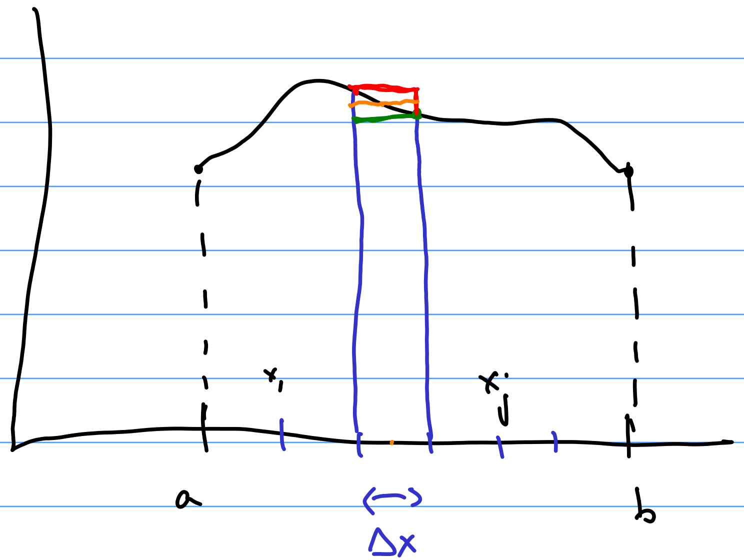
1. CHECK THAT BOUNDS FROM LAST LECTURE APPLY
2. YOU GET THE SAME ANSWER IF YOU USE UPPER RECTANGLES INSTEAD





GENERAL CURVES

$y = f(x) \geq 0$ $\{$ ON $[a,b]$ $\}$



CUT IT INTO n

$$n\Delta x = b - a$$

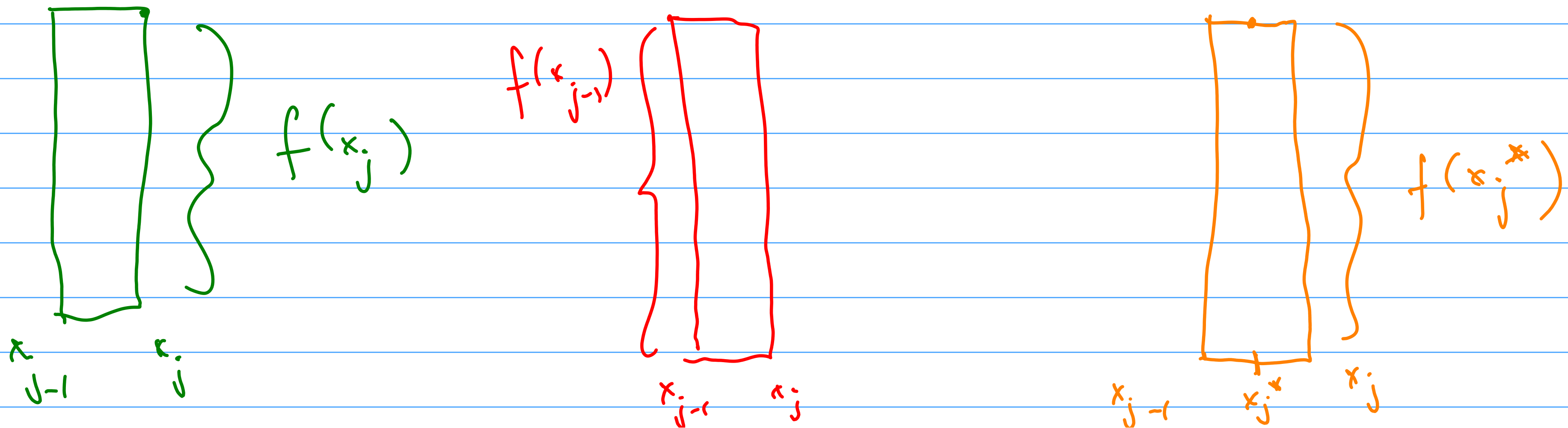
$$\Rightarrow \Delta x = \frac{b-a}{n}$$

