

## MTH 142 Exam 1 Review Sheet

**Curve Sketching:** When asked to sketch the graph of  $f(x)$ ...

**Before Differentiating:** -  $y$ -intercept: Plug in  $x = 0$

-  $x$ -intercept(s): Set  $f(x) = 0$  and solve for  $x$

- Vertical Asymptotes: If  $f(x)$  is written as a quotient  $p(x)/q(x)$ , then  $f$  has a vertical asymptote at  $x = a$  if  $q(a) = 0$  and  $p(a) \neq 0$ .

- Horizontal Asymptotes:  $f$  has horizontal asymptotes at  $y = \lim_{x \rightarrow \infty} f(x)$  (to the right) and  $y = \lim_{x \rightarrow -\infty} f(x)$  (to the left), if these limits exist.

**Sign Chart for  $f'(x)$ :** After computing and simplifying  $f'(x)$ , locate points where it is zero or doesn't exist (including the vertical asymptotes), and plot them on a number line. In each interval between the plotted points, test to see if  $f'$  is positive or negative.

Remember that  $f$  is INCREASING on the intervals where  $f'$  is positive, and DECREASING on the intervals where  $f'$  is negative. You can also identify local minima (valleys) and local maxima (peaks) by drawing arrows on your sign chart as we have in class.

**Sign Chart for  $f''(x)$ :** After computing and simplifying  $f''(x)$ , locate points where it is zero or doesn't exist (including the vertical asymptotes), and plot them on a number line. In each interval between the plotted points, test to see if  $f''$  is positive or negative.

Remember that  $f$  is CONCAVE UP on the intervals where  $f''$  is positive, and CONCAVE DOWN on the intervals where  $f''$  is negative. The points where  $f$  is defined (so vertical asymptotes don't count) and the concavity changes are called **inflection points**.

**Drawing the Curve:** After preparing all of the above information and plotting intercepts, local mins/maxes, inflection points, and asymptotes, draw the curve from left to right, recalling the four possible curve shapes. You want to emphasize local extrema, inflection points, and vertical asymptotes when you plot them, because those are the places that the shape can change.

### Applied Optimization

**Step 0:** Draw a picture if possible and if one is not provided for you.

**Step 1:** Identify variables and constraints.

**Step 2:** Express the quantity to be optimized in terms of the other variables.

**Step 3:** Use the constraints to reduce the function to one variable, and determine the domain.

**Step 4:** Determine the appropriate minimum or maximum of the function on the domain by finding critical points and making a sign chart for the derivative.

**Step 5:** ANSWER THE RIGHT QUESTION!

Best advice: Do lots of examples!

## Antiderivatives/Indefinite Integrals

An **antiderivative** of a function  $f(x)$  is A FUNCTION WHOSE DERIVATIVE IS  $f(x)$ . For example,  $x^3$  is an antiderivative of  $3x^2$ , but so is  $x^3 + 1$ , and  $x^3 + 19$  and  $x^3 - \pi$ .

We use the symbol  $\int f(x) dx$ , called the **indefinite integral** of  $f(x)$ , to represent the COLLECTION OF **ALL** ANTIDERIVATIVES OF  $f(x)$ . It's a consequence of the Mean Value Theorem that if  $F(x)$  is one antiderivative of  $f(x)$ , then all antiderivatives of  $f(x)$  have the form  $F(x) + C$  for some constant  $C$ . This can all be summarized with the following:

$$(1) \quad \int f(x) dx = F(x) + C \quad \text{means} \quad F'(x) = f(x).$$

Going back to our example, we would say  $\int 3x^2 dx = x^3 + C$ , thinking of both sides of this equation as COLLECTIONS OF FUNCTIONS.

**Side Note:** Don't be mentally bound to using specific letters in specific contexts. The choices for what to call variables are totally arbitrary, and if you get too locked in, you might get thrown if you see different letters or symbols. Again referring to the same example, the problem would be no different if it were  $\int 3t^2 dt = t^3 + C$ .

Remember, whenever you're asked to find an indefinite integral, or even just a single antiderivative of a function, you're answering the question "WHOSE DERIVATIVE IS THIS??"

Using (1), any fact that you know about derivatives can be translated into a fact about antiderivatives. In particular, we have the following facts that come instantly from derivatives you already know:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1, \text{ which comes from the power rule for derivatives}$$

$$\int \frac{1}{x} dx = \ln|x| + C, \quad \text{because } \frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$\int e^x dx = e^x + C, \quad \text{because } \frac{d}{dx}(e^x) = e^x$$

$$\int \cos(x) dx = \sin(x) + C, \quad \text{because } \frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\int \sin(x) dx = -\cos(x) + C, \quad \text{because } \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\int \sec^2(x) dx = \tan(x) + C, \quad \text{because } \frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\int \sec(x) \tan(x) dx = \sec(x) + C, \quad \text{because } \frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$$

$$\int \csc(x) \cot(x) dx = -\csc(x) + C, \quad \text{because } \frac{d}{dx}(\csc(x)) = -\csc(x) \cot(x)$$

$$\int \csc^2(x) dx = -\cot(x) + C, \quad \text{because } \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C, \quad \text{because } \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C, \quad \text{because } \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{x^2+1}$$

