MTH 165: Linear Algebra with Differential Equations

Midterm 1

October 13, 2016

NAME (please print legibly): Solutions

Your University ID Number:

Indicate your instructor with a check in the box:

Bobkova	MWF 10:25-11:15	
Lubkin	MWF 9:00-9:50	
Rice	TR 14:00-15:15	
Vidaurre	MW 14:00-15:15	

- You have 75 minutes to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 7 pages.

QUESTION	VALUE	SCORE
1	15	
2	15	
3	20	
4	20	
5	15	
6	15	
TOTAL	100	

1. (15 points) Solve the following initial value problem in explicit form.

$$e^{-x}\frac{dy}{dx} = \frac{6x^2e^{x^3} + 2e^{x^3}}{y}, \quad y(0) = -2.$$

$$y \cdot \frac{dy}{dx} = e^x\left(6x^2e^{x^3} + 2e^{x^3}\right)$$

$$y \cdot \frac{dy}{dx} = e^{x^3+x}\left(6x^2 + 2\right)$$

$$\int y\,dy = \int e^{x^3+x}(6x^2+2)\,dx$$

$u = x^3 + x$

$du = 3x^2 + 1$

$$\int y\,dy = \int 2e^u\,du$$

$$\frac{1}{2}y^2 = 2e^u + C$$

$$\frac{1}{2}y^2 = 2e^{x^3+x} + C$$

$$y^2 = 4e^{x^3+x} + C$$

Solve for C

$(-2)^2 = 4e^0 + C$

$\rightarrow C = 0$

$$y^2 = 4e^{x^3+x}$$

$$y = -\sqrt{4e^{x^3+x}}$$

2. (15 points) Find the general solution to the following differential equation.

$$(t^2 + 1)y' + 6ty = 30t(t^2 + 1)^2.$$

$q(t)$

$$y' + \frac{6t}{t^2+1} y = 30t(t^2+1)$$

$p(t)$

$$\int \frac{6t}{t^2+1}\,dt = \int \frac{3}{u}\,du$$

$$\boxed{\begin{aligned} u &= t^2+1 \\ du &= 2t\,dt \end{aligned}}$$

$$= 3\ln|u|$$
$$= 3\ln(t^2+1),$$

so $I(t) = e^{3\ln(t^2+1)}$
$= (t^2+1)^3$

$$y = \frac{1}{I(t)} \int I(t)q(t)\,dt$$

$$y = \frac{1}{(t^2+1)^3} \int 30t(t^2+1)^4\,dt$$

$$= \frac{1}{(t^2+1)^3} \int 15u^4\,du$$

$$= \frac{1}{(t^2+1)^3} \left(3u^5 + C\right)$$

$$= \frac{1}{(t^2+1)^3} \left(3(t^2+1)^5 + C\right)$$

$$= 3(t^2+1)^2 + \frac{C}{(t^2+1)^3}$$

3. (20 points) Suppose a tank with a 40L capacity is initially filled with 10L of water in which 50g of salt is dissolved. A 3g/L solution is poured into the tank at a rate of 2L/min, while well-mixed solution is drained from the tank at a rate of 1L/min.

(a) How long does it take for the concentration of the solution in the tank to reach 4g/L?

$\frac{dV}{dt} = 1$, $V(0) = 10$ $\rightarrow$ $V = 10 + t$

$A' = r_{in} - r_{out}$

$A' = 3 \cdot 2 - 1 \cdot \frac{A}{V}$

$= 6 - \frac{A}{10+t}$

$A' + \frac{1}{10+t} A = 6$

$\int \frac{1}{10+t}\,dt = \ln(10+t)$,

so $I(t) = e^{\ln(10+t)} = 10+t$

$A = \frac{1}{10+t} \int 6(10+t)\,dt$

$= \frac{1}{10+t}\left(3(10+t)^2 + C\right) = 3(10+t) + \frac{C}{10+t}$

$50 = A(0) = 3 \cdot 10 + \frac{C}{10} \rightarrow C = 200$

$A(t) = 3(10+t) + \frac{200}{10+t}$

$c(t) = \frac{A(t)}{V(t)} = 3 + \frac{200}{(10+t)^2} = 4$

$200 = (10+t)^2$

$t = \sqrt{200} - 10$

(b) What is the concentration of the solution in the tank at the moment the tank begins to overflow?

