

MATH 142 , SUMMER 2021 (A2)

MIDTERM 1
SOLUTIONS

2. (20 points)

(a) Find the vertical and horizontal asymptotes of

$$g(x) = \frac{9x^3}{(x-1)(x^2-5x+6)}$$

VERTICAL : NEED TO CHECK WHERE DENOMINATOR
VANISHES.

$$(x-1)(x^2-5x+6) = 0 \Rightarrow (x-1)(x-2)(x-3) = 0$$

$\Rightarrow x = 1, x = 2$ & $x = 3$ ARE | ASYMPTOTES

HORIZONTAL:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{9x^3}{(x-1)(x^2-5x+6)} &= \lim_{x \rightarrow \infty} \frac{9}{\left(1 - \frac{1}{x}\right)\left(1 - \frac{5}{x} + \frac{6}{x^2}\right)} \\ &= \frac{9}{(1-0)(1-0+0)} = 9\end{aligned}$$

||| by FOR $x \rightarrow -\infty$.

$\therefore y = 9$ IS A HORIZONTAL ASYMPTOTE

(b) Does the following function have any symmetry? If so, what kind?

$$h(x) = e^{x^2} + \frac{\sin x}{x} + \frac{x^6}{x^4 + 1}$$

OBSERVE THAT

$$h(-x) = e^{(-x)^2} + \frac{\sin(-x)}{-x} + \frac{(-x)^6}{(-x)^4 + 1}$$

$$= e^{x^2} + \frac{\sin x}{x} + \frac{x^6}{x^4 + 1} = h(x)$$

FOR
ALL x

$\Rightarrow h$ IS EVEN

- (c) Find the intervals of increase and decrease for the following function. Also, determine the points at which the function has a local maximum and local minimum.

$$f(x) = xe^{-6x^2}$$

$$f'(x) = \frac{d}{dx} (x e^{-6x^2})$$

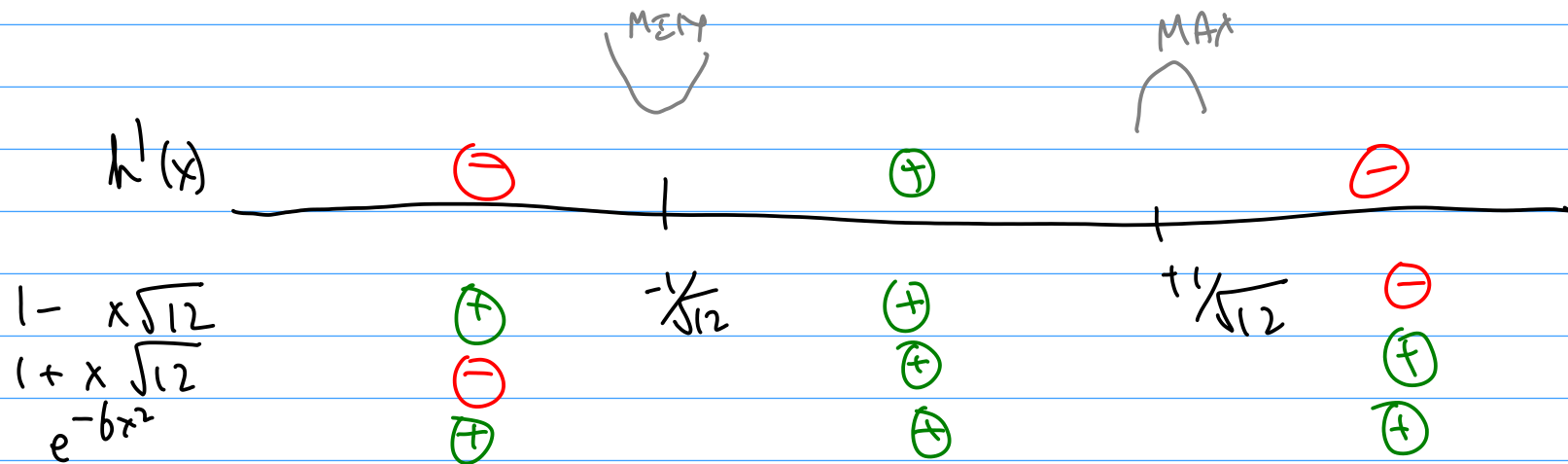
$$= \left[\frac{d}{dx} (x) \right] e^{-6x^2} + x \left[\frac{d}{dx} e^{-6x^2} \right] \quad (\because \text{PRODUCT})$$