**Linearity of Indefinite Integrals:** Indefinite integrals obey the following two properties, because derivatives do. Don't worry too much about these, because if we didn't specifically say that integrals worked this way, you would probably just assume they did:

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx,$$

$$\int cf(x) dx = c \int f(x) dx \quad \text{for any constant } c.$$

These properties allow us to do problems like:

$$\begin{aligned} \int 12x^5 - 7\sqrt{x} + \frac{3}{x} - \frac{9}{x^4} dx &= \int 12x^5 - 7x^{1/2} + \frac{3}{x} - 9x^{-4} dx \\ &= 12\frac{x^6}{6} - 7\frac{x^{3/2}}{(3/2)} + 3\ln|x| - 9\frac{x^{-3}}{-3} + C \\ &= 2x^6 - \frac{14}{3}x^{3/2} + 3\ln|x| + \frac{3}{x^3} + C \end{aligned}$$

Notice how the power rule works for integrals: You first RAISE THE EXPONENT BY 1, then you DIVIDE BY THE NEW EXPONENT.

**REMEMBER:** Antiderivatives or indefinite integrals can be more difficult to find than derivatives, but the good news is that they come with a built-in checking mechanism. If you're trying to find an antiderivative of  $f(x)$ , and you think you have the answer, take your guess and DIFFERENTIATE IT. If the result is  $f(x)$  (the function you started with), then it's right! If not, try again.

## Riemann Sums and Definite Integrals

We use the symbol  $\int_a^b f(x) dx$ , called a **definite integral**, to represent the “signed area” between the curve  $f(x)$  and the  $x$ -axis, over the interval  $[a, b]$ . In other words, looking at the graph of  $f(x)$  on  $[a, b]$ , you take the area that lies ABOVE the  $x$ -axis, and then SUBTRACT the area that is BELOW the  $x$ -axis.

Note that the only difference in the symbols for indefinite integrals and definite integrals is the presence of the upper and lower “limits of integration” indicating the endpoints of the interval, yet (at first glance) the definitions of the two are completely different.

Initially, in order to give this “signed area” a rigorous definition, we defined it as the common limit of different approximations, called **Riemann sums**, using more and more rectangles. Recall the following approximation formulas for the signed area for  $f(x)$  on  $[a, b]$ :

$$R_n = \text{Right endpoint approximation with } n \text{ rectangles} = \frac{b-a}{n} \sum_{i=1}^n f\left(a + \frac{b-a}{n}i\right)$$

$$L_n = \text{Left endpoint approximation with } n \text{ rectangles} = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n}i\right)$$

$$M_n = \text{Midpoint approximation with } n \text{ rectangles} = \frac{b-a}{n} \sum_{i=1}^n f\left(a + \frac{b-a}{n}(i-1/2)\right)$$

We can compute these sums for certain functions using the following summation formulas:

$$\sum_{i=1}^n c = cn \text{ for any constant } c, \quad \sum_{i=1}^n i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

For continuous functions, all these Riemann sums converge to the same thing as  $n \rightarrow \infty$ , which allows us to define the definite integral

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} M_n.$$

### **Properties of the Definite Integral**

Linearity:  $\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx, \int_a^b cf(x) dx = c \int_a^b f(x) dx$

Reversing Limits of Integration:  $\int_b^a f(x) dx = - \int_a^b f(x) dx$

Additivity on Adjacent Intervals: For any numbers  $a, b$ , and  $c$ ,  $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$ .

There turns out to be a great shortcut for computing definite integrals using antiderivatives, which explains why the symbols are so similar. This shortcut comes from the Fundamental Theorem of Calculus, which establishes the connection between differentiation and definite integration.

## Fundamental Theorem of Calculus (FTC)

**Part I:** If  $f(x)$  is continuous and  $F(x) = \int_a^x f(t) dt$  for some number  $a$ , then  $F'(x) = f(x)$ .

Using the chain rule, additivity on adjacent intervals, and reversing limits of integration, we can generalize Part I and obtain the formula:

$$\frac{d}{dx} \left( \int_{h(x)}^{g(x)} f(t) dt \right) = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x).$$

**Examples:**  $\frac{d}{dx} \left( \int_5^x \sqrt{t^3 - 1} dt \right) = \sqrt{x^3 - 1}$ ,  $\frac{d}{ds} \left( \int_0^{\tan(s)} e^{3u} du \right) = e^{3 \tan(s)} \sec^2(s)$ ,

$$\frac{d}{dv} \left( \int_{e^{9v}}^{e^{v^2+1}} \cos(r) dr \right) = \cos(e^{v^2+1})e^{v^2+1}(2v) - \cos(e^{9v})(9).$$

**Part II:** If  $f(x)$  is continuous on  $[a, b]$  and  $F'(x) = f(x)$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Basically, Part II says that in order to compute a definite integral, all you need to do is find one antiderivative, then plug in the two endpoints of the interval and subtract.

