MTH 165: Linear Algebra with Differential Equations

Final Exam

May 6, 2013

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50	
Giorgis Petridis	MWF 13:00 - 13:50	
Eyvindur Ari Palsson	MW 14:00 - 15:15	

- **The presence of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your simplified final answers.**
- **You are responsible for checking that this exam has all 11 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Solve the following initial value problems:

(a) (5 points)

$$(x^2+1)\frac{dy}{dx} + 3x\,y \;=\; 6x, \quad y(0) = 3;$$

(b) (5 points)

$$y' + \frac{y}{x^2} \;=\; \frac{2}{x^2}, \quad y(1) = 1.$$

2. (10 points)

(a) (5 points) Find the value of k which satisfies

$$\det \begin{bmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1 + 5c_1 & 3b_2 + 5c_2 & 3b_3 + 5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{bmatrix} = k \cdot \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

(b) (5 points) Construct two matrices A and B of appropriate dimensions such that

$$\operatorname{rank}(AB) < \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

3. (10 points) For each of the following subsets of P_2 (i.e., the vector space of polynomials with real coefficients and degree at most 2) determine whether it is a subspace. If that is the case, find its dimension.

(a) (5 points) S is the set of polynomials $p \in P_2$ satisfying

$$p'(x) + p(x) \;=\; x^2.$$

(b) (5 points) S is the set of polynomials $p \in P_2$ verifying

$$p(x) \;+\; p(-x) \;=\; 0.$$

4. (10 points) Compute the reduced row-echelon form for the matrix

$$\begin{bmatrix} 7 & 4 & 1 & 7 \\ 4 & 3 & 2 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

and deduce from there a basis and the dimension of its row space.

5. (10 points) Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 square matrices with real entries. Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ c & a+d \end{pmatrix}.$$

(a) (3 points) Show that T is a linear transformation.

(b) (3 points) Determine a basis for $\mathrm{Ker}(T)$. What is $\dim[\mathrm{Ker}(T)]$?

(c) (2 points) Find $\dim[\mathrm{Rng}(T)]$.

(d) (2 points) Is the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ in the range of T? Explain.

6. (10 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) (4 points) Determine its eigenvalues and their multiplicities.

(b) (4 points) Compute the eigenspaces corresponding to each of the eigenvalues and their dimensions.

(c) (2 points) Conclude, with explanation, whether A is a defective or non-defective matrix.

7. (10 points) Solve the initial value problem

$$y''' - 4y'' + 5y' - 2y = 0, \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = 5.$$

8. (10 points) Determine the general solution to

$$y'' + 4y' + 4y = x\,e^{-x}.$$

9. (10 points) Find the general solution to

$$y'' + 2y' = e^{-x} + x.$$

10. (10 points) Solve the initial value problem

$$\begin{cases} x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2, \\ x_1(0) = 3, \quad x_2(0) = 1. \end{cases}$$