$$= e^{-6x^2} + x \left[e^{-6x^2} \frac{d}{dx} (-6x^2) \right] \quad (\because \text{CHAIN})$$

$$= e^{-6x^2} + x (-12x e^{-6x^2}) = (1 - 12x^2) e^{-6x^2}$$

$$h'(x) = 0 \quad \Rightarrow \quad (1 - 12x^2) e^{-6x^2} = 0$$

$$\Rightarrow 1 - 12x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{12}}$$



\therefore LOCAL MAX AT $x = \frac{1}{\sqrt{2}}$

" MIN " $x = -\frac{1}{\sqrt{2}}$

INCREASE : $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

DECREASE : $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

(d) For the same $f(x)$ as in the previous part, find the intervals on which the function is concave up and concave down, and find the points of inflection.

$$f'(x) = (1 - 12x^2) e^{-6x^2}$$

$$\Rightarrow f''(x) = \frac{d}{dx} \left[(1 - 12x^2) e^{-6x^2} \right]$$

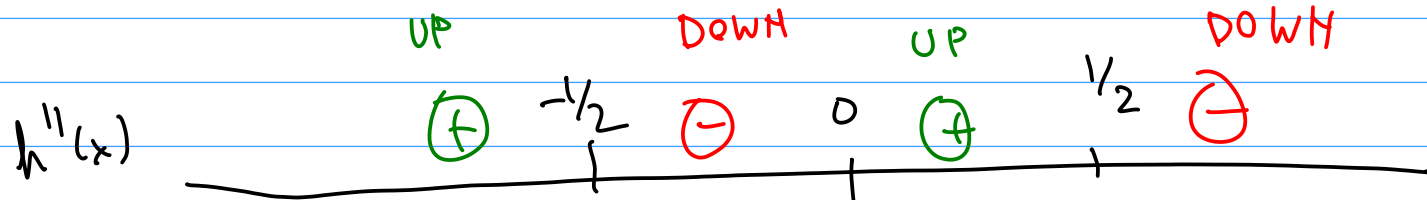
$$= \left[\frac{d}{dx} (1 - 12x^2) \right] e^{-6x^2} + (1 - 12x^2) \left[\frac{d}{dx} e^{-6x^2} \right] \quad [\because \text{PRODUCT}]$$

$$= -24x e^{-6x^2} + (1 - 12x^2) (-12x e^{-6x^2})$$

$$= e^{-6x^2} (-36x + 144x^3) = 36x e^{-6x^2} (1 - 4x^2)$$

$$\Rightarrow f''(x) = 36x(1-2x)(1+2x)e^{-6x^2}$$

$$\therefore f'' = 0 \quad \text{AT} \quad x = 0, \quad x = \frac{1}{2}, \quad x = -\frac{1}{2}$$