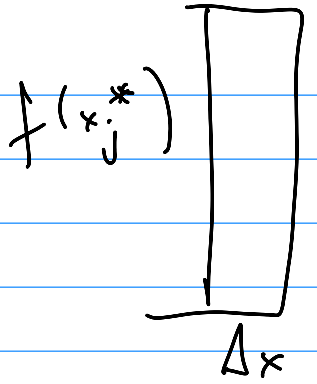
$a \longrightarrow b$ SAMPLE

$$\Delta x = \frac{b-a}{n}$$

$$a = x_0 < x_1 < x_2 \cdots < x_n = b$$

$$x_j = a + j\Delta x$$





$\Rightarrow$ AREA $= f(x_j^*)\,\Delta x$

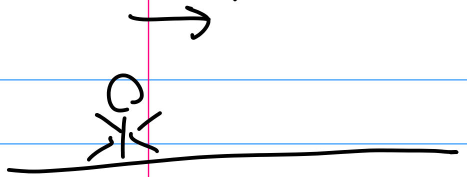
$$\Delta x = \frac{b-a}{n}$$

TOTAL AREA $= \displaystyle\sum_{j=1}^{n} f(x_j^*)\,\Delta x$

DEFN. OF AREA

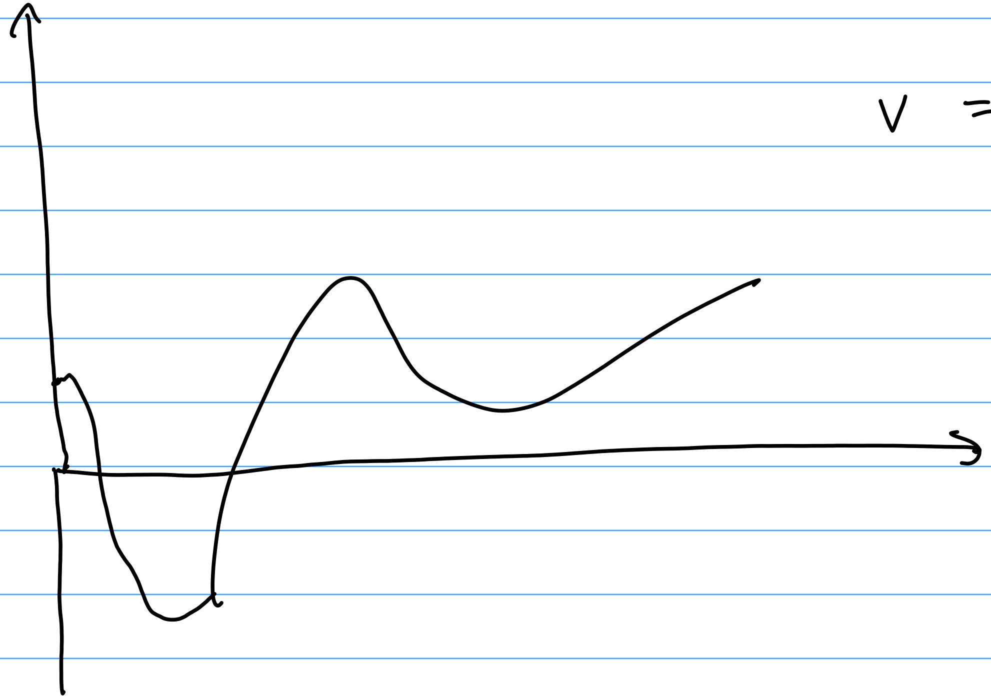
AREA UNDER $f(x)$ $= \displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\,\Delta x$

$v = f(t)$



THE DISTANCE PROBLEM

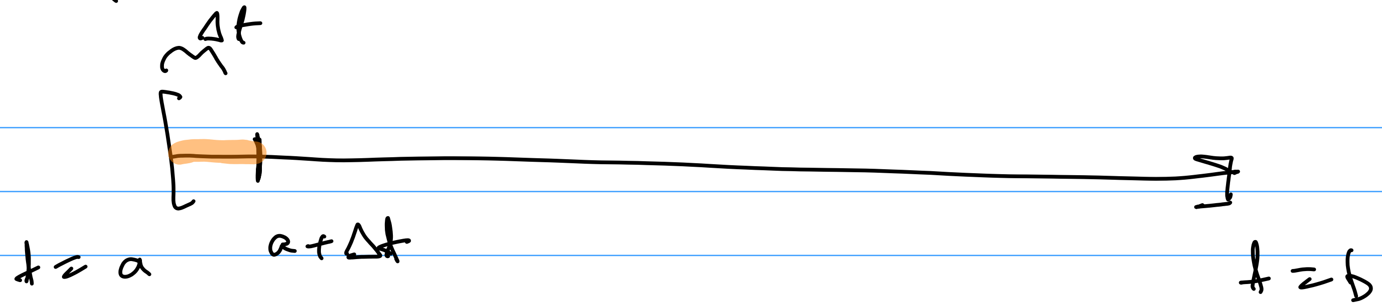
AT TIME $t=0$, YOU ARE AT THE ORIGIN.



