MTH 165: Linear Algebra with Differential Equations

Final Exam

May 7, 2012

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50 AM	
Ang Wei	MW 2:00 - 3:15 PM	

- The presence of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Clearly circle or label your simplified final answers.
- You are responsible for checking that this exam has all 11 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Solve the following initial value problems:

(a) (5 points)

$$(e^t+1)\frac{dy}{dt} + e^t y = 1, \quad y(0) = 1;$$

$$\frac{d}{dt}\left((e^t+1)y(t)\right) = 1$$

$$(e^t+1)y(t) = \int 1\,dt = t + C$$

So $y(t) = \frac{t}{e^t+1} + \frac{C}{e^t+1}$

$$1 = y(0) = \frac{0}{1+1} + \frac{C}{1+1} = \frac{C}{2} \quad \text{so } C = 2.$$

$$\boxed{y(t) = \frac{t}{e^t+1} + \frac{2}{e^t+1}}$$

(b) (5 points)

$$y' = 1 + x + y + xy, \quad y(0) = 0.$$

$$\frac{dy}{dx} = (1+x)(1+y)$$

$$\int \frac{dy}{1+y} = \int (1+x)\,dx$$

$$\ln|1+y| = x + \tfrac{1}{2}x^2 + C$$

Using $y(0)=0$ get $\ln(1+0) = 0 + \frac{1}{2}\cdot 0^2 + C$ or $C = 0$

Thus $1 + y(x) = e^{x+\frac{1}{2}x^2}$ or

$$\boxed{y(x) = e^{x+\frac{1}{2}x^2} - 1}$$

2. **(10 points)** Consider an RC circuit which has $R = 4\,\Omega$, $C = 1/8\,F$, and $E(t) = 12\sin 3t\,V$. If the capacitor is uncharged initially, determine the current in the circuit for $t \geq 0$.

$$\frac{dq}{dt} + 2q = 3\sin(3t) \quad , \; q(0) = 0.$$

Integrating factor $e^{\int 2\,dt} = e^{2t}$

$$\frac{d}{dt}\left(e^{2t} q\right) = 3e^{2t}\sin(3t)$$

$$e^{2t} q(t) = \int 3e^{2t}\sin(3t)\,dt$$

$$I = \int 3e^{2t}\sin(3t)\,dt = \frac{3}{2}e^{2t}\sin(3t) - \int \frac{9}{2}e^{2t}\cos(3t)\,dt$$

$$= \frac{3}{2}e^{2t}\sin(3t) - \frac{9}{2}\left(\frac{1}{2}e^{2t}\cos(3t) + \int \frac{3}{2}e^{2t}\sin(3t)\,dt\right)$$

$$= \frac{3}{2}e^{2t}\sin(3t) - \frac{9}{4}e^{2t}\cos(3t) - \frac{9}{4}I$$

Thus $\int 3e^{2t}\sin(3t)\,dt = \frac{6}{13}e^{2t}\sin(3t) - \frac{9}{13}e^{2t}\cos(3t) + C$

Hence

$$q(t) = \frac{6}{13}\sin(3t) - \frac{9}{13}\cos(3t) + Ce^{-2t}$$

$$0 = q(0) = \frac{6}{13}\cdot 0 - \frac{9}{13}\cdot 1 + C = C - \frac{9}{13} \quad \text{so} \quad C = \frac{9}{13}$$

$$q(t) = \frac{6}{13}\sin(3t) - \frac{9}{13}\cos(3t) + \frac{9}{13}e^{-2t}$$

Thus current

$$i(t) = \frac{dq}{dt} = \frac{18}{13}\cos(3t) + \frac{27}{13}\sin(3t) - \frac{18}{13}e^{-2t}$$

3. (10 points)

(a) (5 points) Construct two matrices A and B of the same dimensions, for which

$$\text{rank}(A+B) \neq \text{rank}(A) + \text{rank}(B).$$

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $\text{rank}(A) = 1$ and $\text{rank}(B) = 1$

But $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has rank 0

and $1+1 \neq 0$.

