MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY
OFF HRS: M, F 4-5PM ; BY APPOINTMENT

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

email: anuragsahay@rochester.edu

COURSE PAGE: bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK DEADLINES : (a) WW 5 → WEDNESDAY, 11 PM
   (b) WW 6 → FRIDAY, 11 PM

2. MIDTERM 1 HAS BEEN GRADED. PLEASE CONSULT THE ANSWER KEY AND MAKE REGRADE REQUESTS.

3. ONE-ON-ONE MEETINGS (CHECK EMAILS!)

4. PARTICIPATION POINTS FOR LAST WEEK ARE UP.

$$\int_{-a}^{a} = \int_{-a}^{0} + \int_{0}^{a}$$

SYMMETRY

$$\boxed{\int_{-a}^{0} \quad u = -x}$$

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

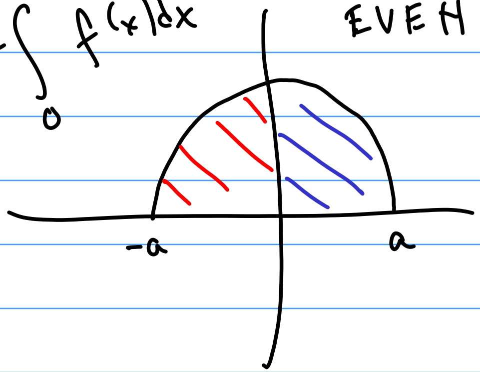
(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

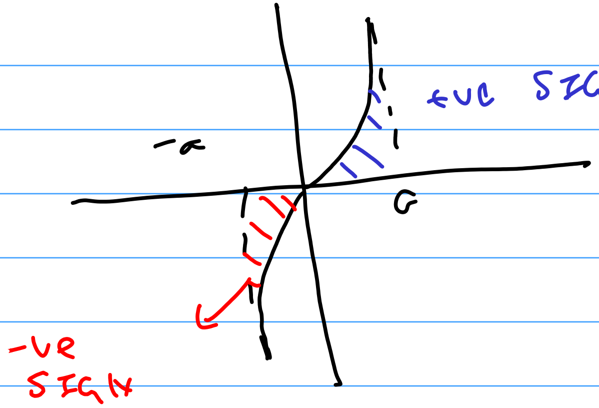
(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