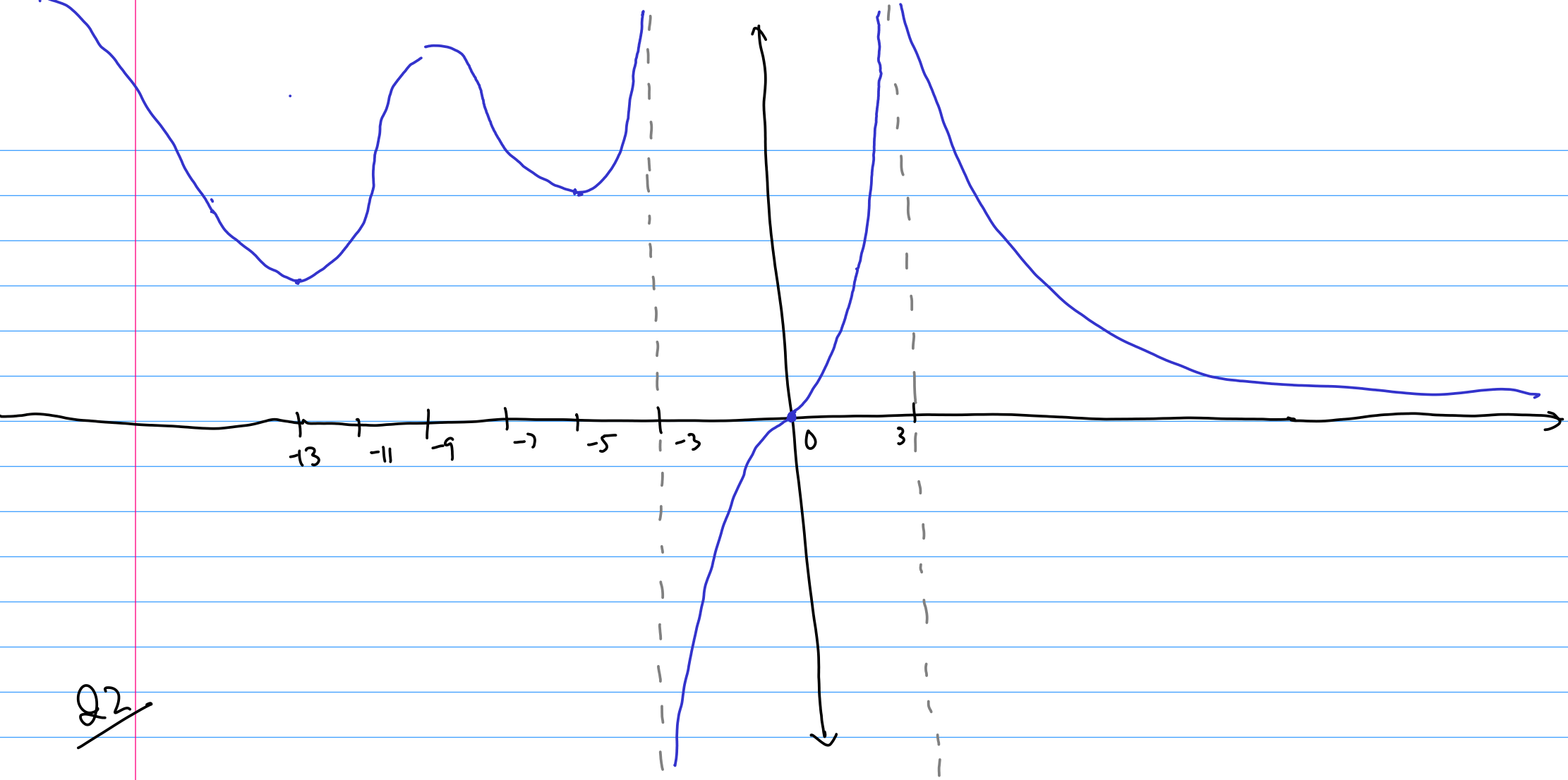


$36x$	$-$	$-$	$+$	$+$
$1-2x$	$+$	$+$	$+$	$-$
$1+2x$	$-$	$+$	$+$	$+$
e^{-6x^2}	$+$	$+$	$+$	$+$

