MTH 165: Linear Algebra with Differential Equations

Final Exam

May 4, 2015

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Indicate your instructor with a check in the box:

Dummit	TR 16:50-18:05	
Friedmann	MW 16:50-18:05	
Petridis	MWF 10:25-11:15	
Rice	MW 14:00-15:15	

- You have 3 hours to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 14 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Find a solution (implicit solutions are acceptable) for the following initial value problems on the domain $(0, \infty)$:

(a) $2y + xy' = x^{-1}$, $y(1) = A$.

(b) $2x + yy' = x^{-1}$, $y(1) = B$.

2. (10 points) Find a basis for the nullspace of each matrix.

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

(b) $B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$

(c) $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$

3. (10 points) Let $M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix}$, where k is a parameter.

(a) Find $\det(M)$.

(b) Find all value(s) of k such that M is not an invertible matrix.

We continue taking $M = \begin{bmatrix} k & 0 & k \\ 0 & 1 & 0 \\ 1 & 0 & k \end{bmatrix}$, where k is a parameter.

(c) Find all value(s) of k such that $\lambda = 2$ is an eigenvalue of A.

4. (10 points) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, explain why not.

(a) $V = \mathbb{R}^3$ and $S = \{(x, y, z) \in V \mid x + y = z\}$.

(b) $V = M_2(\mathbb{R})$, the set of 2×2 matrices, and $S = \{A \in V \mid A^2 = 0\}$.

5. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and determine (with justification) whether A is a defective matrix. (In other words, determine whether $\mathbb{R}^4$ has a basis consisting of eigenvectors of A.)

(b) Find the eigenvalues of A^2, and determine (with justification) whether A^2 is a defective matrix. (In other words, determine whether $\mathbb{R}^4$ has a basis consisting of eigenvectors of A^2.)

6. (10 points) Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 real matrices. Consider the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

(a) Find a basis for the kernel of T, and the dimension of the kernel.

Recall that
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

(b) Find the dimension of the range of T.

(c) Is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the range of T? Justify why or why not.

7. (10 points) Find the general solution for each differential equation:

(a) $y'' + 4y' + 4y = 0.$

(b) $y^{(4)} - y = 0.$

(c) $y''' - 2y'' + 5y' = 0.$

8. (10 points) Solve the equation

$$y'' + 4y = 4\cos(2x) + 8e^{2x}$$

with initial conditions $y(0) = 3$, $y'(0) = 4$.

9. (10 points) Consider a spring-mass system with spring constant $k = 4\,\mathrm{N/m}$ and a mass $m = 1\,\mathrm{kg}$.

(a) Suppose there is no friction (or damping), and an external driving force of $6\sin(4t)\,\mathrm{N}$ is applied to the mass (in the positive direction). If at time $t = 0$ the mass is at rest in the equilibrium position, find the position $y(t)$ of the mass at time t for $t \geq 0$.

Continue to consider the spring-mass system with spring constant $k = 4\,\text{N/m}$, a mass $m = 1\,\text{kg}$, and an external driving force of $6\sin(4t)\,\text{N}$ and no friction (or damping).

(b) What is the earliest time that the mass returns to its equilibrium position? (Hint: you may need to use the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.)

10. (10 points) Solve the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 + 2x_2 \\ x_2' &= -x_1 + 4x_2 \end{aligned}$$

subject to the initial conditions $x_1(0) = 1$ and $x_2(0) = 1$.