f EVEN

$\downarrow$

$$\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)\,dx$$





f ODD

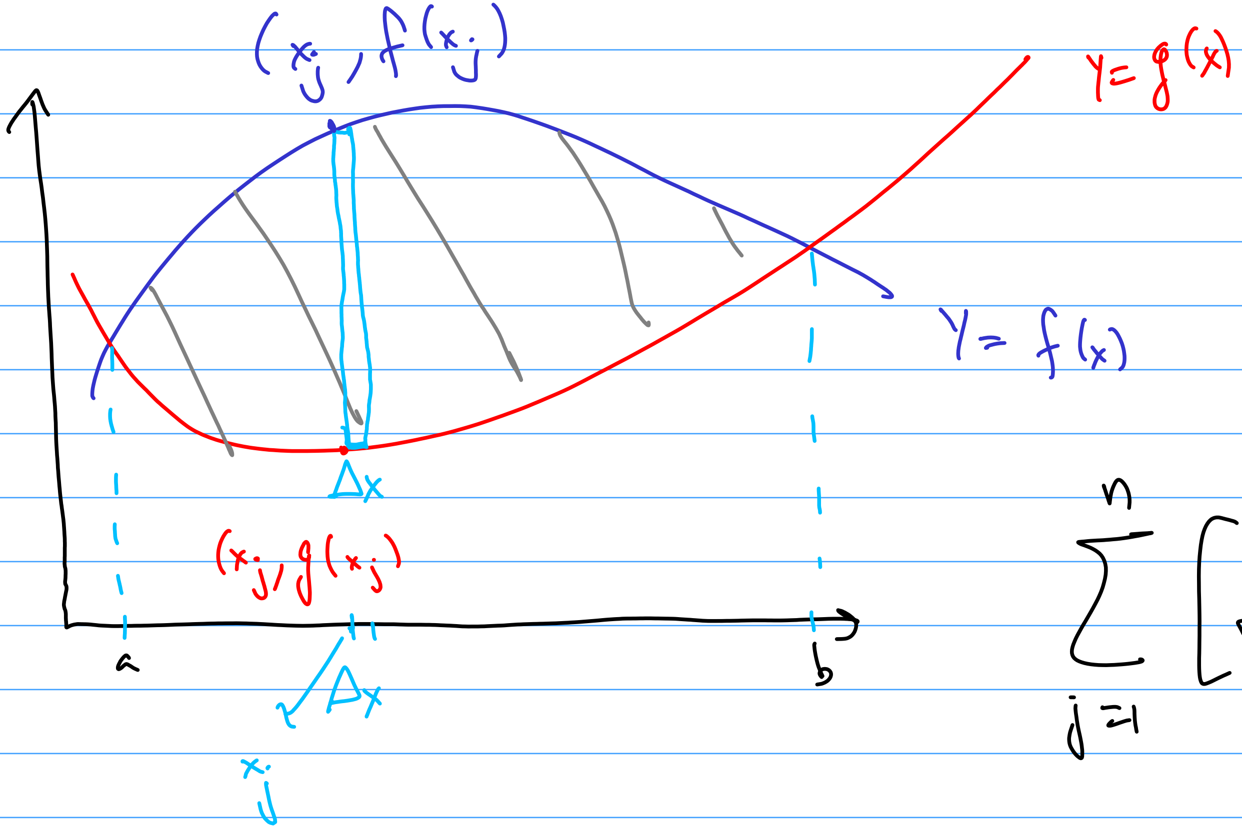
$\downarrow$

$$\int_{-a}^{a} f(x)\,dx = 0$$

$g(x) = x$

$f(x) = x^2$

§6.1 AREAS B/W CURVES



$$\sum_{j=1}^{n} \left[f(x_j) - g(x_j) \right] \Delta x_j$$

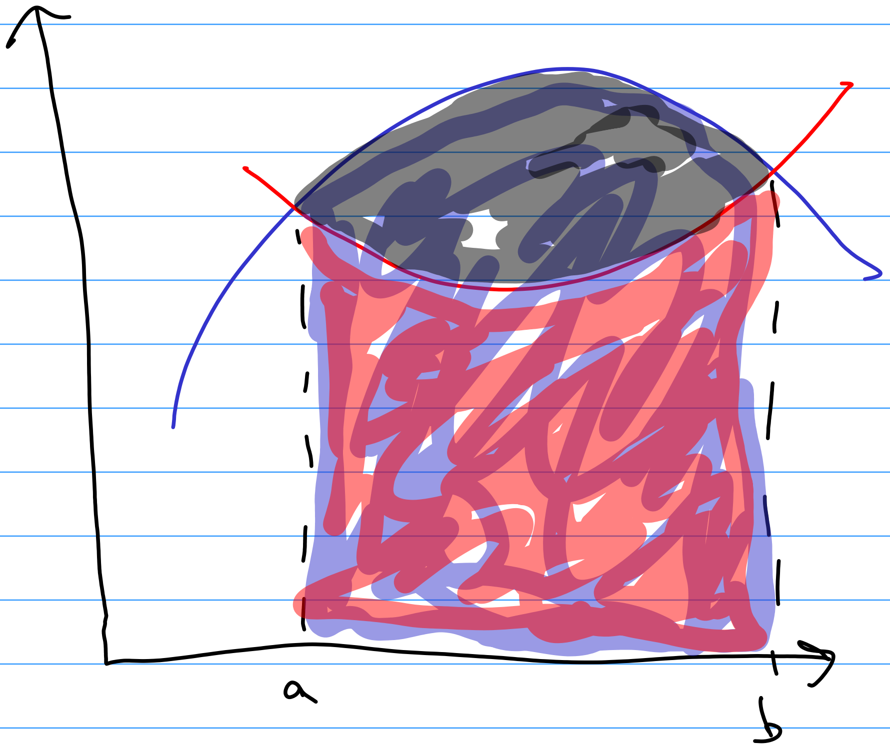
$$\sum_{j=1}^{n} \left[f(x_j) - g(x_j) \right] \Delta x_j \xrightarrow[\Delta x \to \infty]{n \to \infty} \int_a^b (f(x) - g(x))\, dx$$