**Examples:**  $\int_1^8 4x + 1 - \frac{6}{\sqrt[3]{x}} dx = \int_1^8 4x + 1 - 6x^{-1/3} dx = \left[ 4 \frac{x^2}{2} + x - 6 \frac{x^{2/3}}{(2/3)} \right]_1^8 = \left[ 2x^2 + x - 9x^{2/3} \right]_1^8$   
 $= \left( 2(8)^2 + 8 - 9(8)^{2/3} \right) - \left( 2(1)^2 + 1 - 9(1)^{2/3} \right) = 128 + 8 - 36 - 2 - 1 + 9 = 106$

$$\int_0^{\pi/12} 6 \sec^2(3\theta) d\theta = \left[ 6 \frac{\tan(3\theta)}{3} \right]_0^{\pi/12} = \left[ 2 \tan(3\theta) \right]_0^{\pi/12} = 2 \tan(\pi/4) - 2 \tan(0) = 2.$$

$$\int_1^5 \frac{1}{s} ds = \left[ \ln |s| \right]_1^5 = \ln(5) - \ln(1) = \ln(5).$$

## Definite Integrals by Elementary Means

Recall that for some functions, definite integrals can be computed just by using basic geometry to compute the signed area. For example, if  $f(x) = mx + b$  is a straight line, then the area between the graph and the  $x$ -axis can be decomposed into triangles and/or rectangles. Also if  $f(x) = \sqrt{r^2 - x^2}$ , the graph is a semicircle of radius  $r$ . Even more simple is the definite integral of a constant function, which amounts to a single rectangle, in general

$$\int_a^b C dx = C(b - a).$$

## Initial Value Problems

Recall that since all the antiderivatives for a specific function only differ by a constant, you can completely determine a function from its derivative and its value at one point.

For example, if we know that  $f'(x) = 3x^2$ , then we know that  $f(x) = x^3 + C$  for some constant  $C$ . If, in addition, we know that  $f(2) = 5$ , then we can use that information to SOLVE FOR  $C$ , since  $f(2) = 8 + C = 5$ , hence  $C = -3$  and  $f(x) = x^3 - 3$ .

We also did examples where we determined a function from its second derivative, the value of its derivative at one point, and the value of the function at one point, which essentially allows us to do the process outlined above twice in a row to get the answer. For example, if we know  $g''(x) = \cos(x)$ ,  $g'(0) = 3$ ,  $g(0) = 4$ , then we integrate once to get  $g'(x) = \sin(x) + C$ , use that  $g'(0) = 3$  to find that  $C = 3$ , integrate again to get  $g(x) = -\cos(x) + 3x + C$ , then use that  $g(0) = 4$  to find that  $C = 5$ , yielding the final answer  $g(x) = -\cos(x) + 3x + 5$ .

## Physics Problems

If the position of an object traveling on a straight line (such as the height of a rising/falling object) is denoted by  $s(t)$ , then the derivative  $s'(t)$  is the **velocity** and is denoted by  $v(t)$ , and the second derivative  $s''(t) = v'(t)$  is the **acceleration** and is denoted by  $a(t)$ .

We paid special attention to the case of a rising/falling object, where we made the assumption that the only force acting on the object is gravity, and hence the acceleration is constant  $a(t) = -g$ , where  $g \approx 32\text{ft/s}^2 \approx 9.8\text{m/s}^2$ . Integrating twice like in the section above, we get the formula

$$s(t) = -\frac{g}{2}t^2 + v_0t + s_0,$$

where  $v_0 = v(0)$  is initial velocity and  $s_0 = s(0)$  is initial height. (Note: you could also let  $g$  denote the corresponding negative quantity, e.g.  $g \approx -32\text{ft/s}^2$ , and leave the negative sign out of the formula, it's no different, just remember that for a rising/falling object the acceleration should be a negative number).

More generally, in any situation where the acceleration is constant  $a(t) = a$ , we have the formula

$$s(t) = \frac{a}{2}t^2 + v_0t + s_0.$$

Also recall that if an object moves in one dimensional motion with velocity  $v(t)$  from  $t = a$  to  $t = b$ , then we have the formulas

$$\text{Displacement} = \int_a^b v(t) dt$$

and

$$\text{Total Distance Traveled} = \int_a^b |v(t)| dt.$$

For the latter calculation, you need to determine where  $v(t)$  is positive versus negative, and integrate each piece individually.

## Techniques of Integration

### - **Substitution** (Reverse Chain Rule)

$$\text{Formula: } \int f(u(x))u'(x) dx = \int f(u) du$$

When determining what to pick for  $u$ , look for a function that is *inside* of another function, and whose derivative is present somewhere in the integrand. Remember that “inside” can mean a lot of things, including in a denominator or an exponent.

Also, remember that when using substitution for a DEFINITE integral, you should CHANGE YOUR LIMITS OF INTEGRATION by plugging in your original limits into your formula for  $u$ . The nice thing about definite integrals is that the answer is just a number, so once you properly substitute and change your limits, you can proceed with your new integral and totally leave the original function and variable behind.