(b) (5 points) Construct two matrices A and B of appropriate dimensions, such that

$$\text{rank}(AB) \neq \text{rank}(A) \cdot \text{rank}(B).$$

Pick $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\text{rank}(A) = 1$ and $\text{rank}(B) = 1$

But $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has rank 0

and $0 \neq 1 \cdot 1$

4. (10 points) For each of the following subsets of $M_{3\times 3}(\mathbb{R})$, determine whether it is a subspace. If that is the case, find its dimension.

(a) (5 points) S is the set of matrices $A \in M_{3\times 3}(\mathbb{R})$ satisfying

$$A^T = 2A.$$

First check $0^T = 0 = 2 \cdot 0$ so 0 is in the set. Take A, B from the set. Then $A^T = 2A$ and $B^T = 2B$. $(A+B)^T = A^T + B^T = 2A + 2B = 2(A+B)$ so $A+B$ also in the set. Closed under addition. Let k be a scalar. $(kA)^T = kA^T = k2A = 2(kA)$ so kA also in the set. Closed under scalar Multiplication. $\boxed{\text{Is a subspace}}$

To find the dimension take $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$ from the set. Then

$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix} = A^T = 2A = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2j \end{bmatrix}$. Thus $a=2a$, $e=2e$, $j=2j$, $d=2b$, $b=2d$, $g=2c$, $c=2g$, $h=2f$, $f=2h$ which yields $a=b=c=d=e=f=g=h=j=0$

Thus $S=\{0\}$ and by definition the $\boxed{\text{dimension is } 0}$

(b) (5 points) S is the set of matrices $B \in M_{3\times 3}(\mathbb{R})$ verifying

$$B^T + 3B = 5\,I_{3\times 3}.$$

$0^T + 3 \cdot 0 = 0 + 0 = 0 \neq 5 \cdot I_{3\times 3}$

Thus 0 not in the set so S is not a subspace.

5. **(10 points)** Show that the vectors $v_1 = (2, -3, 5)$, $v_2 = (8, -12, 20)$, $v_3 = (1, 0, -2)$, $v_4 = (0, 2, -1)$, and $v_5 = (7, 2, 0)$ span $\mathbb{R}^3$. Find a subset of the set $\{v_1, v_2, v_3, v_4, v_5\}$ which is a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix}$$

$$\overset{A_{21}(1)}{\sim} \begin{bmatrix} -1 & -4 & 1 & 2 & 9 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix}$$

$$\overset{M_1(-1)}{\sim} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix}$$

$$\underset{A_{13}(-5)}{\overset{A_{12}(3)}{\sim}} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & -3 & -4 & -25 \\ 0 & 0 & 3 & 9 & 45 \end{bmatrix}$$

$$\overset{A_{32}(1)}{\sim} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & 0 & 5 & 20 \\ 0 & 0 & 3 & 9 & 45 \end{bmatrix}$$

$$\underset{M_3(\frac{1}{3})}{\overset{M_2(\frac{1}{5})}{\sim}} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 15 \end{bmatrix}$$

$$\overset{P_{23}}{\sim} \begin{bmatrix} 1 & 4 & -1 & -2 & -9 \\ 0 & 0 & 1 & 3 & 15 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

rank is 3 so span $\mathbb{R}^3$

A subset which is a basis

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

6. (10 points) Let P_2 denote the vector space of polynomials with real coefficients and degree at most 2. Define $T : P_2 \to P_2$ by

$$T(f) = f(0)\,x + f'(2)\,x^2.$$

(a) (3 points) Show that T is a linear transformation.

$$T(c_1 f + c_2 g) = (c_1 f + c_2 g)(0)x + (c_1 f + c_2 g)'(2)x^2$$
$$= c_1 f(0)x + c_2 g(0)x + c_1 f'(2)x^2 + c_2 g'(2)x^2$$
$$= c_1(f(0)x + f'(2)x^2) + c_2(g(0)x + g'(2)x^2)$$
$$= c_1 T(f) + c_2 T(g)$$ Thus T linear transf.