<u>Thm</u> : $f(x) \geq g(x)$ FOR ALL $x \in [a, b]$

THEN AREA b/w f & g FROM $x = a$ TO $x = b$

$$= \int_a^b \left(f(x) - g(x) \right) dx$$

BLUE SHADED $= \int_a^b f(x)\,dx$

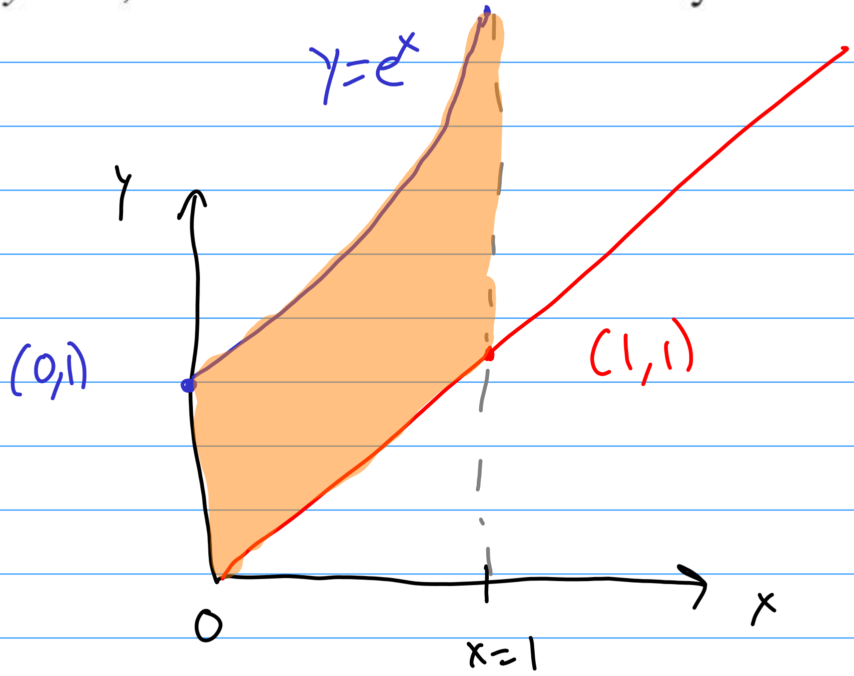


$y = f(x)$

RED SHADED $= \int_a^b g(x)$

AREA b/w CURVES = BLACK SHADED AREA = BLUE − RED $= \int_a^b \left(f(x) - g(x)\right) dx$

EXAMPLE 1 Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



$y = x$

$y = e^x$

$e^x - x$

INCREASING

$a = 0$

$b = 1$

$f(x) = e^x$ $\quad g(x) = x$

$$\text{AREA} = \int_0^1 (e^x - x)\,dx$$

(F.T.C.)

$$\int_0^1 (e^x - x)\,dx = e^x - \frac{x^2}{2}\Big]_0^1$$

$$= \left(e^1 - \frac{1^2}{2}\right) - \left(e^0 - \frac{0^2}{2}\right)$$

$$= e - \frac{1}{2} - 1 = \boxed{e - \frac{3}{2}}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

AREA

STEPS:

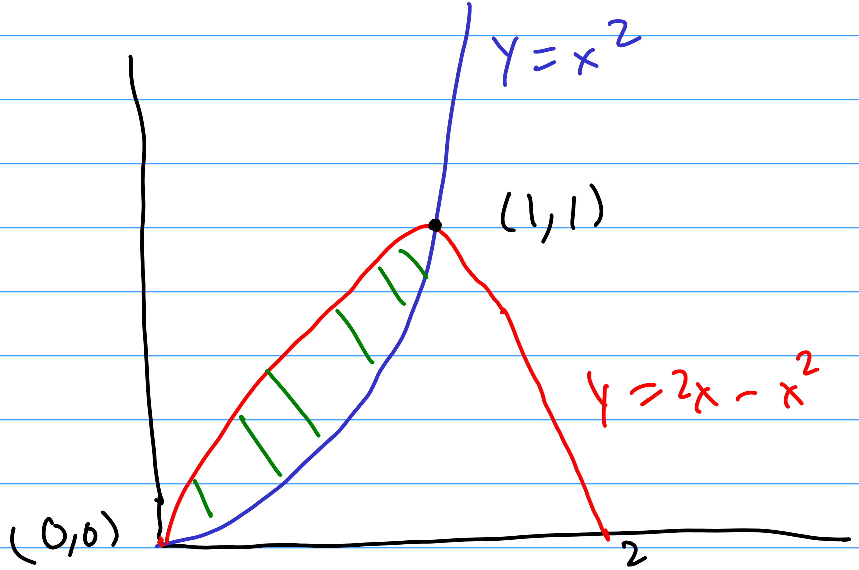
1. DRAW A PICTURE
2. IDENTIFY AREA
3. SET UP INTEGRAL *
4. SOLVE THE INTEGRAL

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

TO FIND LIMITS OF INTEGRATION

$x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \qquad 2x(x-1)=0$

$x = 0 \qquad x = 1$



STEPS:

1. DRAW A PICTURE
2. IDENTIFY AREA
3. SET UP INTEGRAL (*)
4. SOLVE THE INTEGRAL

$f(x) = 2x - x^2$

$g(x) = x^2$

$a = 0$

$b = 1$

AREA $\int_0^1 (2x - x^2) - x^2 \, dx$

$$\int_0^1 (2x - x^2) - x^2 \, dx$$

$$= \int_0^1 \left(2x - 2x^2\right) dx = x^2 - \frac{2x^3}{3} \Big]_0^1$$

$$= 1 - \frac{2 \cdot 1^3}{3}$$

$$= 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

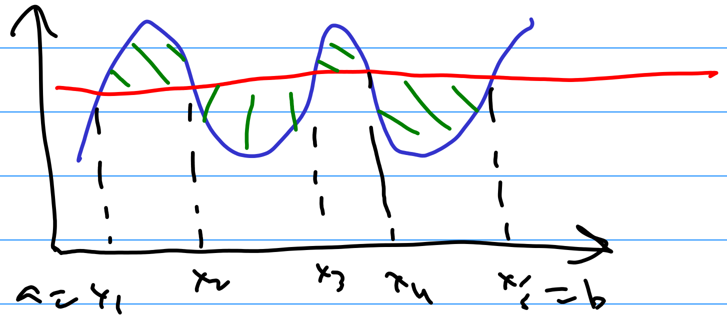
AREA =

N.B.: ALL PREVIOUS EXAMPLES HAD $f(x) \geq g(x)$ ON $[a,b]$

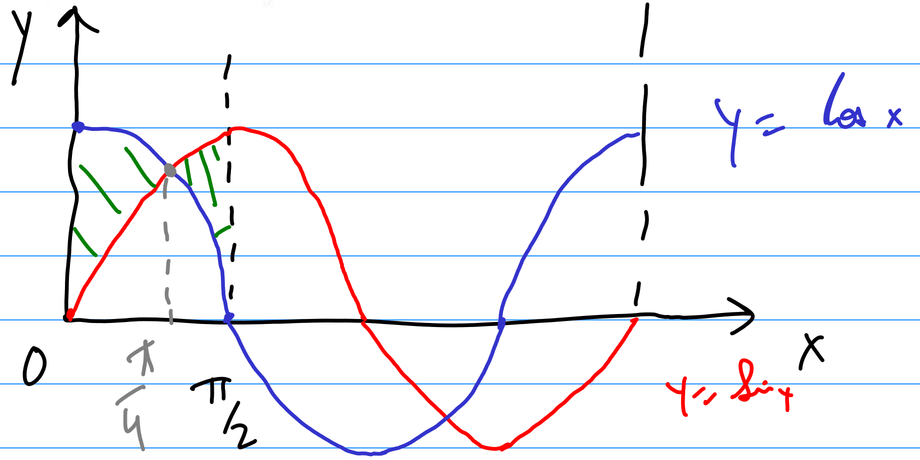
$$\int_a^b (f(x) - g(x))\,dx$$

3 The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$



EXAMPLE 4 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.



$x = \pi/4$, SIGN FLIPS

$$\int_a^b |f(x) - g(x)|\,dx$$

$f(x) = \cos x$

$g(x) = \sin x$

$a = 0,\ b = \pi/2$

$$\int_0^{\pi/2} |\cos x - \sin x|\,dx$$

b/w $\pi/4$ & $\pi/2$

? $\cos x - \sin x < 0$

? $\cos x - \sin x > 0$

b/w 0 & $\pi/4$

$$\int_0^{\pi/2} |\cos x - \sin x| \, dx = \int_0^{\pi/4} |\cos x - \sin x| \, dx + \int_{\pi/4}^{\pi/2} |\cos x - \sin x| \, dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$$

$$= \Big(\sin x + \cos x\Big]_0^{\pi/4} + \Big(-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left(\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right) + \left(-\cos \tfrac{\pi}{2} - \sin \tfrac{\pi}{2}\right) + \left(\cos \tfrac{\pi}{4} + \sin \tfrac{\pi}{4}\right)$$

b/w $\pi/4$ & $\pi/2$

? $\cos x - \sin x < 0$

? $\cos x - \sin x > 0$

b/w 0 & $\pi/4$

$v > 0$ $\quad |v| = v$

$v < 0$ $\quad |v| = -v$

$$\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right) + \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) + \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

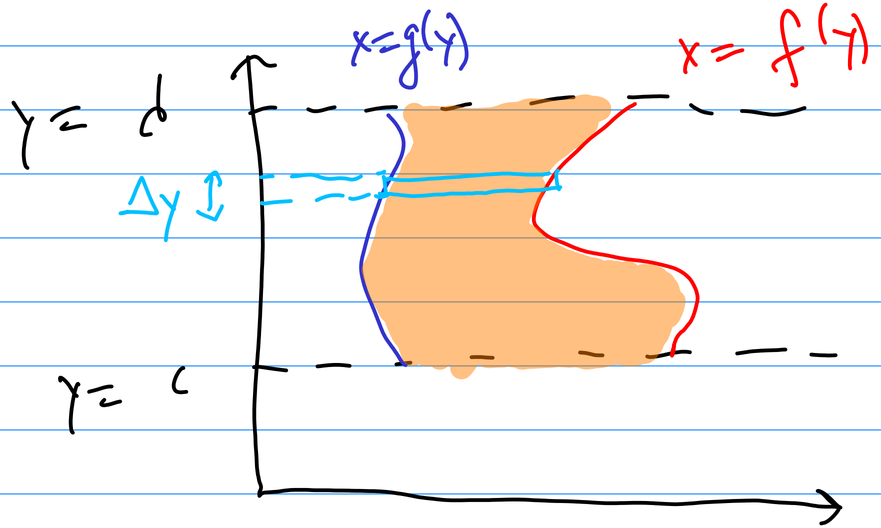
$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(0 + 1\right) + \left(-0 - 1\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\frac{4}{\sqrt{2}} - 2 = \frac{2 \cdot 2}{\sqrt{2}} - 2 = 2\sqrt{2} - 2$$

$$= 2(\sqrt{2} - 1)$$

AREA

INTEGRATING WITH RESPECT TO y



$$\sum_{j=1}^{n} \left[f(y_j) - g(y_j) \right] \Delta y$$

$$\xrightarrow[\Delta y \to 0]{n \to \infty} \int_{c}^{d} \left[f(y) - g(y) \right] dy$$