POINTS OF INFLECTION = $-\frac{1}{2}, 0, \frac{1}{2}$

CONCAVE UP $\rightsquigarrow (-\infty, -\frac{1}{2}) \cup (0, \frac{1}{2})$

CONCAVE DOWN $\rightsquigarrow (-\frac{1}{2}, 0) \cup (\frac{1}{2}, \infty)$



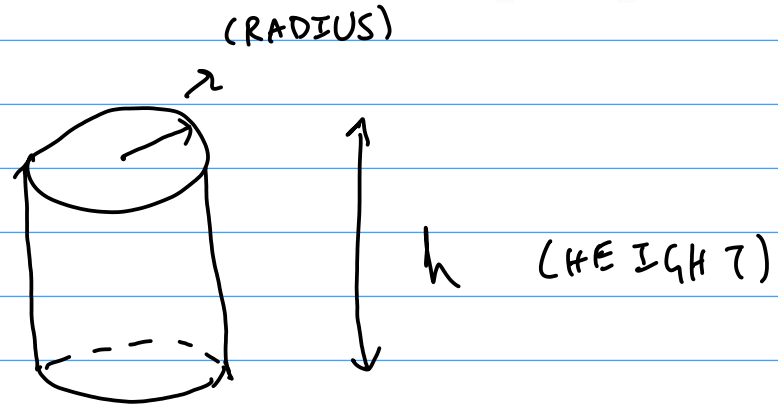
Q2

4. (15 points) The sides of a cylindrical container with an open top are to be made of metal, while the (circular) base is to be made of wood. The sheet of metal used costs $\$3\pi$ per square centimeter while the wood costs $\$1$ per square centimeter.

If the container needs to have volume 9 cubic centimeters, what dimensions (i.e. what radius and what height) should one choose to minimize the cost of manufacturing the cylinder?

DIAGRAM :

C (cost)



$$\text{VOLUME} = \pi r^2 h = 9 \Rightarrow h = \frac{9}{\pi r^2} ;$$

$$C = (3\pi) * \text{CURVED AREA} + 1 * (\text{BASE AREA})$$

$$= (3\pi) (2\pi r h) + \pi r^2$$

$$= 6\pi^2 r h + \pi r^2$$

$$= (6\pi^2 r) \left(\frac{9}{\pi r^2} \right) + \pi r^2 \quad \left[\because h = \frac{9}{\pi r^2} \right]$$

$$C(r) = \frac{54\pi}{r} + \pi r^2$$

$$\Rightarrow C'(\lambda) = -\frac{54\pi}{\lambda^2} + 2\pi\lambda = 0$$

$$\Rightarrow 2\pi\lambda = \frac{54\pi}{\lambda^2} \Rightarrow \lambda^3 = 27 \Rightarrow \lambda = 3$$

CLEARLY, $C'(\lambda) < 0$ WHEN $\lambda < 3$

& $C'(\lambda) > 0$ WHEN $\lambda > 3$

$\Rightarrow \lambda = 3$ IS THE GLOBAL MINIMIZER

$$\Rightarrow h = \frac{9}{\pi\lambda^2} = \frac{9}{\pi 3^2} = \frac{1}{\pi} \Rightarrow \lambda = 3 \text{ cm}, h = \frac{1}{\pi} \text{ cm}$$

5. (18 points) Compute the following integrals by interpreting them geometrically and using areas from high school geometry. You will get no points if you use Riemann summation or the fundamental theorem of calculus.

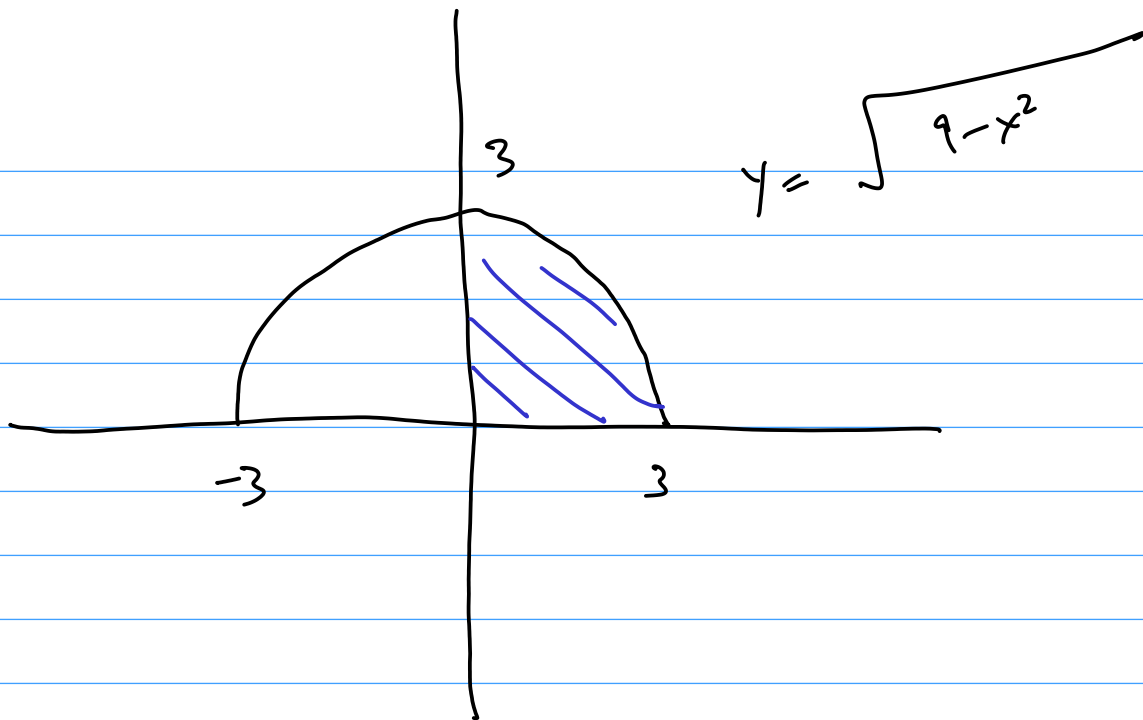
(a)