$v = f(t)$

$f(t) = \sqrt{t}$

TIME AXIS

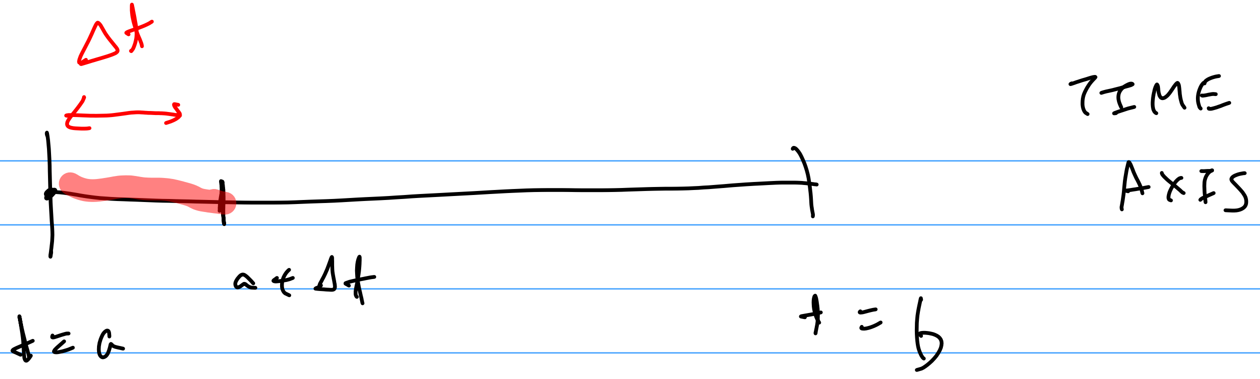


$v = f(t) \approx f(a)$

$v = 5\ m/s$

$5\ s$

$$\underbrace{(5\ m/s)}_{v}\ \underbrace{(5\ s)}_{\Delta t} = 25\ m$$



$1^2 \to 2^2$

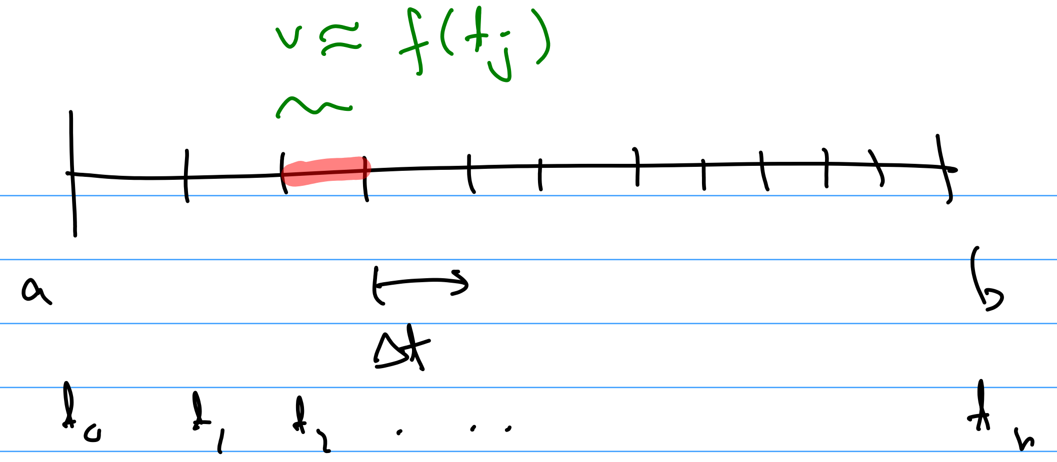
$f(t) = t^2$

$\approx (1\ m/s) \times 1s \approx 1\ m$

AT TIME t, $v = t^2$

$v \approx f(a)$

TIME TAKEN $= \Delta t$ → DISTANCE $\approx f(a)\Delta t$



$$a = t_0 < t_1 < t_2 \cdots < t_n = b$$

$$n\,\Delta t = b - a$$

$$\Rightarrow \Delta t = \frac{b-a}{n}$$

$$\left[\begin{array}{l} a = 0\,s \\ b = 60\,s \\ \Delta t = 1\,s \end{array}\right]$$

DISTANCE IN jth INTERVAL $\approx f(t_j)\,\Delta t$

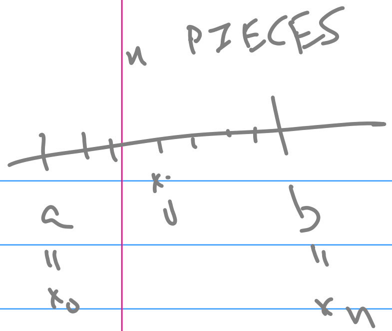
TOTAL DISTANCE $\simeq \sum_{j=1}^{n} f(t_j)\,\Delta t$

$= \quad (\text{IF} \quad n \longrightarrow \infty)$

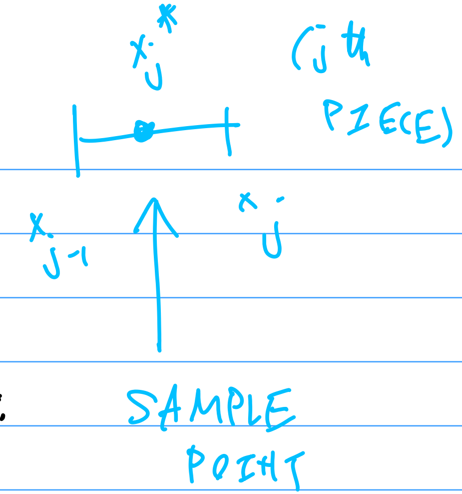
TOTAL DISTANCE $= \lim_{n \to \infty} \sum_{j=1}^{n} f(t_j)\,\Delta t$

RESTART AT

6:45 PM



§ 5.2 THE DEFINITE INTERAL



THIS LEADS TO THE FOLLOWING:

__DEFN__: FOR A FUNCTION f DEFINED ON AN INTERVAL $[a,b]$, LET

$\Delta x = \frac{b-a}{n}$, $a = x_0, x_1, \dots, x_{n-1}, x_n = b$ $\{x_j = a + j\Delta x\}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$ $\{x_j^* \in [x_{j-1}, x_j]\}$

IF THE LIMIT EXISTS, f IS SAID TO BE (RIEMANN) INTEGRABLE.