EXAMPLE 5 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

c/d

$y = x - 1$ & $y^2 = 2x + 6$

$x = y + 1$

(x, y) LIES ON BOTH

$y^2 = 2(y+1) + 6$

$\Rightarrow \quad y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0 \quad \Rightarrow \quad y = -2, \; y = 4$

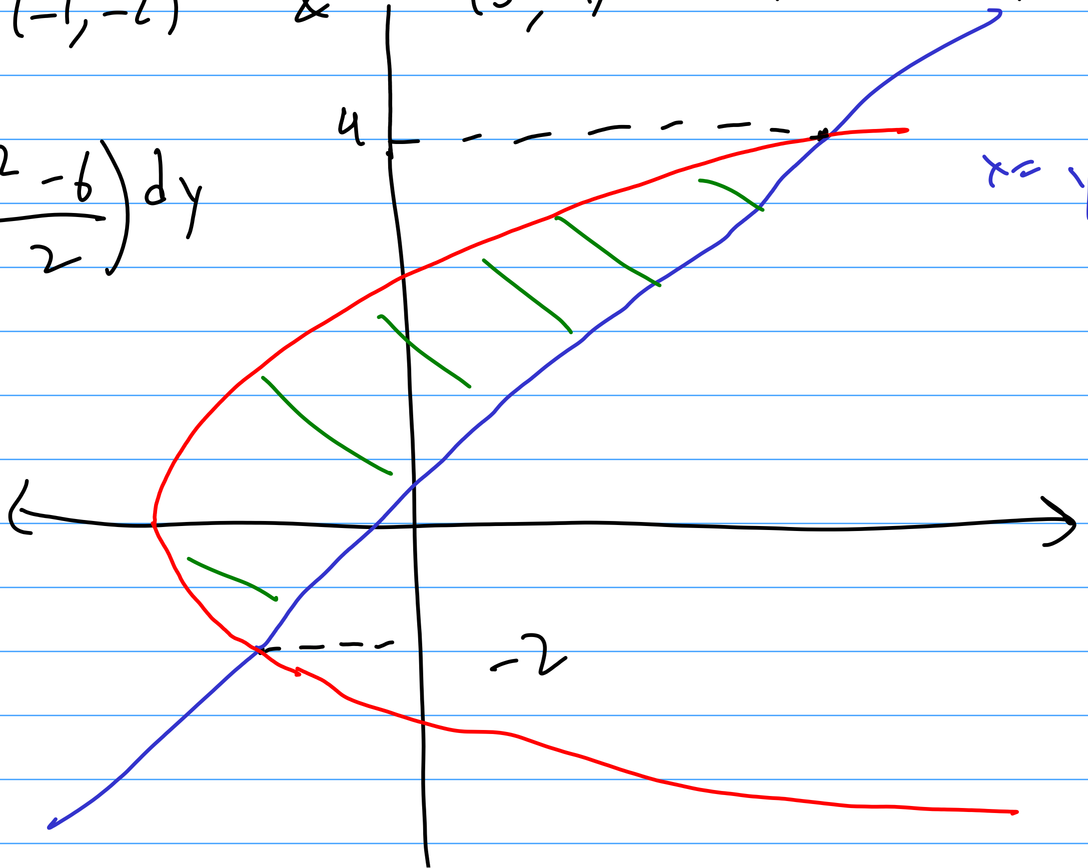
EXAMPLE 5 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$(-1,-2)$ & $(5,4)$ ARE ON BOTH CURVES