(b) (3 points) Determine a basis for Ker(T). What is dim[Ker(T)]?

If ~~p(x)~~ $a + bx + cx^2$ is in $Ker(T)$ then $T(p) = 0$

Note $p(0) = a$ and $p'(x) = b + 2cx$ so $p'(2) = b + 4c$.

Thus have $0 = ax + (b+4c)x^2$ which shows that $a = 0$ and $b = -4c$. Thus $p \in Ker(T)$ if $p(x) = c(-4x + x^2) \in span\{-4x + x^2\}$. Hence $\{-4x + x^2\}$ a basis for $Ker(T)$ and $dim(Ker(T)) = 1$

(c) (2 points) Find dim[Rng(T)].

$dim(P_2) = 3$ so by the generalized rank-nullity theorem

$$dim(Rng(T)) = dim(P_2) - dim(Ker(T)) = 3 - 1 = 2$$

(d) (2 points) Is the polynomial $5x + 3$ in the range of T? Explain.

$5x + 3$ is not in the range of T because else we would have

$$5x + 3 = f(0)x + f'(2)x^2 \text{ for some } f \in P_2$$

but for two polynomials to be equal then all coefficients must be equal so since the constant coefficients are 3 and 0, $3 \neq 0$, then that equality can never hold.

7. (10 points) Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

(a) (4 points) Determine its eigenvalues and their multiplicities.

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 1 & 4-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 3-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda)(4-\lambda) + (\lambda-3) = (\lambda-3)(1-(\lambda-4)^2) = (-1)(\lambda-3)^2(\lambda-5)$$

Eigenvalues are $\underline{3}$ with $\underline{\text{multiplicity } 2}$ and $\underline{5}$ with $\underline{\text{multiplicity } 1}$.

(b) (4 points) Compute the eigenspaces corresponding to each of the eigenvalues and their dimensions.

$\underline{\lambda=3}$: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \underset{A_{13}(-1)}{\overset{A_{12}(-2)}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ v_2, v_3 free variables

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ so $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ basis for eigenspace and dimension thus 2.

$\lambda=5$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \underset{A_{13}(1)}{\overset{A_{12}(2)}{\sim}} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{M_2(-\frac{1}{2})}{\overset{M_1(-1)}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ v_3 free variable

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 \\ 2v_3 \\ v_3 \end{bmatrix} = v_3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ so $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ basis for eigenspace and dimension thus 1

(c) (2 points) Conclude, with explanation, whether A is a defective or non-defective matrix.

A is non-defective because the geometric multiplicity matches the algebraic multiplicity for each of the eigenvalues.

8. (10 points) Solve the initial value problem

typo in problem

$$y'' - 4y' + 8y = 0, \quad y(0) = 3, \quad y'(0) = 10.$$

$$P(r) = r^2 - 4r + 8$$

roots $\dfrac{4 \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 8}}{2} = \dfrac{4 \pm 4i}{2} = 2 \pm 2i$

$2 \pm 2i$: $e^{2t}\cos(2t)$
$e^{2t}\sin(2t)$

Gen. sol. $y(t) = c_1 \cdot e^{2t}\cos(2t) + c_2 e^{2t}\sin(2t)$

$3 = y(0) = c_1 \cdot 1 \cdot 1 + c_2 \cdot 1 \cdot 0 = c_1$

$y(t) = 3e^{2t}\cos(2t) + c_2 e^{2t}\sin(2t)$

$y'(t) = 6e^{2t}\cos(2t) - 6e^{2t}\sin(2t) + 2c_2 e^{2t}\sin(2t) + 2c_2 e^{2t}\cos(2t)$

$10 = y'(0) = 6\cdot 1\cdot 1 - 6\cdot 1\cdot 0 + 2\cdot c_2 \cdot 1 \cdot 0 + 2\cdot c_2\cdot 1\cdot 1 = 6 + 2c_2$

and thus $c_2 = 2$.