UPPER LIMIT OF INT

$\int$ (INTEGRAL SYMBOL)

LIEBNIZ

STYLIZED S

$\int \leftrightarrow \sum$

$$\int_a^b \overbrace{f(x)}^{\text{INTEGRAND}} \, dx$$

(~SUMMAND)

HAS NO MEANING IN AND OF ITSELF*

LOWER LIMIT OF INTEGRATION

ANATOMY OF AN INTEGRAL

* IN 142

NOTE 1 : x IS A DUMMY VARIABLE

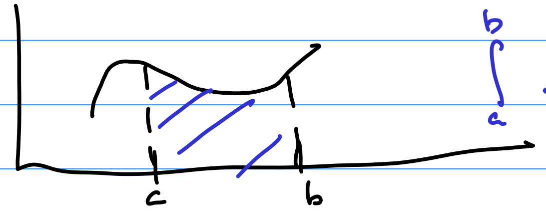
$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt$$

NOTE 2 : RIEMANN SUMS

$$\sum_{j=1}^{n} f(x_j^*)\,\Delta x$$ (RIEMANN SUM)

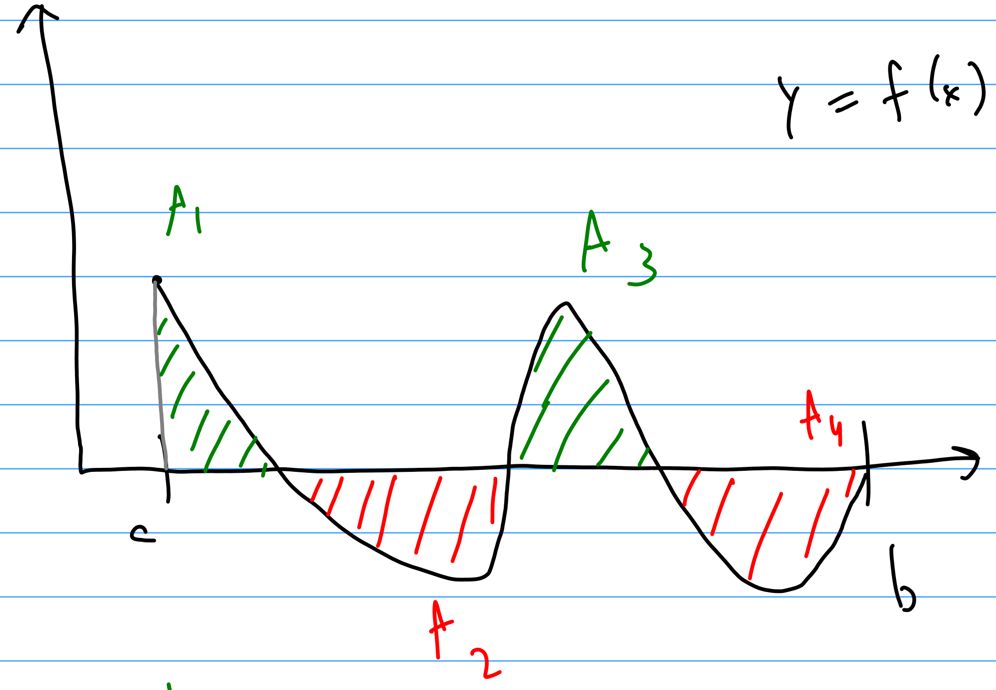
NOTE 3 : $f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx$ IS AN AREA

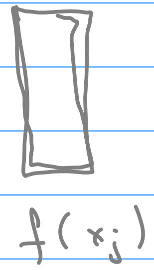
(AREA UNDER THE CURVE $y = f(x)$ B/W $x = a$ & $x = b$)



$\int_a^b f(x)\,dx$

NOTE 4: WHAT IF f IS SOMETIMES NEGATIVE?





$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$$

EXAMPLE 1 Evaluate the Riemann sum for $f(x) = x^3 - 6x$, $0 \leqslant x \leqslant 3$, with $n = 6$ subintervals and taking the sample endpoints to be right endpoints.

