MTH 165: Linear Algebra with Differential Equations

Midterm 2

November 22, 2016

NAME (please print legibly): Solutions

Your University ID Number: ________

Indicate your instructor with a check in the box:

Bobkova	MWF 10:25-11:15	
Lubkin	MWF 9:00-9:50	
Rice	TR 14:00-15:15	
Vidaurre	MW 14:00-15:15	

- You have 75 minutes to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 11 pages.

QUESTION	VALUE	SCORE
1	21	
2	18	
3	21	
4	20	
5	20	
TOTAL	100	

1. (21 points) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, provide a counterexample.

(a) $V = P_2(\mathbb{R})$, the set of polynomials of degree at most 2, and $S = \{p \in V : p'(0) = 1\}$.

X ① The polynomial $p(t) = 0$ is NOT in S,
so S is not a subspace.

NO

(b) $V = M_2(\mathbb{R})$, the set of 2×2 matrices, and

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V \mid a - 4b = c - 5d \right\}.$$

YES

√ ① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in S since $0 - 4 \cdot 0 = 0 - 5 \cdot 0$.

√ ② Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ are in S,

i.e. $a - 4b = c - 5d$ and $w - 4x = y - 5z$.

$A + B = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$, and
$$\begin{aligned}(a+w) - 4(b+x) &= (a-4b) + (w-4x) \\ &= (c-5d) + (y-5z) \\ &= (c+y) - 5(d+z),\end{aligned}$$
so $A + B \in S$.

√ ③ Further, if $\lambda \in \mathbb{R}$, $\lambda A = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$, and
$$\begin{aligned}\lambda a - 4\lambda b &= \lambda(a - 4b) \\ &= \lambda(c - 5d) \\ &= \lambda c - 5\lambda d,\end{aligned}$$
so $\lambda A \in S$.

(c) $V = \mathbb{R}^2$, and $S = \{(x, y) \in V : |y| = |x|\}$.

√ ① $(0,0) \in S$ since $|0| = |0|$.

NO

X ② $(1,1)$ and $(1,-1)$ are in S, but $(1,1) + (1,-1) = (2,0)$ is NOT in S. Therefore, S is not a subspace.

2. (18 points) Answer the following questions, with justification, about a given collection of vectors in a given vector space.

(a) Do the polynomials $p_1(t) = 1 + t^2$, $p_2(t) = t^3$, and $p_3(t) = 4 - t$ span all of $V = P_3(\mathbb{R})$, the set of polynomials of degree at most 3?

The dimension of $P_3(\mathbb{R})$ is $3+1=4$,
so at least 4 vectors are required to span it,
hence the answer is NO.

Alternative solutions: Demonstrate that a specific polynomial cannot be expressed as a linear combination of p_1, p_2, p_3, or, set up a system of equations for writing a general polynomial p as a linear combination of p_1, p_2, p_3, and observe that it's possible for the system to have no solution.

(b) Are the functions $f(t) = e^t$, $g(t) = t^2$, and $h(t) = \sin(t)$ linearly independent in $V = C^2(\mathbb{R})$, the set of functions with everywhere-continuous second derivatives?

$$W(t) = \begin{vmatrix} e^t & t^2 & \sin(t) \\ e^t & 2t & \cos(t) \\ e^t & 2 & -\sin(t) \end{vmatrix}$$

YES

$$= e^t \Big[\big[-2t\sin(t) - 2\cos(t)\big] - \big[-t^2\sin(t) - 2\sin(t)\big] + \big[t^2\cos(t) - 2t\sin(t)\big] \Big]$$

For $t=0$, all terms except $e^0 = 1$ and $-2\cos(0) = -2$ are 0,
so $W(0) = -2$. Since the Wronskian isn't always 0,
the functions are linearly independent.

(c) Do the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$$

form a basis for $V = \mathbb{R}^3$?

Since the number of vectors equals the dimension, the conditions of spanning, linear independence, and being a basis are all equivalent, and all hold if and only if the rank of $A = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{bmatrix}$ is 3,

which holds if $\det(A) \neq 0$.