Solution

$$y(t) = 3e^{2t}\cos(2t) + 2e^{2t}\sin(2t)$$

9. **(10 points)** Determine the general solution to

$$y'' - 5y' + 6y = 4xe^{2x}.$$

First solve $y''-5y'+6y=0$

$P(r)=r^2-5r+6 = (r-3)(r-2)$

$r=2: \; e^{2x}$

$r=3: \; e^{3x}$

solution $y_c(x) = c_1 e^{2x} + c_2 e^{3x}$

Trial solution $y_p(x) = x(a+bx)e^{2x} = axe^{2x} + bx^2e^{2x}$

Note

$y_p'(x) = ae^{2x} + 2axe^{2x} + 2bxe^{2x} + 2bx^2e^{2x}$

$y_p''(x) = 2ae^{2x} + (2a+2b)e^{2x} + 2(2a+2b)xe^{2x} + 4bxe^{2x} + 4bx^2e^{2x}$

$= \cancel{4ae^{2x}} \; (4a+2b)e^{2x} + (4a+8b)xe^{2x} + 4bx^2e^{2x}$

Plugging in yields

$(4a+2b)e^{2x} + (4a+8b)xe^{2x} + 4bx^2e^{2x} - 5\left(ae^{2x} + (2a+2b)xe^{2x} + 2bx^2e^{2x}\right)$

$+ 6\left(axe^{2x} + bx^2e^{2x}\right) = 4xe^{2x}$

$(2b-a)e^{2x} - 2bxe^{2x} = 4xe^{2x}$

Thus $-2b=4$ and $(2b-a)=0$ so $b=-2$ and $a=-4$

Gen. sol.

$y(x) = c_1e^{2x} + c_2e^{3x} - 4xe^{2x} - 2x^2e^{2x}$

10. (10 points)

(a) (5 points) Determine the annihilator for the function

$$F(x) = x\,e^{-x} + 2\cos x.$$

$(D+1)^2$ annihilates xe^{-x}

D^2+1 annihilates $2\cos(x)$

Thus $A(D) = (D+1)^2(D^2+1)$ annihilates $F(x)$

(b) (5 points) Using the information obtained previously, find the general solution to

$$y''' - y' = x\,e^{-x} + 2\cos x.$$

$P(r) = r^3 - r = r(r^2-1) = r(r-1)(r+1)$

$r=-1$: e^{-x}

$r=0$: $e^{0\cdot x} = 1$

$r=1$: e^{x}

Trial solution $y_p(x) = x(a+bx)e^{-x} + c\cos(x) + d\sin(x)$

$= axe^{-x} + bx^2e^{-x} + c\cdot\cos(x) + d\sin(x)$

$y_p'(x) = ae^{-x} - axe^{-x} + 2bxe^{-x} - bx^2e^{-x} - c\sin(x) + d\cos(x)$

$y_p''(x) = -ae^{-x} + (2b-a)e^{-x} + (a-2b)xe^{-x} - 2bxe^{-x} + bx^2e^{-x} - c\cos(x) - d\sin(x)$

$= (2b-2a)e^{-x} + (a-4b)xe^{-x} + bx^2e^{-x} - c\cos(x) - d\sin(x)$

$y_p'''(x) = (2a-2b)e^{-x} + (a-4b)e^{-x} + (4b-a)xe^{-x} + 2bxe^{-x} - bx^2e^{-x} + c\sin(x) - d\cos(x)$

Plug into equation and get $2a-6b=0$, $4b=1$, $2c=0$, $-2d=2$. Thus gen. sol.

$y(x) = c_1e^{-x} + c_2 + c_3e^{x} + \frac{3}{4}xe^{-x} + \frac{1}{4}x^2e^{-x} - \sin(x)$