$f(x) = x^3 - 6x$

$a = 0, \ b = 3$

$0 \le x \le 3$

$n = 6$

$x_j^* \to$ RIGHT END POINT

$$\Delta x = \frac{b-a}{n}$$

$x_j = a + j\Delta x$

$= 0 + j\frac{1}{2}$

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$

$x_0 = a = 0$ $\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 = b = 3$

$x_j^* = x_j = j/2$

x_{j-1} & x_j $\Rightarrow$ SAMPLE POINT IS $x_j^* = x_j$

$f(x) = x^3 - 6x$

RIEMANN SUM

$$= \sum_{j=1}^{6} f(x_j^*)\,\Delta x$$

$$= \sum_{j=1}^{6} f\left(\frac{j}{2}\right)\left(\frac{1}{2}\right)$$

$$= \sum_{j=1}^{6} \left[\left(\frac{j}{2}\right)^3 - \frac{6j}{2}\right] \times \frac{1}{2}$$

$$= \sum_{j=1}^{6} \left[\frac{j^3}{16} - \frac{3j}{2}\right]$$

$$\sum_{j=1}^{6} \left[\frac{j^3}{16} - \frac{3j}{2} \right] = \left(\sum_{j=1}^{6} \frac{j^3}{16} \right) - \left(\sum_{j=1}^{6} \frac{3j}{2} \right) = \frac{1}{16} \sum_{j=1}^{6} j^3 - \frac{3}{2} \sum_{j=1}^{6} j$$

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \qquad \left\{ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \right\}$$

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2} \right]^2 \qquad \left\{ 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^3 \right\}$$

$$\left(\frac{1}{16}\sum_{j=1}^{6} j^3\right) - \left(\frac{3}{2}\sum_{j=1}^{6} j\right) = \frac{1}{16}\left[\frac{6(6+1)}{2}\right]^2 - \frac{3}{2}\left[\frac{6(6+1)}{2}\right]$$

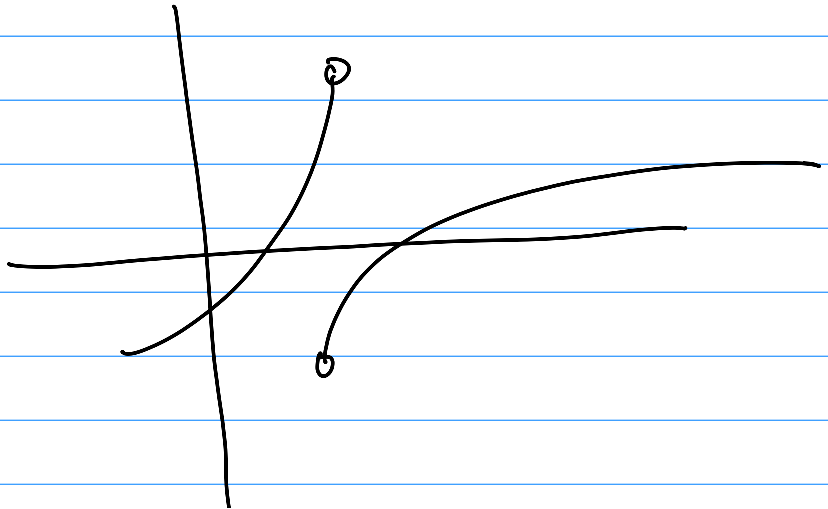
$$= \frac{1}{16}\left[21^2\right] - \frac{3}{2}\left[21\right]$$

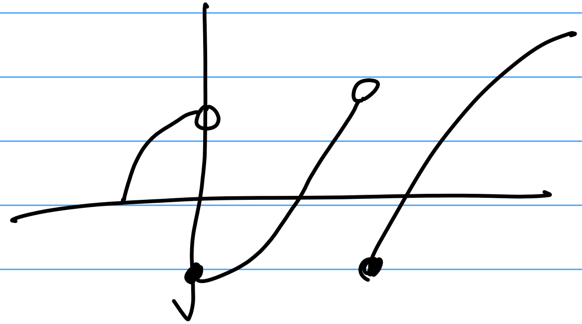
$\boxtimes$

(DON'T WORRY ABOUT "IS THIS INTEGRABLE"!

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

(BEYOND THE SCOPE OF THE COURSE)





EXAMPLE 4

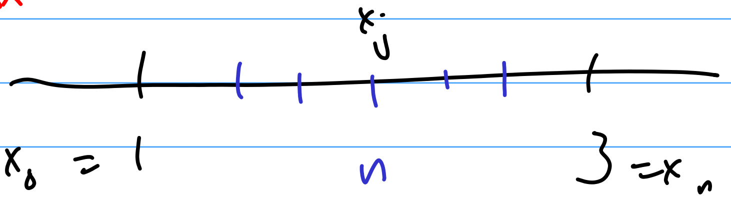
(a) Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.