Using column 2,

$\det(A) = 4(1-5) = -16$,

so the answer is YES.

3. (21 points) Let

$$A = \begin{bmatrix} 1 & 2 & 7 & 9 \\ 3 & 7 & 26 & 28 \\ 5 & 11 & 40 & 46 \end{bmatrix}$$

(a) Determine a basis for the row space of A.

$$\begin{bmatrix} 1 & 2 & 7 & 9 \\ 3 & 7 & 26 & 28 \\ 5 & 11 & 40 & 46 \end{bmatrix} \xrightarrow[-5R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 7 & 9 \\ 0 & 1 & 5 & 1 \\ 0 & 1 & 5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 7 & 9 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for rowspace

= nonzero rows of row echelon form

$[1 \ \ 2 \ \ 7 \ \ 9]$

$[0 \ \ 1 \ \ 5 \ \ 1]$

(b) Determine a basis for the column space of A.

Basis for column space

= columns of original matrix corresponding to pivots of row echelon form

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 11 \end{bmatrix}$$

(c) Determine a basis for the nullspace of A.

$$\left[\begin{array}{cccc|c} 1 & 2 & 7 & 9 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 7x_3 + 9x_4 = 0$$
$$x_2 + 5x_3 + x_4 = 0$$

Let $x_3 = s$, $x_4 = t$, so

$$x_2 + 5s + t = 0$$
$$x_2 = -5s - t$$

$$x_1 + 2(-5s - t) + 7s + 9t = 0$$

$$x_1 = 3s - 7t$$

$$\text{null}(A) = \left\{ \begin{bmatrix} 3s-7t \\ -5s-t \\ s \\ t \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} 3 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis: $\begin{bmatrix} 3 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

4. (20 points) Answer the following about a 31×14 matrix A (that is, a matrix with 31 rows and 14 columns) with $\text{rank}(A) = 14$. No justification is required for parts (a)-(c).

(a) rowspace(A) is a 14 -dimensional subspace of $\mathbb{R}^d$ with $d =$ 14

(b) colspace(A) is a 14 -dimensional subspace of $\mathbb{R}^d$ with $d =$ 31

(c) null(A) is a 0 -dimensional subspace of $\mathbb{R}^d$ with $d =$ 14

(d) Are the rows of A linearly independent? Why or why not?

No The dimension of the span of the rows (14), is less than the number of rows (31).

(e) Are the columns of A linearly independent? Why or why not?

Yes The dimension of the span of the columns (14), is equal to the number of columns.

5. (20 points) Answer the following, with justification, about the function

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a+d \\ b+c & a+b+c+d \end{bmatrix}.$$

(a) Show that T is a linear transformation.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

$$T(A+B) = T\left(\begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}\right) = \begin{bmatrix} a+w & a+w+d+z \\ b+x+c+y & a+w+b+x+c+y+d+z \end{bmatrix}$$

$$= \begin{bmatrix} a+w & (a+d)+(w+z) \\ (b+c)+(x+y) & (a+b+c+d)+(w+x+y+z) \end{bmatrix} = T(A) + T(B).$$

Further, if $\lambda \in \mathbb{R}$, $T(\lambda A) = \begin{bmatrix} \lambda a & \lambda a + \lambda d \\ \lambda b + \lambda c & \lambda a + \lambda b + \lambda c + \lambda d \end{bmatrix} = \lambda \begin{bmatrix} a & a+d \\ b+c & a+b+c+d \end{bmatrix}$

$= \lambda \cdot T(A)$.

(b) Is $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$ in the kernel of T?

$$T\left(\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0+0 \\ 5+-5 & 0+5-5+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

YES

(c) Is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the range of T?

The question is if $\begin{bmatrix} a & a+d \\ b+c & a+b+c+d \end{bmatrix}$

could ever equal $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

But, if $a+d = b+c = 0$, then $a+b+c+d = (a+d)+(b+c) = 0$,

so this is impossible.

(d) What is $\dim(\ker(T)) + \dim(\text{Rng}(T))$?

By the Rank-Nullity Theorem,

this is the dimension of $M_2(\mathbb{R})$,

which is 4.