$$\int_0^3 \sqrt{9 - x^2} dx.$$

(b)

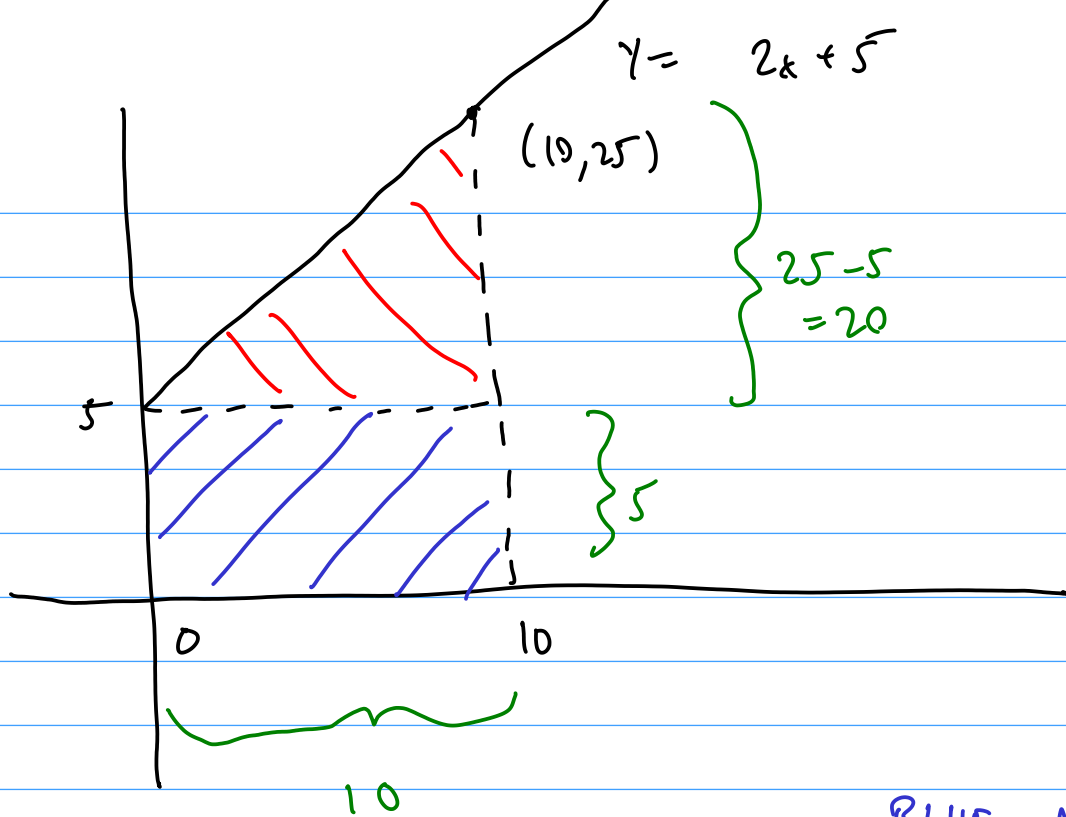
$$\int_0^{10} (2x + 5) dx.$$

a).