$$\int_1^3 e^x\,dx \qquad f(x) = e^x$$

$b = 3 \qquad a = 1$

$$x_j = a + j\,\Delta x$$

$$\underbrace{= 1 + \frac{2j}{n}}_{x_j = x_j^*}$$



$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$x_j^* = x_j$ [RIGHT END-POINT]

$$\sum_{j=1}^{n} f(x_j^*)\,\Delta x = \sum_{j=1}^{n} f\left(1 + \frac{2j}{n}\right)\cdot\left(\frac{2}{n}\right)$$

$$\sum_{j=1}^{n} f\left(1+\frac{2j}{n}\right)\cdot\left(\frac{2}{n}\right) = \sum_{j=1}^{n}\left\{e^{\left(1+\frac{2j}{n}\right)}\cdot\frac{2}{n}\right\}$$

$$\int_{1}^{3} e^{x}\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}\left[e^{\left(1+\frac{2j}{n}\right)}\cdot\frac{2}{n}\right]$$

$\int_a^b f(x)\,dx \rightarrow$ REAL NUMBER

PROPERTIES OF THE DEFINITE INTEGRAL

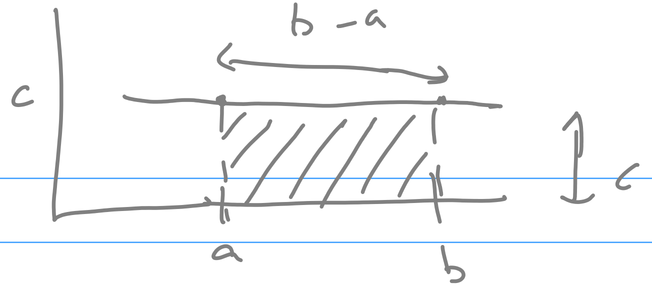
$$\Delta x = \frac{a-b}{n} = -\left[\frac{b-a}{n}\right]$$

1. $a < b$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

2. $\int_a^a f(x)\,dx = 0$

$$\Delta x = \frac{a-a}{n} = 0$$



3. $$\int_a^b c \, dx = c(b-a)$$

$(c \to \text{CONSTANT})$

4. $$\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

$$\left(\int \sum = \sum \int \right)$$

5. $\displaystyle\int_a^b [cf(x)]\,dx = c\int_a^b f(x)\,dx$

[$c \to$ CONSTANT]

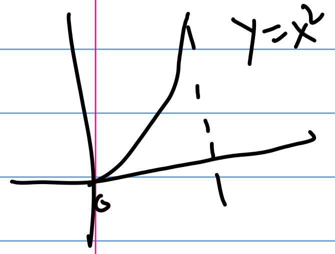
$ca + cb = c(a+b)$

6. $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

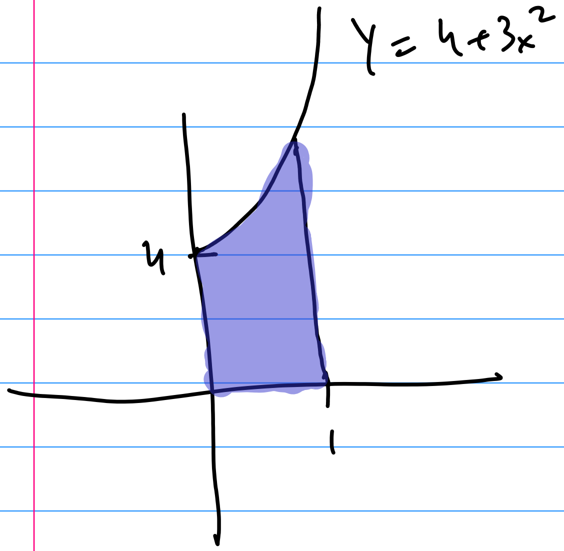
$\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b [f(x) + (-g(x))]\,dx = \int_a^b f(x)\,dx + \int_a^b (-g(x))\,dx$

$$\int_0^1 x^2\,dx = 1/3$$

EXAMPLE 7 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.



