MTH 165: Linear Algebra with Differential Equations

Final Exam

May 5, 2014

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Indicate your instructor with a check in the box:

Friedmann	MW 16:50 - 18:05	
Karapetyan	MW 14:00-15:15	
Petridis	MWF 10:00 - 10:50	

- You have 3 hours to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed.
- Show all your work and simplify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 11 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: Obtaining an examination prior to its administration. Using unauthorized aid during an examination or having such aid visible to you during an examination. Knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.

1. (10 points) Find the explicit solution to the following initial value problem showing all your work

$$y' - x\,y^2 = x\,,\ y(0) = 1\,.$$

2. (10 points) (i) Let A be an invertible $n \times n$. Is it true that the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$? It it is, carefully explain why. If it is not provide an explicit example that disproves the claim.

(ii) Is there a system of three distinct linear equations in two unknowns that has a unique solution? If there is, provide an explicit example. If there is not, explain carefully why this is the case.

(iii) Is there a system of two distinct linear equations in three unknowns that has a unique solution? If there is, provide an explicit example. If there is not, carefully explain why this is the case.

3. (10 points) Let A and B be the following matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ and } B = -2 \begin{bmatrix} a_1 + c_1 & c_1 & -1 & b_1 \\ a_2 + c_2 & c_2 & 3 & b_2 \\ 0 & 0 & 2 & 0 \\ a_3 + c_3 & c_3 & 7 & b_3 \end{bmatrix}.$$

Find $\det(B)$ in terms of $\det(A)$ showing all your work.

4. (10 points) Let $M_2(\mathbb{R})$ be the vector space of 2×2 matrices with real components, 0_2 be the zero 2×2 matrix, B be the following matrix

$$B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix},$$

and S be the following subset of $M_2(\mathbb{R})$

$$S = \{A \in M_2(\mathbb{R}) : AB = 0_2\}.$$

Prove that S is a subspace of $M_2(\mathbb{R})$; find a basis for it; and determine its dimension.

5. (10 points) Let A be the following matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

Define a linear transformation T by $T\mathbf{x} = A\mathbf{x}$.

(i) The kernel of T is a subspace of $\mathbb{R}^d$ for what value of d?

$d =$

(ii) Find a basis for the kernel of T showing all your work.

(iii) The range (or image) of T is a subspace of $\mathbb{R}^d$ for what value of d?

$d =$

(iv) What is the dimension of the range of T? Justify your answer.

6. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(i) Determine its eigenvalues and their (algebraic) multiplicities showing all your work.

(ii) Find a basis for each eigenspace showing all your work.

(iii) Conclude, with explanation, whether A is a defective or non-defective matrix.

7. (10 points) Find the general solution to each of the following differential equations showing all your work.

(i) $t^2\, y'' \,-\, 8\,t\,y' \,+\, 20\,y \,=\, 0\,.$

[Hint: try powers of t as solutions.]

(ii) $y^{(4)} \,+\, 4\,y^{(3)} \,+\, 10\,y'' \,+\, 12\,y' \,+\, 9\,y \,=\, 0\,.$

[Hint: $r^4 + 4r^3 + 10r^2 + 12r + 9 = (r^2 + 2r + 3)^2$.]

8. (10 points) Find the general solution to the following differential equation showing all your work.

$$y''' - y'' = 12\,x^2\,.$$

9. (10 points) A spring with spring constant 8 N/m is loaded with a 2 kg mass and allowed to reach equilibrium. It is then displaced and released. Suppose that after $\pi/2$ seconds the mass is 1 m below the equilibrium position and moving upward with speed $2\sqrt{3}$ m/s.

Find the equation of motion of the displacement $y(t)$ from the equilibrium position, the amplitude, and the phase. Neglect friction. Show all your work.

10. (10 points) Solve the following initial value problem showing all your work.

$$\begin{aligned} x_1' &= x_1 + 3x_2, \quad x_1(0) = 2 \\ x_2' &= -3x_1 + x_2, \quad x_2(0) = 3. \end{aligned}$$