$$\int_0^3 \sqrt{9-x^2} = \text{BLUE AREA} = \frac{1}{4} \text{ (AREA OF CIRCLE OF RADIUS } r=3)$$
$$= \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (3^2) = \frac{9\pi}{4}$$

b)



RED AREA

$$= \frac{1}{2} (\text{BASE}) (\text{HEIGHT})$$

$$= \frac{1}{2} (10)(20)$$

$$= 100$$

BLUE AREA = (BASE)(HEIGHT)

$$= (10)(5) = 50$$

$$\int_0^{10} (2x+5) dx = \text{BLUE} + \text{RED}$$

$$= 50 + 100 = 150$$

6. (12 points) The velocity of a particle moving in a straight line at time t is given in m/s by the formula

$$v(t) = t^2 - 4t - 12$$

- (a) What is the displacement of the particle between $t = 0$ and $t = 9$?
- (b) What is the distance travelled by the particle between $t = 0$ and $t = 9$?

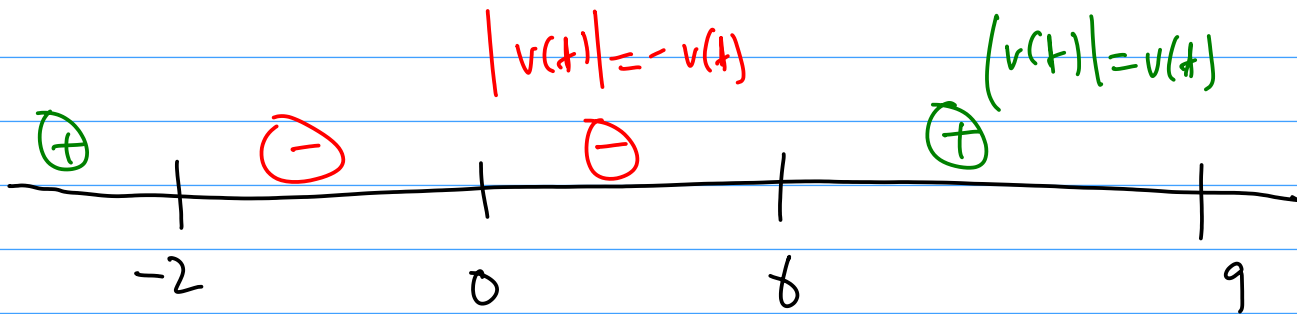
Remember to include the units in your final answer. Time is measured in s .

$$\text{DISPLACEMENT} = \int_0^9 v(t) dt = \int_0^9 (t^2 - 4t - 12) dt$$

$$= \left[\frac{t^3}{3} - \frac{4t^2}{2} - 12t \right]_0^9 = \frac{9^3}{3} - 2 \times 9^2 - 12 \times 9 = -27$$

$$\text{DISTANCE} = \int_0^9 |v(t)| dt = \int_0^9 |t^2 - 4t - 12| dt$$

$$v(t) = t^2 - 4t - 12 = (t - 6)(t + 2)$$



$$\therefore \text{DISTANCE} = \int_0^9 |v(t)| dt$$

$$= \int_0^6 |v(t)| dt + \int_6^9 |v(t)| dt$$

$$= \int_0^6 (-v(t)) dt + \int_6^9 v(t) dt$$

$$= \int_0^6 (-t^2 + 4t + 12) dt + \int_6^9 (t^2 - 4t + 12) dt$$

$$= \int_0^6 (-t^2 + 4t + 12) dt + \int_6^9 (t^2 - 4t + 12) dt$$

$$= \left[\frac{-t^3}{3} + 2t^2 + 12t \right]_0^6 + \left[\frac{t^3}{3} - 2t^2 + 12t \right]_6^9$$

$$= \frac{-6^3}{3} + 2 \times 6^2 + 12 \times 6 + \left[\frac{9^3}{3} - 2 \times 9^2 + 12 \times 9 \right] - \left[\frac{6^3}{3} - 2 \times 6^2 + 12 \times 6 \right]$$

$$= 117$$

7. (20 points) Compute the antiderivatives for the following functions.