$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

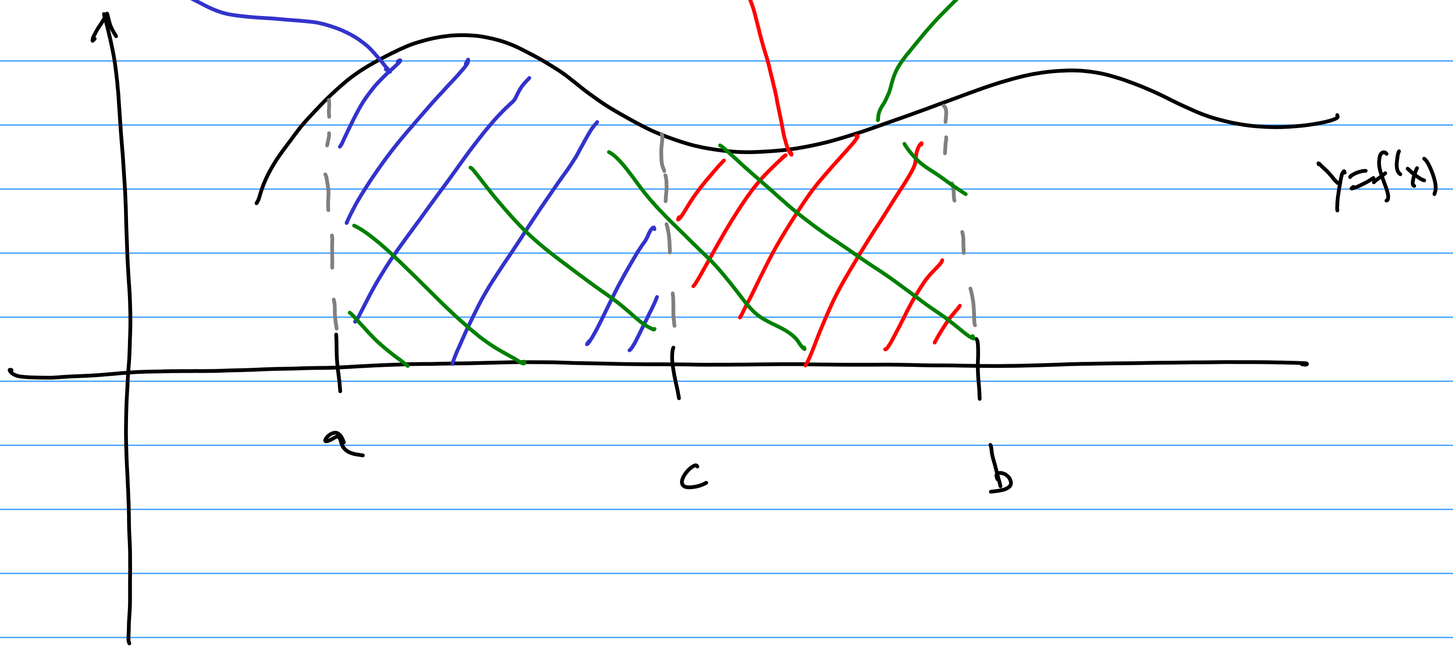


$$= \underbrace{\int_0^1 4\,dx} + 3\int_0^1 x^2\,dx$$

$$= 4(1-0) + 3\left(1/3\right)$$

$$= 4 + 1 = 5$$

7. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



EXAMPLE 8 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

$$\int_0^{10} f(x)\,d = 17$$

$$\int_0^{8} f(x) = 12$$

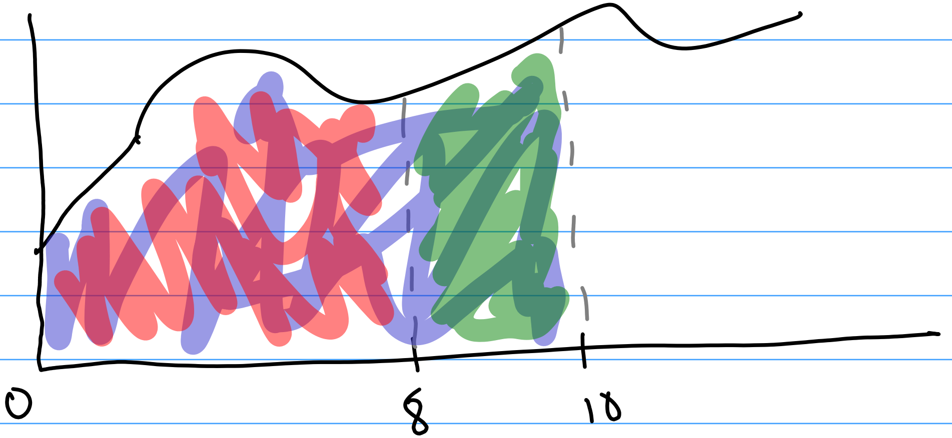
$$\int_8^{10} f(x)\,dx = 17-12$$

$$= 5$$

17

12

$$\int_0^{10} = \int_0^{8} + \int_8^{10}$$



$$\int_2^5 \frac{1}{x^2}\,dx \longrightarrow +ve/-ve$$

COMPARISON PROPERTIES

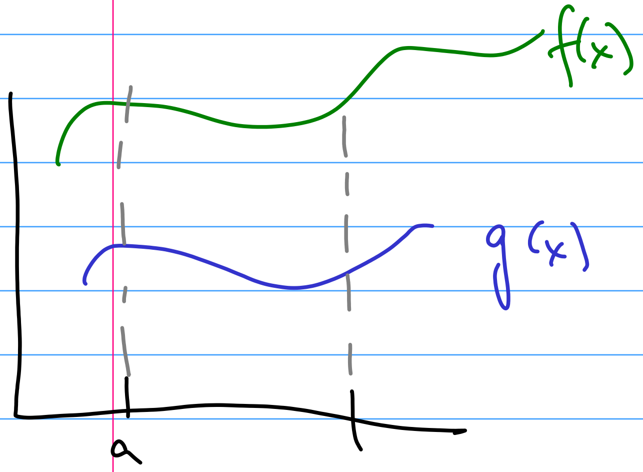
1. $f(x) \geq 0$ FOR ALL $a \leq x \leq b$