Tank begins to overflow at $t = 30$,

and $c(30) = 3 + \frac{200}{40^2} = 3.125\ g/L$

4. (20 points) Consider the following system of equations, where x, y, z are the variables and k is a real constant.

$$x + 4y + 5z = 1$$

$$3x - y + z = 4$$

$$13y + kz = 2$$

(a) Determine which values of k cause the system to have one solution, no solutions, and infinitely many solutions, respectively.

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 3 & -1 & 1 & 4 \\ 0 & 13 & k & 2 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -13 & -14 & 1 \\ 0 & 13 & k & 2 \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -13 & -14 & 1 \\ 0 & 0 & k-14 & 3 \end{array}\right]$$

If $k \neq 14$: one solution

If $k = 14$: inconsistent row, no solution

Infinitely many solutions is not possible for this system.

(b) Solve the system with $k = 17$.

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -13 & -14 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$x + 4y + 5z = 1$$
$$-13y - 14z = 1$$
$$3z = 3$$

$z = 1$ $\longrightarrow$ $-13y - 14 = 1$ $\longrightarrow$ $x - \frac{60}{13} + 5 = 1$

$y = -\frac{15}{13}$ $\qquad$ $x = \frac{8}{13}$

5. (15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 2 & 5 & 7 \end{bmatrix}$$

(a) Find A^{-1}, or conclude that it does not exist.

Gauss - Jordan Procedure

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 4 & 5 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-4R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 8 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[-3R_3+R_1 \to R_1]{-2R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -23 & -3 & 12 \\ 0 & 1 & 0 & -10 & -1 & 5 \\ 0 & 0 & 1 & 8 & 1 & -4 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & -10 & -1 & 5 \\ 0 & 0 & 1 & 8 & 1 & -4 \end{array}\right]$$

A^{-1}

$$\begin{bmatrix} -3 & -1 & 2 \\ -10 & -1 & 5 \\ 8 & 1 & -4 \end{bmatrix}$$

(b) Find the matrix B that satisfies

$$BA - \begin{bmatrix} 1 & -1 & 2 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = 2 \begin{bmatrix} 0 & 0 & -2 \\ 1 & -1 & -3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$BA - \begin{bmatrix} 1 & -1 & 2 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 \\ 2 & -2 & -6 \\ -2 & 0 & -2 \end{bmatrix}$$

$$BA = \overbrace{\begin{bmatrix} 1 & -1 & -2 \\ 4 & 2 & 0 \\ -1 & 3 & 3 \end{bmatrix}}^{M} \longrightarrow B = MA^{-1}$$

$$B = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 2 & 0 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & -1 & 2 \\ -10 & -1 & 5 \\ 8 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -9 & -2 & 5 \\ -32 & -6 & 18 \\ -3 & 1 & 1 \end{bmatrix}$$

6. (15 points) Let $A = \begin{bmatrix} 3 & 7 & 1 \\ 0 & 5 & 2 \\ 3 & k & 5 \end{bmatrix}$, where k is a real number.

(a) Compute $\det(A)$ in terms of k.

Using column 1,

$$\det(A) = 3(25-2k) - 0 + 3(14-5)$$
$$= 102 - 6k$$

(b) Determine all possible values of $\text{rank}(A)$, along with which values of k cause those values to occur.

The first two rows guarantee $\text{rank}(A) \geq 2$,

and we know $\text{rank}(A) = 3$ if and only if $\det(A) \neq 0$,

so:

$K = 17$: $\text{rank}(A) = 2$

$k \neq 17$: $\text{rank}(A) = 3$