$y = x - 1$

$x = y + 1$

$$\int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy$$



$d = 4$

$c = -2$

$f(y) = y + 1$

$g(y) = \frac{y^2-6}{2}$

$y^2 = 2x + 6$

$x = \frac{y^2-6}{2}$

AREA DESIRED $= \displaystyle\int_{-2}^{4} \left[(y+1) - \left(\frac{y^2 - 6}{2} \right) \right] dy$

STEPS:

1. DRAW A PICTURE
2. IDENTIFY AREA
3. SET UP INTEGRAL (*)
4. SOLVE THE INTEGRAL

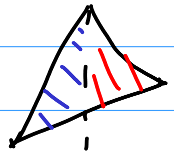
TRY YOURSELF

BACK AT

7 PM ET.

$$\int_a^b (f(x) - g(x))\,dx$$

BREAKOUT ROOM #1



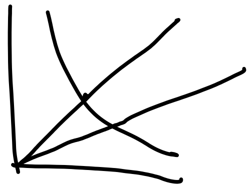
EXAMPLE 6 Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = \frac{1}{4}x$, using (a) x as the variable of integration and (b) y as the variable of integration.

[IN THE FIRST QUADRANT]

→ DRAW THE RIGHT PICTURE

→ IDENTIFY THE RIGHT AREA

$x \to$

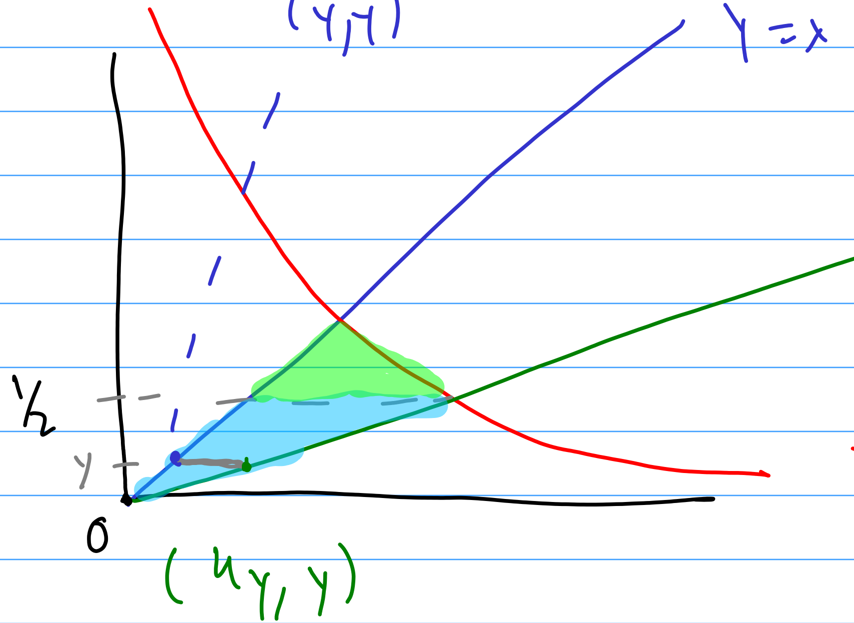


$$\int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

$$\int_{1/2}^{1} \left(\frac{1}{y} - y\right) dy$$

$$\int_0^{1/2} \left(\underset{\substack{\uparrow \\ f(y)}}{4y} - \underset{\substack{\uparrow \\ g(y)}}{y}\right) dy$$

$c = 0,\ d = 1/2$



$y = \frac{x}{4}, \quad x = 4y$

$xy = 1, \quad y = \frac{1}{x}, \quad x = \frac{1}{y}$

$$\text{ANSWER} = \frac{3}{8} + \ln 2 - \frac{3}{8} = \ln 2$$

$$\int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

$$= \int_0^1 \frac{3x}{4}\, dx + \underbrace{\int_1^2 \frac{1}{x}\, dx}_{\ln x \Big]_1^2 = \ln 2} - \int_1^2 \frac{x}{4}\, dx$$

$$\frac{3x^2}{8}\Big]_0^1 = \frac{3}{8}$$

$$\frac{x^2}{8}\Big]_1^2 = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$$