$$\int_a^b f(x)\,dx \geq 0$$

2. $g(x) \leq f(x)$ FOR ALL $a \leq x \leq b$

$\{ \Leftrightarrow f(x) \geq g(x) \}$

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$$

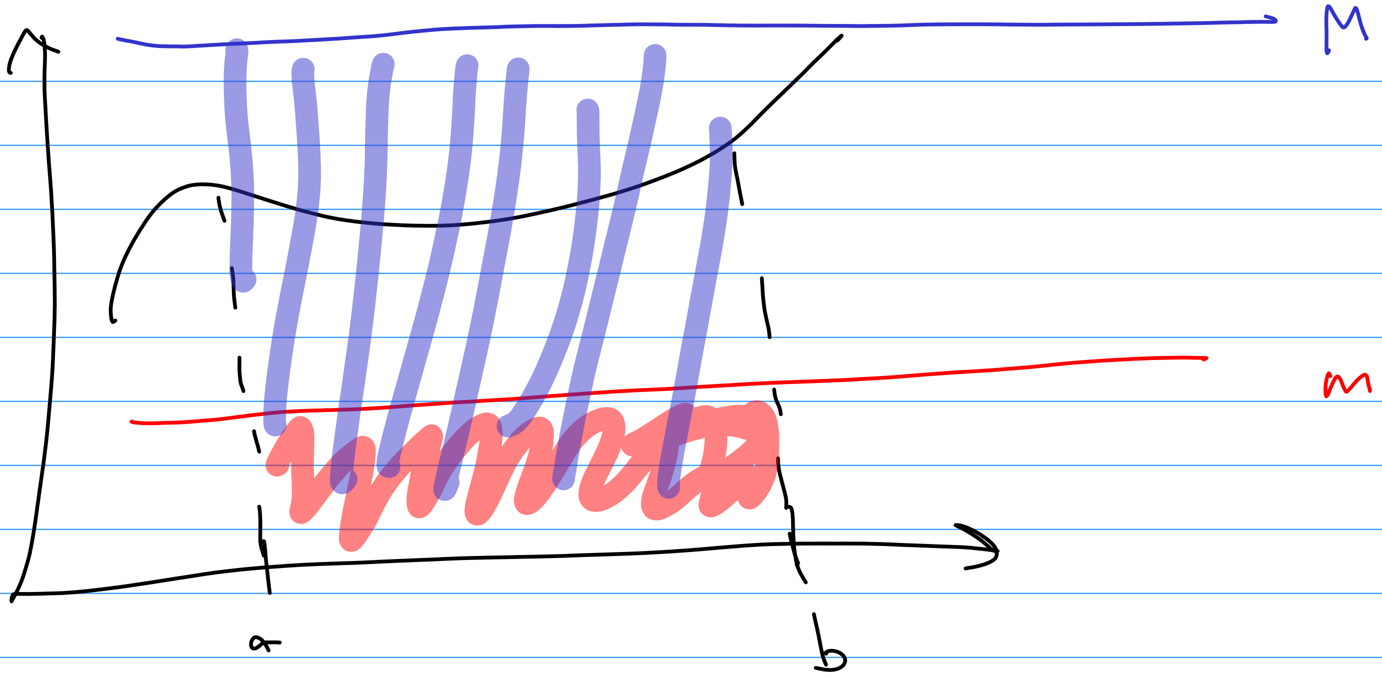


$$\boxed{\begin{array}{l} F(x) = f(x) - g(x) \Rightarrow F(x) \geq 0 \\ \Rightarrow \int F(x) \geq 0 \Rightarrow \int f \geq \int g \end{array}}$$

3. $m \leq f(x) \leq M$ FOR ALL $a \leq x \leq b$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Pf $\quad m \leq f(x) \leq M \Rightarrow \underbrace{\int_a^b m\,dx}_{m(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b M\,dx}_{M(b-a)}$



$$\overset{m(b-a)}{\boxed{}} \le \int_a^b f(x)\,dx \le \boxed{} \to M(b-a)$$

EXAMPLE 9 ~~Use Property 8~~ to estimate $\int_0^1 e^{-x^2}\,dx$.

e^{-x^2}

$$m \le e^{-x^2} \le M \quad \text{FOR } x \in [0,1]$$

DECREASING FUNCTION

(MAX AT 0
MIN AT 1)

$$M = e^{-0^2} = e^{-0} = 1$$

$$m = e^{-1^2} = e^{-1} = 1/e$$

$$1(1-0) = 1$$

$$\left(\tfrac{1}{e}\right)(1-0) = 1/e$$

$$m(b-a) \le \int_0^1 e^{-x^2} \le M(b-a)$$

$$\frac{1}{e} \leq \int_0^1 e^{-x^2}\,dx \leq 1$$