(a)

$$\tan^2 x. = \sec^2 x - 1 \Rightarrow \int \tan^2 x dx = \int (\sec^2 x - 1) dx$$

(b)

$$\sqrt[3]{x}(x^2 + x + 1). = x^{1/3}(x^2 + x + 1) = (\ln x - x + C)$$

(c)

$$2^x + x^2.$$

$$\int (2^x + x^2) dx = \frac{2^x}{\ln 2} + \frac{x^3}{3} + C$$

$$\Rightarrow \int x^{2/3} + x^{4/3} + x^{1/3} dx = \frac{3x^{10/3}}{10} + \frac{3x^{7/3}}{7} + \frac{3x^{4/3}}{4} + C$$

(d)

$$\int \left(\frac{1}{\sqrt{1-x^2}} + \cos x \right) dx = \arcsin x + \sin x + C$$

(e)

$$\sin^2 x + \cos^2 x$$



$$= 1$$

\Rightarrow

$$\int (\sin^2 x + \cos^2 x) dx = \int dx = x + C$$

8. (20 points) Consider the following integral:

$$\int_0^3 (x-1)^2 3^x dx.$$

Write a Riemann sum (using Σ notation) for this integral in which the partition has $n = 3$ subintervals of equal length and the sample points x_j^* are the right endpoints of the subintervals. Evaluate this sum, doing all the numerical and algebraic simplifications necessary.

$$n = 3, \quad b = 3, \quad a = 0, \quad \Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$$

$$x_j^* = a + j \Delta x = 0 + j \cdot 1 = j$$

$$\therefore \text{RIGHT END-POINT APPROX} = \frac{b-a}{n} \sum_{j=1}^n f\left(a + j\left(\frac{b-a}{n}\right)\right) = \sum_{j=1}^3 f(j)$$

RIEMANN
SUM

$$= \sum_{j=1}^n (j-1)^2 3^j$$

$$= ((-1)^2 \cdot 3^1 + (2-1)^2 \cdot 3^2 + (3-1)^2 \cdot 3^3)$$

$$= 0 + 9 + 108$$

$$= 117$$

9. (20 points) Compute the following. You are allowed to use the fundamental theorem of calculus.

(a)

$$\frac{d}{dx} \int_2^{1-x^2} e^{(1-t)^2} dt. \quad \rightarrow \text{(NEXT PAGE)}$$

(b)

$$\int_0^4 (x^3 + x^{15}) dx = \left[\frac{x^4}{4} + \frac{x^{16}}{16} \right]_0^4 = \frac{4^4}{4} + \frac{4^{16}}{16} = 4^3 + 4^{14}$$

(c)

$$\int_5^{10} \frac{dx}{x}$$

$$\begin{aligned} &= \left[\ln x \right]_5^{10} = \ln 10 - \ln 5 \\ &= \ln \left(\frac{10}{5} \right) = \ln 2 \end{aligned}$$

$$\text{Let } g(x) = \int_2^x e^{(1-t)^2} dt$$

$$\therefore \text{By FTC, } g'(x) = e^{(1-x)^2}$$

WE WANT

$$\begin{aligned} \frac{d}{dx} \int_2^{1-x^2} e^{(1-t)^2} dt &= \frac{d}{dx} g(1-x^2) = \frac{d}{dx}(1-x^2) g'(1-x^2) \\ &\quad [\because \text{CHAIN}] \\ &= -2x g'(1-x^2) = -2x e^{[1-(1-x^2)]^2} = -2x e^{x^4} \end{aligned}$$

(c)

$$\int_5^{10} \frac{dx}{x} \quad (\text{SEE PREVIOUS})$$

(d)

$$\frac{d}{dx} \int_2^x (t \log t) dt. = x \log x \quad (\text{BY F.T.C.})$$